Eliciting Coordination with Rebates

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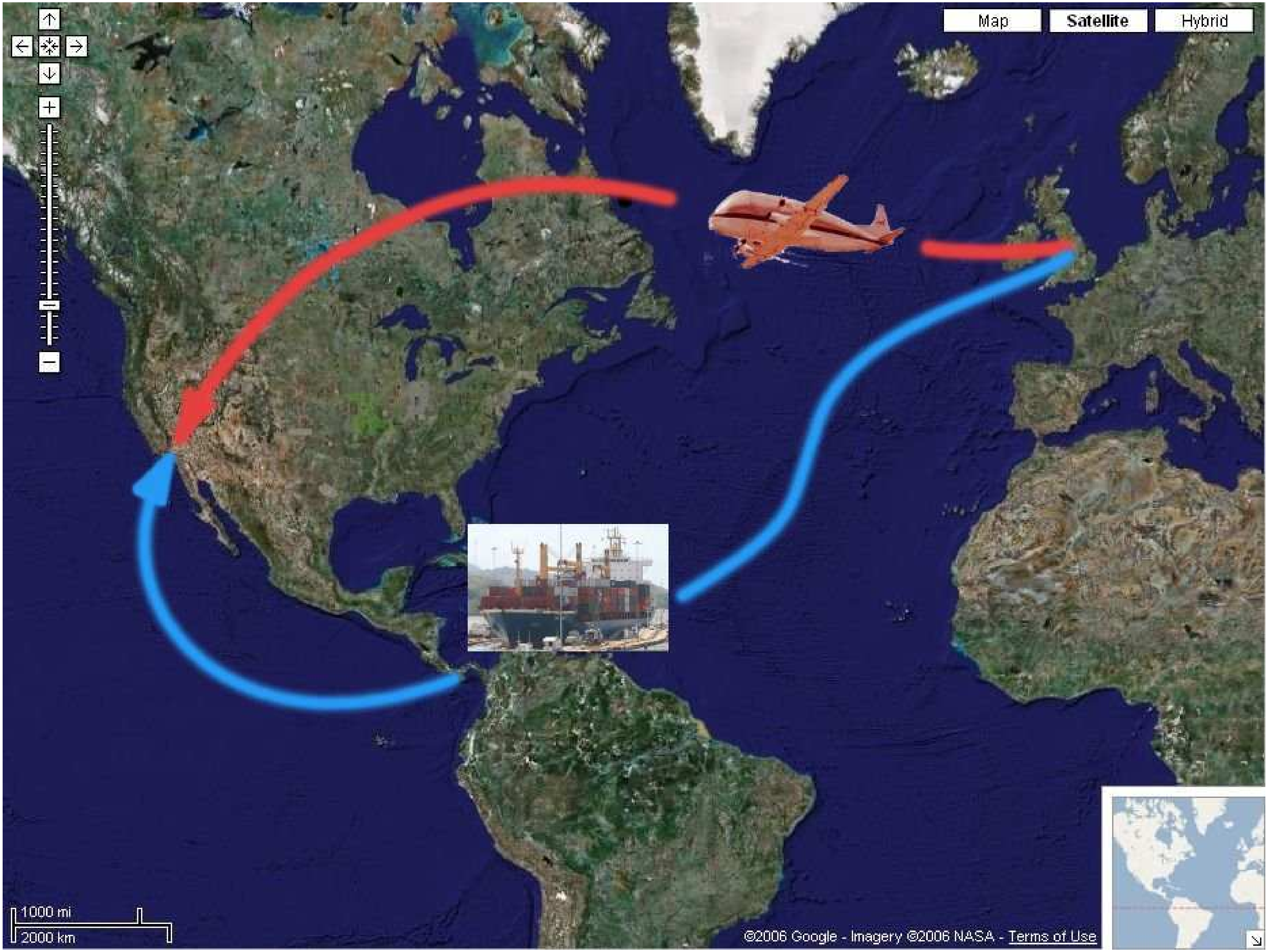
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Competitors make selfish decisions

How efficient is the system?

centralized	**competition**
global perspective	selfish
social optimum =	**Nash equilibrium** =
min total cost s.t. solution is feasible	$\forall$ user: min individual cost s.t. solution is feasible
efficient, unfair	not efficient, no regret


HSBC

Price of Anarchy

- Ideas and techniques: in the interface of Management Science, Economics and Computer Science

- Use worst-case analysis to measure efficiency of equilibria

- Compare equilibria to good upper bounds on social surplus

$\longrightarrow$ Usually social optimum

POA=worst-case performance ratio of equilibrium to optimum

POA measures the "price" of not having central coordination in system

Price of Anarchy: Consequences

If **POA** small, owner may want to let participants choose:
there is not much to gain from dictating what people should do!

If **POA** large, owner may want to re-design system or to give incentives to achieve more efficient results:

- Mechanism Design
  e.g., if certain design guidelines are used, then equil. $\approx$ opt.

- Pricing and Stackelberg games
  e.g., system's owner prices resources s.t. equil. $=$ opt.

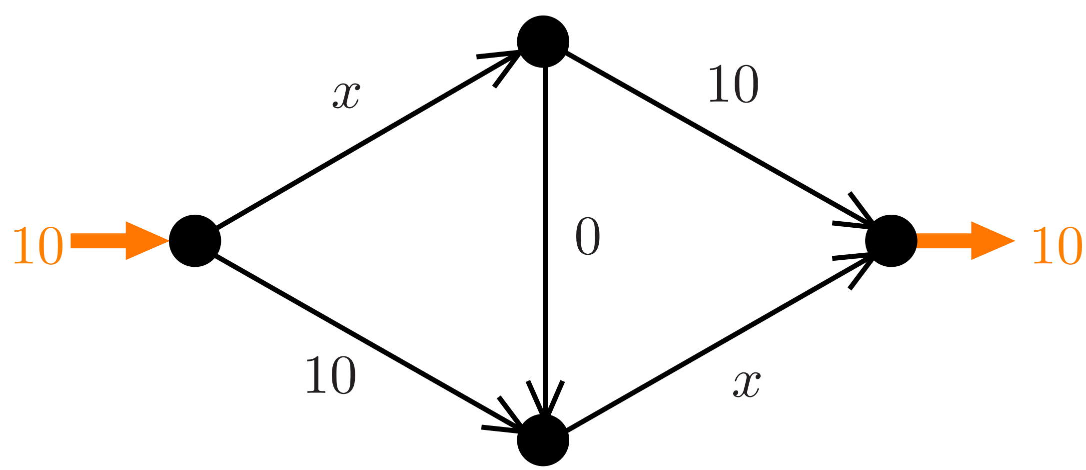
This talk: system's owner offers rebates

Outline

- Competition among Price-Taking Participants
  Nash Equilibrium & Social Optimum
- The Price of Anarchy
- Eliciting Coordination with Rebates

The Competitive Model

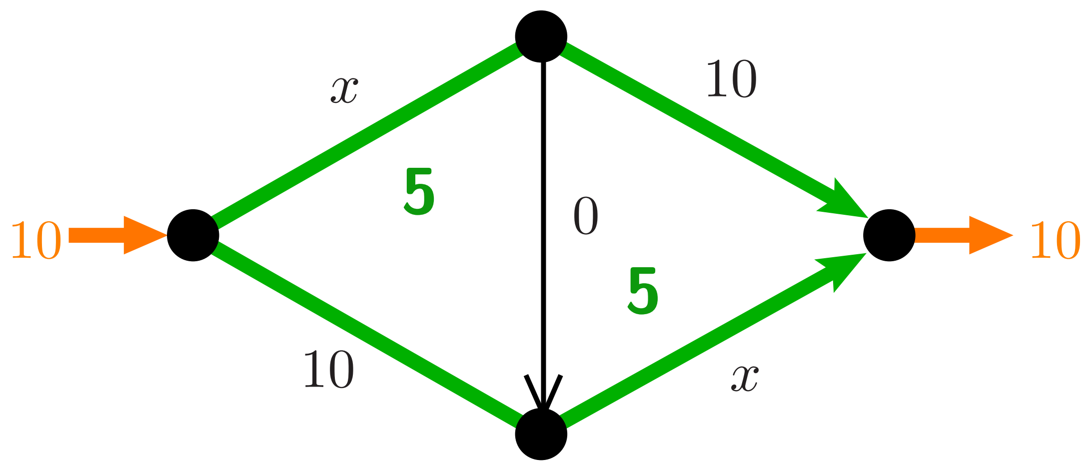
- Network $G = (N, A)$ with OD pairs of rate r_k, $k \in K$
- Resources have cost functions c_a depending on total demand x_a
- Participants are price-takers
- Example: Braess' Instance



Social Optimum

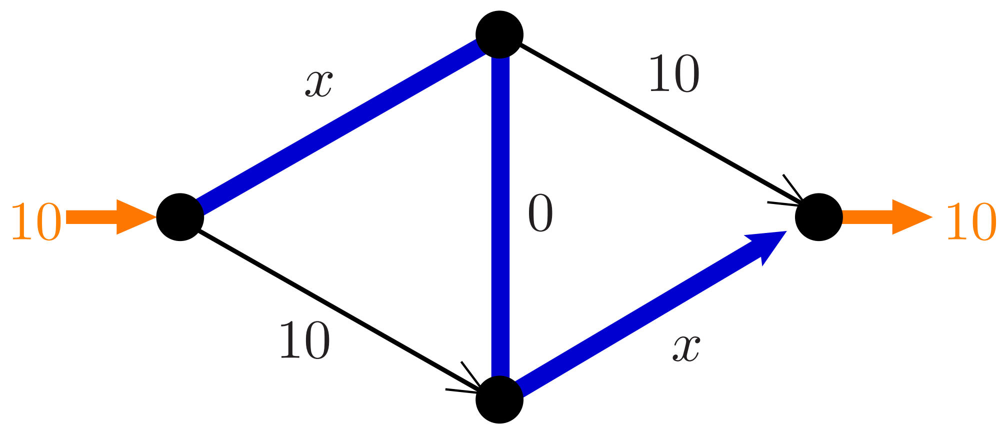
- Participants' Cost $C(x) \ := \ \sum_{a \in A} c_a(x_a)x_a$

- A **SO** flow x^{SO} is a feasible flow that minimizes $C(x)$



Nash Equilibrium

Definition: A flow x^{NE} is a **NE** of the network game if no participant can switch to a path with smaller cost — Wardrop'52



NE exists & essentially unique — Beckmann, McGuire & Winsten'56

Price of Anarchy

Price of Anarchy measures impact of lack of central coordination

Papadimitriou STOC'01

$$\textbf{POA} := \max_{\text{instances}} \frac{C(\textbf{NE})}{C(\textbf{SO})}$$

- For unrestricted cost functions, **POA** is unbounded
- We will assume a fixed set of latency functions $\mathcal{C}$

Bounds on the Price of Anarchy

Roughgarden & Tardos '02, Roughgarden '03

Correa, Schulz & S.M. '04,'05

Theorem. For polynomials of maximum degree p,
$C(\textbf{NE})/C(\textbf{SO})$ is bounded by

degree	1	2	3	4	...	p
POA	$4/3$	1.626	1.896	2.151	...	$\Omega(p/\ln p)$

Corollary. Braess' Instance is worst possible

Game and system objectives are 'aligned'.
Then, selfishness drives the system close to optimality

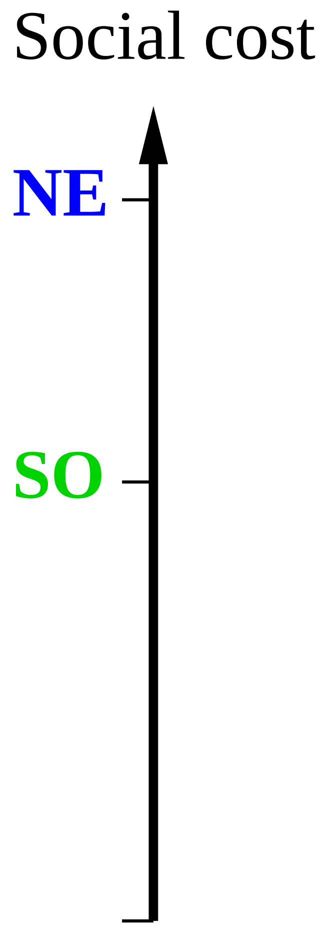
Is the SO a relevant reference?

- We wanted to quantify the cost of lacking coordination
- By comparing **NE** to **SO**, we implicitly acknowledge that the latter is attainable

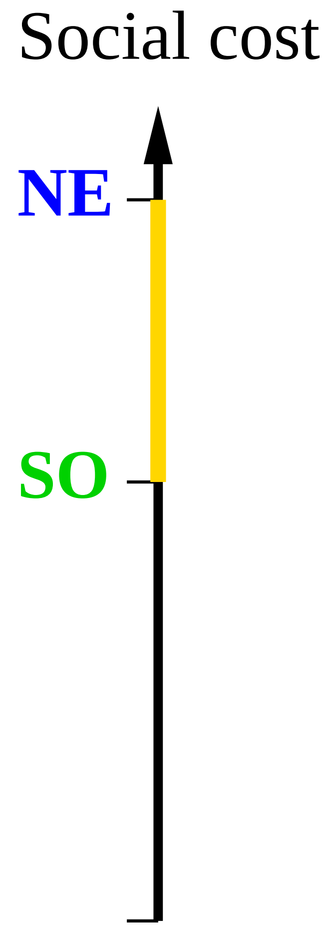
> But this is not necessarily true!

- E.g., prices for different participants under a **SO** may vary:
  its unfairness may prevent its implementation
- We propose to compare to an achievable coordinated solution

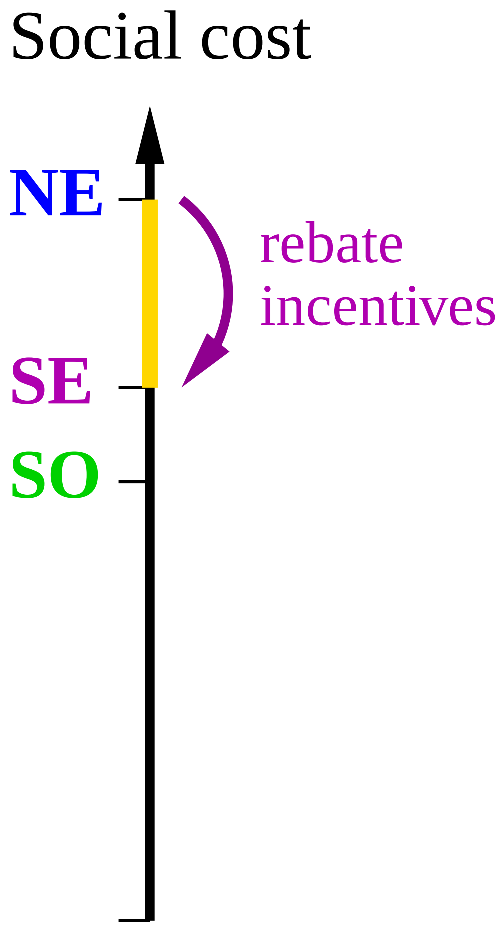
The Owner Could Save by Eliciting Coordination



The Owner Could Save by Eliciting Coordination



The Owner Could Save by Eliciting Coordination



Stackelberg Game

We focus on monetary incentives

A mechanism where participants pay extra for popular resources may not be acceptable

It may be more fair to offer rebates for unpopular ones

- system's owner offers rebate s_a for resource a
- participants perceive cost $[c_a(x_a) - s_a]^+$
- actual rebate paid $= \min(s_a, c_a(x_a))$
  (perceived costs can't be negative: rebates cannot exceed costs)

Eliciting Coordination with Rebates

- The system's owner (Stackelberg leader) offers rebates to participants

- Owner computes optimal rebates predicting participants' reactions

- The vector of rebates s together with the corresponding 2^{nd}-stage solution x^s represent a Stackelberg equilibrium (**SE**)

The leader **modifies the participants' game** to elicit a more desirable outcome

Network Pricing

The following articles use Stackelberg games and pricing to

- min. participants' cost — Dupuit 1849, Pigou'20, Knight'24, . . .
- max. owner's profit — Acemoglu & Ozdaglar'03
- min. social cost (including prices paid) — Cole, Dodis & Roughgarden'06

Although in some articles negative values for prices (rebates) were allowed, full efficiency was a requirement.

Instead, we go for the solution that minimizes the social cost

Social Cost to be Minimized

$$C_\rho(s) := \underbrace{\sum x_a^{\mathsf{s}}\,[c_a(x_a^{\mathsf{s}}) - s_a]^+}_{\text{participants' perceived cost}} + \rho \underbrace{\sum x_a^{\mathsf{s}} \min(c_a(x_a^{\mathsf{s}}), s_a)}_{\text{rebate cost}}$$

$$= \underbrace{\sum x_a^{\mathsf{s}} c_a(x_a^{\mathsf{s}})}_{\text{participants' real cost}} + (\rho - 1) \underbrace{\sum x_a^{\mathsf{s}} \min(c_a(x_a^{\mathsf{s}}), s_a)}_{\text{rebate cost}}$$

The parameter ρ is the sensitivity of the leader to the cost of rebates

- $\rho \to 0$: unlimited rebate budget $\Rightarrow s = (c_a(x_a^{\mathsf{SO}}))_{a \in A}$ and $x^s = x^{\mathsf{SO}}$
- $\rho = 1$: two terms of social cost equally important $\Rightarrow C(\textbf{SO}) \leq C_\rho(s) \leq C(\textbf{NE})$
- $\rho \to \infty$: the leader cannot afford rebates $\Rightarrow s = 0$ and $x^s = x^{\mathsf{NE}}$

Price of Anarchy Revisited

We redefine the price of anarchy in the Stackelberg game as:

$$\textbf{SPOA} := \frac{C(\textbf{NE})}{C(\textbf{SE})} = \frac{C_\rho(0)}{\min_{s \geq 0} C_\rho(s)}$$

As this compares an equilibrium to what can be achieved with rebates, **SPOA** is a less pessimistic estimate for the efficiency-loss

If **SPOA** is $\begin{cases} \text{large, then rebates improve the system} \\ \text{small, then additional coordination is beneficial} \end{cases}$

Case $\rho \leq 1$

Participants' perceived cost is more important than rebate cost to owner

$$C_\rho(s) = \underbrace{\sum x_a^{\mathsf{s}} \left[c_a(x_a^{\mathsf{s}}) - s_a\right]^+}_{\text{participants' perceived cost}} + \rho \underbrace{\sum x_a^{\mathsf{s}} \min(c_a(x_a^{\mathsf{s}}), s_a)}_{\text{rebate cost}}$$

- The optimal rebate is to offer $c_a(x_a^{\mathsf{SO}})$
- In the 2^{nd}-stage game, the **SO** is the unique equilibrium
- The inefficiency is **SPOA** $= \dfrac{C_\rho(0)}{\min_s C_\rho(s)} = \dfrac{C(\mathbf{NE})}{\rho C(\mathbf{SO})} = \dfrac{\mathbf{POA}}{\rho}$

Case $\rho \geq 1$

Theorem. For substitutable resources and affine costs:

We find optimal rebates and a corresponding equilibrium

Also, **SPOA** $\leq \dfrac{4\rho}{4\rho - 1}$ (tight bound)

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Theorem. For substitutable resources and affine costs:

We find optimal rebates and a corresponding equilibrium

Also, **SPOA** $\leq \dfrac{4\rho}{4\rho - 1}$ (tight bound)

- If $\rho \to 1$, then **SPOA** $\to$ **POA** $= 4/3$

  small ρ, system benefits from rebate mechanism

- If $\rho \to \infty$, then **SPOA** $\to 1$

  large ρ, coordination not useful

Final Remarks: Other Situations To Consider

Model Characteristics:

- participants: **price-takers** / price-setters
- market structure: **substitutable resources** / single / multiple commodities
- cost functions: **affine** / general
- Participants have different price sensitivities

Goals:

(i) the computation of optimal rebates

(ii) the study of the price of anarchy