VISUAL DETECTION OF APERIODIC SPATIAL STIMULI BY PROBABILITY SUMMATION AMONG NARROWBAND CHANNELS[1]

NORMA GRAHAM

Department of Psychology, Columbia University, New York, NY 10027, U.S.A.

Abstract—Recent psychophysical results of Shapley and Tolhurst and of Kulikowski and King-Smith have suggested that the visual system contains broadband channels like "edge detectors" and "line detectors" as well as relatively narrowband "spatial frequency" channels. These recent results (including thresholds for aperiodic stimuli) can be completely explained using only relatively narrowband channels with probability summation among them. This explanation requires many fewer free parameters than the original explanation based on both broadband and narrowband channels. The bandwidths of the individual narrowband channels can be estimated and are similar to those previously estimated from sine-wave summation experiments.

INTRODUCTION

A number of early attempts to model the human visual system used single-channel models. In single-channel models, the important stage of the visual system is a collection of receptive fields all of the same kind (e.g. size). Although this kind of model can account for a number of visual phenomena, it is inadequate for many others. In particular, this kind of model has not been able to account for the results of some threshold-summation experiments in which the detectability of a compound stimulus is compared to the detectability of each of the components of the compound stimulus (e.g. Campbell and Robson, 1968; Thomas, 1970; Graham and Nachmias, 1971; Sachs, Nachmias and Robson, 1971).

To account for the results of these threshold-summation experiments as well as for other phenomena, multiple-channels models have been suggested (Campbell and Robson, 1968; Pantle and Sekular, 1968; Thomas, 1970; Kulikowski and King-Smith, 1973). In multiple-channels models there are receptive fields of many different kinds (of, for example, many

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[2] The word "detector" will be used, as Shapley and Tolhurst (1973) and Kulikowski and King-Smith (1973) used it, to refer to a single receptive field. Thus a channel is made up of many detectors. Notice that both a particular channel and the receptor fields (detectors) making up that particular channel will respond to the same range of spatial frequencies.

[3] Jacob Nachmias first pointed out to me the possible implications of probability summation for explaining the experimental results of Shapley and Tolhurst (1973) and Kulikowski and King-Smith (1973). I am grateful to him and to Robert Shapley, John Robson, Dave Tolhurst, Ewen King-Smith and Janusz Kulikowski for many interesting discussions. Don Hood and Bernice Rogowitz read a draft of this paper and made many useful suggestions.

different sizes and orientations). In the terminology to be used in this paper, each channel of a multiple-channels model contains a collection of receptive fields of the *same* kind (e.g. size and orientation) with the positions of the receptive fields distributed uniformly across the visual field.

Within the framework of multiple-channels models, the question naturally arises as to what the characteristics of the various channels are. Shapley and Tolhurst (1973) and Kulikowski and King-Smith (1973) did threshold-summation experiments using compound stimuli composed of sinusoidal (narrowband) and aperiodic (broadband) spatial stimuli. They interpreted their results in terms of various classes of channels or detectors,[2] some responding to a relatively narrow band of spatial frequencies and some to a broad band. As will be shown here, however, these same experimental results might be equally well interpreted in terms of only one class of detector or channel, those responding to relatively narrow bands of spatial frequencies.

The reason for the difference between the conclusion of Shapley and Tolhurst (1973) and Kulikowski and King-Smith (1973) and the conclusion reached here is in the assumption about variability. The original investigators assumed that there is no independent variability in the responses of different receptive fields—therefore, a given stimulus is always detected by the same receptive field, the most sensitive one. One might equally well assume, however, that there is independent variability in the responses of different receptive fields or, at least, of receptive fields in different channels. Therefore, on one trial one channel may detect a given stimulus while on another trial, another channel may. This alternate assumption will be called the assumption of probability summation among channels.[3]

How probability summation among only relatively narrowband channels might account for the experimental results of Shapley and Tolhurst (1973) and Kulikowski and King-Smith (1973) is explained without using mathematics in Graham (1977). A rather simple but approximate quantitative prediction of some of their experimental results is also explained

there. An exact quantitative prediction of most of their experimental results (including thresholds for aperiodic stimuli) is reported in the present paper. Use of the convenient mathematical form for psychometric functions that was suggested by Quick (1974) allowed these exact predictions to be derived. In the process, an estimate of channel bandwidth and some information about how many channels respond to a pure sine was obtained.

ASSUMPTIONS OF THE PROBABILITY SUMMATION MODEL

The seven major assumptions used in this paper are introduced and discussed in the following sections. The first three of the seven assumptions are basic to the model being tested, the model of probability summation among multiple, relatively narrowband channels. The second four assumptions were the minimal reasonable set of assumptions that could be found that would allow predictions to be calculated.

A glossary of the symbols and special terms used in this paper is presented as Appendix 2 for reference. The symbols will also be defined as they are introduced in the following. In any case, the reader can skip all equations and most symbols and still understand the main points.

A few general introductory comments may help make the symbols less confusing. c will refer to the contrast of a stimulus, f to a spatial frequency, P to a probability, and R to a response magnitude. S refers to a sensitivity where a sensitivity is, as is conventional, the reciprocal of a threshold measure.

Many symbols will contain both an argument in parentheses and a subscript—for example, $R_i(f)$ designates the magnitude of the response of the ith spatial frequency channel to a sine of frequency f. The argument in parentheses always refers to the stimulus involved either by name or, if the stimulus is a sine-wave, by frequency. When the subscript is a single character, as in $R_i(f)$, the whole symbol will refer to some property of a channel and the subscript indicates which of the channels it is. In two cases, the subscript will be the word *test* and the precise definition will be given later. When there is no subscript, the symbol will refer to a property of the visual system as a whole. For example, $S(f)$ is psychophysical contrast sensitivity, that is, the reciprocal of the observer's contrast threshold for a sine-wave of frequency f.

Assumption 1. Probability summation among multiple channels

It will be assumed (a) that an observer detects a stimulus whenever at least one of the multiple channels detects a stimulus, and (b) that the variability in different channels is completely uncorrelated. Or, formally, letting $P(\text{stim})$ be the probability that the observer detects the stimulus, $P_i(\text{stim})$ be the probability that the ith channel detects the stimulus, and N be the total number of channels, it is assumed that

$$P(\text{stim}) = 1 - \prod_{i=1}^{N} [1 - P_i(\text{stim})]. \tag{1}$$

In words, the probability of the observer's detecting a stimulus is just one minus the probability that no

channel detects it. And the probability that no channel detects it is just the product of the probabilities of each channel's not detecting it.

There is considerable experimental support for this assumption about the multiple channels, at least for the channels involved in sine-plus-sine experiments (Sachs *et al.*, 1971).

Assumption 2. Linear channels

To be able to calculate the response of individual channels to different stimuli, one must know something about the properties of individual channels. Three simple, linear properties will be assumed here.

First property: linearity with contrast. The response of a channel will be assumed to vary linearly with contrast of the stimulus. Let $c(\text{stim})$ be the contrast in the stimulus and $S_i(\text{stim})$ be the sensitivity of the ith channel to that stimulus (the reciprocal of the contrast threshold of that channel for that stimulus), then the response of the ith channel to that stimulus is a number that equals contrast time sensitivity, i.e.

$$R_i(\text{stim}) = c(\text{stim}) \cdot S_i(\text{stim}). \tag{2}$$

Second property: additivity of responses to test and sine. The test-plus-sine experiments of concern here always involved combinations of a test stimulus and sine-wave grating in *additive* or *subtractive* phase. That is, the sine-wave was always in either exactly the same phase or exactly the opposite phase as the component in the test-stimulus of the same frequency. A channel's response to such a test-plus-sine combination will be assumed to equal either the sum (if additive phase) or the difference (if subtractive phase) of the channel's responses to the test and to the sine alone. Writing the responses to the test and to the sine as in (2) and using the convention of allowing the sign of the sine-wave contrast to indicate phase (sign is positive if additive phase and negative if subtractive phase), this second property can be expressed as

$$R_i(\text{test with sine}) = c(\text{test}) \cdot S_i(\text{test}) + c(\text{sine}) \cdot S_i(\text{sine}). \tag{3}$$

Third property: response to aperiodic test stimuli. To calculate the responses of a linear channel to any arbitrary stimulus, one can use the methods of Fourier analysis. For the case of relatively narrowband channels and broadband stimuli (e.g. aperiodic stimuli), an excellent approximate solution can be simply obtained.[4] This approximation, within the framework of the other assumptions, will allow the derivation of a relationship from which one can estimate the best-fitting parameters, in particular, the bandwidths of the channels. Thus, the bandwidths will not need to be specified in advance.

According to this approximation, the sensitivity of the ith channel to a broadband test stimulus equals the sensitivity of the channel to its best frequency [$S_i(f_i)$, where f_i is the best frequency of channel i] multiplied by the amount of its best frequency contained in the test stimulus [measured by $F_{\text{test}}(f_i)$ which is discussed further below] multiplied by the channel's equivalent bandwidth (W_i, further defined below). Or, formally

$$S_i(\text{test}) = W_i \cdot S_i(f_i) \cdot F_{\text{test}}(f_i). \tag{4}$$

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[4] Many of the technical arguments briefly described here are described more fully in an unpublished manuscript available from the author.

The measure of amount of frequency contained in the test stimulus, $F_{test}(f_i)$, is closely related to the Fourier transform of the test stimulus. In particular, if $L(x)$ is the luminance profile of the test stimulus adjusted so that the peak minus trough luminance equals 1.0, then $F_{test}(f)$ is four times the absolute value of the Fourier transform of $L(x)$ at frequency f.[5]

The equivalent bandwidth W_i is a common measure of bandwidth and equals the area under the channel's sensitivity function divided by the sensitivity of the channel at its best frequency, or

$$W_i = \frac{\int_{-\infty}^{+\infty} S_i(f)\,df}{S_i(f_i)}.$$

If the channel has a rectangular sensitivity function (is uniformly sensitive to all frequencies it has non-zero sensitivity to), then the equivalent bandwidth equals the full channel bandwidth. If the channel has a triangular sensitivity function, the equivalent bandwidth is the width of the frequency range for which the channel's sensitivity is greater than one-half the peak sensitivity.

Since (4) is strictly true only if the Fourier spectrum of the test stimulus is absolutely flat within the channel's frequency range and if the phase characteristics of stimulus and channel are appropriately matched, some calculations (see footnote 4) were done (using Fast Fourier Transforms) to check the accuracy of this expression for realistic channels and test stimuli. The accuracy was extremely good. The property (4), therefore, will be assumed to be true for all the channels and aperiodic stimuli discussed below.

The three properties described above are strictly true (see footnote 4) for a particularly simple kind of linear channel. (The second property is true only for limited ranges of contrast but that is sufficient.) This simple kind of channel consists of receptive fields that act linearly (i.e. that follow the superposition rule, as has been conventionally assumed). In addition, (a) there is no uncorrelated variability in the responses of receptive fields located at different spatial positions within the channel (so there is no *probability summation across space*) and (b) the response of the channel equals the response of the maximally responding receptive field.

The three properties described above are also approximately true for other kinds of linear channels (see footnote 4). In particular, as is described further in the Discussion section, they are approximately true of a channel in which there is uncorrelated variability in the responses of receptive fields located at different positions so there is probability summation across space.

[5] This factor of four comes from two sources, each of which contributes a factor of two. The first factor comes from the definition of contrast used by Shapley and Tolhurst and Kulikowski and King-Smith for aperiodic stimuli. The standard aperiodic stimulus when computing Fourier transforms has a peak–trough difference of one, but the standard sine-wave wave has a peak–trough difference of two. Therefore, the standard aperiodic stimulus has only half the contrast of the standard sine-wave. The second source of a factor of 2 is that, in general, the Fourier transform values for both positive and negative frequencies must be added together, but in the above expression, only the positive frequency is used.

Assumption 3. One bandwidth per frequency

All channels having the same best frequency will be assumed to have the same bandwidth. This is in contrast to Shapley and Tolhurst's and Kulikowski and King-Smith's approach that allowed both narrowband channels (for detecting sine-waves) and broadband channels (for detecting aperiodic stimuli) at the same best frequency.

Assumption 4. Quick's psychometric function

Assumption 2 specifies the relationship between magnitude of a channel's response and stimulus contrast. In order to use assumption 1, however, the relationship between probability of detection and stimulus contrast must be specified. Frequently, this relationship is described by a cumulative Gaussian, a function that is very difficult to work with. Quick (1974) suggested the following form which is close to a cumulative Gaussian and much easier to use. Let $P_i(\text{stim})$ be the probability of the ith channel's detecting the stimulus, then it will be assumed that

$$P_i(\text{stim}) = 1 - 2^{-R_i(\text{stim})^k}$$
$$= 1 - 2^{-[c(\text{stim})\cdot S_i(\text{stim})]^k}, \quad (5)$$

where k is the parameter that determines the steepness of the function and is assumed to be the same for all channels. Notice that $P_i(\text{stim})$ equals one-half when the response of the channel, $R_i(\text{stim})$, equals 1.0. This is consistent with equation (2) in making $S_i(\text{stim})$ equal to the reciprocal of the contrast necessary for $P_i(\text{stim})$ to equal one-half.

Invariance property. Using the function specified in (5), the psychophysical psychometric functions for all stimuli are predicted to have exactly the same shape on the log contrast axis. This invariance can be shown easily by combining the equation specifying probability summation (1) with the equation specifying Quick's function (5) and doing simple algebraic manipulations. Remember that $P(\text{stim})$ is the probability of an observer's detecting the stimulus. One finds that

$$P(\text{stim}) = 1 - 2^{-[c(\text{stim})S(\text{stim})]^k}, \quad (6)$$

where

$$S(\text{stim}) = \left\{ \sum_{i=1}^{N} S_i(\text{stim})^k \right\}^{1/k}. \quad (7)$$

Notice that $S(\text{stim})$ equals the observer's sensitivity to the stimulus, i.e. the reciprocal of the contrast producing a probability of detection by the observer of 50%.

Green and Luce (1975) have proved that functions of the above form (the base need not be 2) are the only ones having this invariance property.

The predicted psychophysical psychometric function (6) has the same shape on the log contrast axis for every stimulus since only a multiplicative constant is affected by the stimulus. This invariance is at least approximately true in the available data. In fact, the predicted psychophysical function has the same shape as that of the function (5) for an individual channel. The parameter k can be estimated, therefore, from the observer's psychometric function for any stimulus. A k of 4 or 5 provides a good fit to many functions, to those reported by Sachs *et al.* (1971), for example.

Alternative interpretation in terms of nonlinear summation. As Quick (1974) discusses, the expression in (7) suggests an alternative but, within the present context, equivalent

assumption to replace the assumption of probability summation. Perhaps the variability in the responses of different channels is perfectly correlated but the responses are actually added to each other in a nonlinear summation consistent with the expression (7). In this alternative interpretation, the variability in the observer's responses would come from a later stage of visual processing. Since this alternative interpretation makes the same predictions as the probability summation assumption coupled with Quick's psychometric function, the two interpretations cannot be distinguished from each other by the kind of data to be discussed here.

Assumptions 5 and 6. Continuity in channel characteristics

Fortunately, in making predictions for the test-plus-sine experiments, it was not necessary to make assumptions about exactly how many channels exist or about the sensitivities of the whole set of channels to the test stimulus. Some simplifying assumptions had to be used, however, in order to make the equations tractable. It is reasonable, on the basis of existing evidence, to assume that nothing about the channels or their responses changes too quickly as you move from one frequency to another, and that channels are close enough together in best frequency so that there is no obvious discreteness. Two assumptions specifying particular aspects of such continuity proved sufficient. Their rigorous mathematical formulations are given below.

Assumption 5. Nearby channels' response to test stimulus. Roughly, nearby channels are channels having best frequencies close to each other. Rigorously, two channels i and j are *nearby* if they both have non-zero sensitivity to the same sine, i.e. if there exists some frequency f for which both $S_i(f)$ and $S_j(f)$ are greater than zero.

The assumption made about nearby channels is that they are equally sensitive to a broadband test stimulus. Formally, assumption 5 states that

$$S_i(test) = S_j(test) \qquad (8)$$

if channels i and j are nearby.

As is clear from equation (4), this assumption would be exactly true if (a) nearby channels had the same bandwidth and peak sensitivity and (b) the spectrum of the test stimulus were flat throughout the frequency range occupied by the nearby channels. Since both (a) and (b) are approximately true for nearby channels, the assumption is reasonable. One might expect, however, that this assumption would be violated in frequency regions where the psychophysical contrast sensitivity function is changing quickly [so (a) will not be true] and in frequency regions where the spectrum of the test stimulus is changing quickly [so (b) will not be true].

Assumption 6. Close enough channel best frequencies. Is it necessary to assume that, for any arbitrary frequency f_a, there is a channel which has its best frequency exactly at f_a? Fortunately not. Instead it can be assumed that the most sensitive channel to the arbitrary frequency f_a has its best frequency near that frequency. Or, formally, let a^* be the channel which, of all channels, is most sensitive to f_a. Remember that, by previously introduced notation, f_{a^*} is the best frequency of channel a^* (that is, it is the frequency which, of all frequencies, is responded to best by channel a^*). Assumption 6 simply states that the frequencies f_a and f_{a^*} are close to each other. How close is close? Close enough that the sensitivity of channel a^* to both frequencies f_a and f_{a^*} is approximately equal, and close enough that test stimulus contains the same amount of both frequencies. These conditions will certainly hold, with room to spare, if (a) and (b) mentioned under assump-

tion 5 hold. In general, assumption 6 is reasonable for a model having multiple narrowband channels.

Assumption 7. Identical minimal sets of sensitivities to a sine

Sachs *et al.* (1971), and others, simplified their theoretical calculations by assuming that only one channel responded to a pure sinusoidal grating. That assumption, which was initially used here, was rejected later because it was shown to be in conflict with some test-plus-sine data (see below). Assumptions 7, 5, and 3 were used instead because they were the simplest substitute that could be found. Assumption 7 is like an assumption that sensitivity functions for different channels are the same "shape" although not necessarily of the same bandwidth or the same peak sensitivity. "Shape", however, is not a precise description of what is assumed constant, and assumption 7 does not actually deal with the sensitivity functions of individual channels. Instead, it somewhat restricts the sensitivities of different channels to a single sine.

To present assumption 7, the notion of a *minimal set of sensitivities* is convenient. First consider the set of numbers which is a list of the sensitivities of all the channels to a particular frequency. Then normalize all those numbers so they express sensitivity relative to the most sensitive channel, i.e. the numbers in the list are multiplied by the factor necessary to make the largest number equal to 1.0. Remove all the zeros from the list. Now the set is a list of all the non-zero relative sensitivities to the particular frequency. In some cases, this is the minimal set. However, if this set of non-zero relative sensitivities consists of a sub-set which is repeated twice or more, the subset is actually the minimal set. For example, if the set of non-zero relative sensitivities consisted of three 1.0s, three 0.5s and three 0.3s, then the minimal set would be the three number subset containing 1.0, 0.5 and 0.3. This minimal set captures the information that one-third of the channels responding at all respond maximally, one-third respond at one-half maximum, and one-third respond at three-tenths maximum. In short, it throws away information about the absolute level of sensitivities and about the actual number of sensitive channels, and keeps information about the pattern of sensitivities across channels.

Assumption 7 states that the pattern of sensitivities across different channels to a single sine is the same for all frequencies of sine, or, formally, that the minimal set is the same for all frequencies of sine.

Four possible minimal sets were considered in the development of the model. Examples of each are illustrated in Fig. 1 by plotting the relative sensitivity of twenty channels to a single sine wave. Relative sensitivity is plotted on the vertical axis. The twenty different channels are represented by different positions on the horizontal axis. For illustrative purposes, the twenty channels were arranged in a sensible order on the horizontal axis (although their order does not matter) and a smooth curve drawn through the individual points. The reason for the names attached to each case should be apparent in the shapes of these curves.

A minimal set of relative sensitivities is indicated for each case by the filled-in circles. For the three
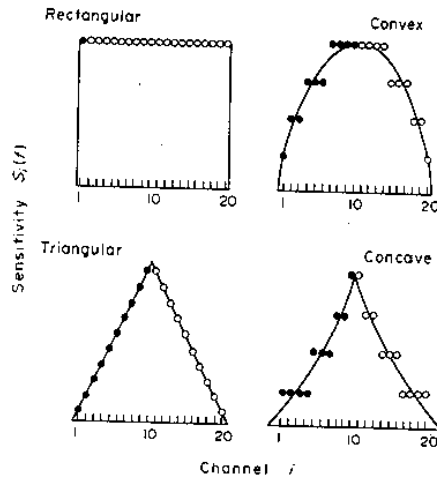
Fig. 1. Illustrations of four possible distributions of different channels' relative sensitivities to a single sine wave. In each case, 20 channels (indicated along the horizontal axis) have non-zero sensitivity to the sine wave. The sensitivity of each channel to the sine is given on the vertical axis (normalized so that the most sensitive channel's sensitivity equals one). In each of the four cases, a minimal set of relative sensitivities (defined in text) is indicated by closed circles; the duplicates of the minimal set are indicated by open circles. The minimal set for the rectangular case is {1}. For the convex case it is {1/4, 1/2, 1/2, 3/4, 3/4, 3/4, 1, 1, 1, 1}. For the triangular case it is {1/10, 2/10, 3/10, 4/10, 5/10, 6/10, 7/10, 8/10, 9/10, 1}. For the concave case it is {1/4, 1/4, 1/4, 1/4, 1/2, 1/2, 1/2, 3/4, 3/4, 1}.

cases other than the rectangular case, the minimal set consists of half of the 20 relative sensitivities. For the rectangular case, however, since all 20 numbers are the same, the minimal set is the one-member set containing the number 1.0. The case where only one channel responds to a sine, therefore, is an instance of the rectangular case.

Whether assumption 7 is reasonable or not is unclear. However, the effects of assumed minimal set on the predictions of the model are easy to keep track of and, further, are not great. The only effect will turn out to be on a parameter α, the value of which is linearly related to predicted sensitivities. The only effect on α, at least for the range of cases shown in Fig. 1, is less than a factor of 2. A factor of 2 is a very small portion of the range of sensitivities predicted by the model.

Assumption 7 implies that α is the same for all frequencies. A more formal presentation of minimal sets and related material is given in Appendix 1.

DEVELOPMENT OF THE PROBABILITY SUMMATION MODEL

Predictions for test-plus-sine contrast interrelationship functions

The bulk of Shapley and Tolhurst's (1973) and Kulikowski and King-Smith's (1973) results came from *test-plus-sine* experiments, experiments in which thresholds were measured for combinations of aperiodic test stimuli and sinusoidal gratings. The results

for a particular test stimulus and particular frequency of sine wave can be given as a *contrast interrelationship function*. For each contrast in the sine, this function gives the contrast needed in the test stimulus for the combination of test and sine to be at threshold. In order to present the predictions of the probability summation model for these contrast interrelationship functions, it will be useful to define two new concepts.

Equivalent number of channels. If all channels having non-zero sensitivity to a sine of frequency f_a have equal sensitivity (the rectangular case), then the equivalent number of channels just equals the number of sensitive channels. If the channels sensitive to the sine are unequally sensitive, however, then the equivalent number of channels will be less than the number of channels sensitive to the sine. Rigorously, M_a^E, the equivalent number of channels sensitive to a sine of frequency f_a, is defined as

$$M_a^E = \left(\frac{S(f_a)}{S_{a^*}(f_a)} \right)^k.$$ (9)

Remember that a^* is the channel, of all channels, that is most sensitive to frequency f_a.

Importance index. Each test-sine pair can be characterized by a single number which proves to be very useful in calculating predictions from the probability summation model. This is a number which is, on an intuitive level, a measure of how important the channels sensitive to the sine are in the detection of the test stimulus. This number will be called the importance index, $I(f_a, \text{test})$ and is defined as follows

$$I(f_a, \text{test}) = \frac{M_a^E \cdot S_{a^*}(\text{test})^k}{\sum_{j=1}^N S_j(\text{test})^k}.$$ (10)

A particularly easy interpretation of this importance index exists for the special case where (a) all channels that respond to the sine respond equally to it (i.e. the rectangular case) and (b) all channels that respond to the test stimulus respond equally to it. For example, suppose that channels 1–100 respond equally to the *test* stimulus and no other channels respond to it at all. Suppose that the first 10 channels respond to the sine ($M_a^E = 10$) and no others respond to it at all. The denominator of the importance index will then be the sum of 100 terms, each equal to the $S_{a^*}(\text{test})^k$ term in the numerator. So the value of the importance index will be 10/100. Or, in general, the importance index for this special case will equal the fraction of the total number of channels sensitive to the test stimulus that is also sensitive to the sine.

The predicted function. To calculate the predictions of the probability summation model for test-plus-sine contrast interrelationship functions requires only straightforward algebra. One finds an expression giving the contrast in the best stimulus necessary for the test-and-sine combination to be at threshold [i.e. for P(test-and-sine) to be 0.50] as a function of contrast in the sine wave. The derivation of the general formula is given in Appendix 1 and depends only on assumptions 1, 2 (the first two properties), 4, and 5.

As it turns out, any contrast interrelationship function predicted by the model is specified completely

if three pieces of information are known: (a) k, the parameter determining the steepness of the psychometric function, (b) the minimal set for the sine frequency, and (c) $I(f_a,$ test), the importance index for the particular sine and test stimulus involved. To put this fact in another way, once the steepness of the psychometric function and minimal set have been specified, all possible contrast interrelationship functions predicted by the model form a one-parameter family. The parameter is the importance index. This reduction to a one-parameter family is a consequence of the nearby-channels assumption 5 and is what makes that assumption so useful. There are probably other assumptions that would lead to a similar reduction, but this one is sufficient.

The solid lines in Fig. 2 are part of the family of functions for k equal to 5 and the rectangular case. The axes in Fig. 2 give relative contrast. The relative contrast of a stimulus is just contrast divided by psychophysical threshold for the stimulus or, equivalently, contrast multiplied by psychophysical sensitivity for the stimulus, $c(\text{stim})\ S(\text{stim})$. Four functions are shown in Fig. 2, those for importance indices of 1.0, 0.5, 0.33 and 0.2. An infinite number of other functions exist for other values of the importance index, of course. The significant point, however, is that only one function exists for a given value of the importance index.

The curvature apparent in the solid lines of Fig. 2 is not an inherent property of the general probability summation model, but is a result of the assumption of rectangular minimal set. The predicted functions straighten out when several channels are sensitive to the sine but sensitive to different extents, as is true for the concave case and an importance index of 0.157 (dotted line in Fig. 2).

Predictions for the sensitivity of the test-stimulus detector, $S_{\text{test}}(f)$

For most combinations of test stimuli and sines, Shapley and Tolhurst and Kulikowski and King-Smith did not collect full contrast interrelationship functions. Rather, they collected only two or three points on these functions (i.e. used only two or three contrasts of sine wave) and then did a linear extrapolation. The result of this linear extrapolation was a quantity they called the "sensitivity of the test-stimulus detector to the sine" for reasons which are explained in Kulikowski and King-Smith (1973) and in Graham (1977). Operationally, this quantity is a measure of how much the threshold for the test-sti-

mulus was lowered by the presence of the sine-wave. It will sometimes be referred to here, therefore, as "the effectiveness of the sine-wave in lowering the threshold of the test-stimulus". The symbol that will be used for this quantity is $S_{\text{test}}(f)$ where the subscript test refers to the particular test-stimulus and the argument f is the frequency of the particular sine.

The predicted power law. Families of contrast interrelationship functions like the family of solid lines in Fig. 2 were calculated using a computer for various minimal sets and steepnesses of psychometric functions (values of k). Linear extrapolation was then done from a pair of points on these functions in the manner of Shapley and Tolhurst and Kulikowski and King-Smith. This extrapolation yields the predicted value of $S_{\text{test}}(f)$, the predicted sensitivity of the test-stimulus detector to a sine (i.e. the effectiveness of the sine in lowering the threshold for the test stimulus).

An extremely convenient relationship emerged from these calculations. In particular, according to the probability summation model, the test-stimulus detector's sensitivity to a sine (relative to the observer's sensitivity to that sine) is a power function of the importance index for the particular test stimulus and sine involved:[6]

$$\frac{S_{\text{test}}(f_a)}{S(f_a)} = \alpha[I(f_a, \text{test})]^\beta. \tag{11}$$

In words, the more important the channels sensitive to the sine are in the detection of the test stimulus, the more effective the sine is in lowering the threshold
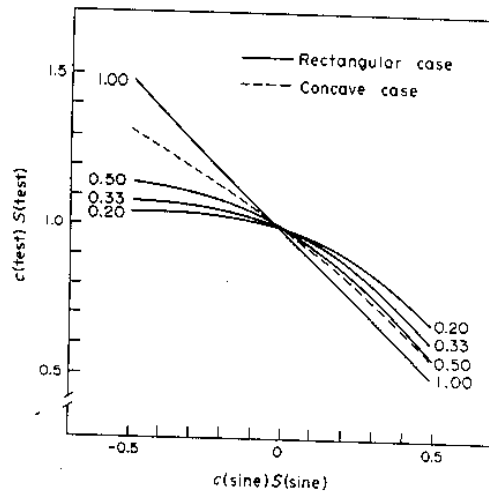
Fig. 2. Examples of contrast interrelationship functions predicted by the probability summation among channels model. Contrast in the test and sine stimuli relative to their own thresholds are given on the vertical and horizontal axes, respectively. Each point on a curve represents a combination of test stimulus and sine stimulus that is just at threshold. The solid lines are predictions from the rectangular case of the model; the dotted line from the concave case. The number next to each curve gives the importance index of the test-plus-sine combination producing that curve. The parameter k determining the steepness of the psychometric function was always 5.

Table 1. Values of multiplicative constant α and exponent β in power law for different minimal sets, values of the steepness parameter k, and extrapolation pairs

| | Extrapolation pair | | | | | |
| | (0, +0.5) | | (0, +0.2) | | (−0.5, +0.5) | |
	α	β	α	β	α	β
Rectangular						
$k = 5$	1.1	0.33	1.0	0.70	1.0	0.47
$k = 4$	1.1	0.33	1.0	0.58	Not calculated	
$k = 3$	1.1	0.33	1.0	0.49	Not calculated	
Convex						
$k = 5$	1.3	0.33	Not calculated		1.25	0.47
Triangular						
$k = 5$	1.6	0.33	Not calculated		1.6	0.47
Concave						
$k = 5$	1.7	0.33	Not calculated		1.9	0.47

for the test stimulus. The exact relationship between the two quantities is a power law.

The values of the constants α and β are given in Table 1 for particular minimal sets, steepnesses of psychometric function (k's) and pairs of points used in the linear extrapolation. Three possible pairs of points were used in the linear extrapolations: the pair (in the third column of Table 1) usually used by Shapley and Tolhurst and Kulikowski and King-Smith containing the two points where the relative contrast of the sine was either −0.5 (negative phase) or +0.5 (positive phase); a pair that may have been used by the original investigators where the relative sine contrast was zero or 0.5; and a third pair tried out of curiosity where the relative sine contrast was zero or 0.2. These pairs will be called extrapolation pairs. As can be seen in Table 1, both α and β depend on the extrapolation pair and on the steepness parameter. For a given extrapolation pair and steepness parameter, the value of β does not depend on which minimal set is assumed; the value of α does but varies by less than a factor of 2 as the minimal set is changed.

Making the power law testable. In order to use the power law in (11) to make quantitative predictions for the data, one must be able to calculate the importance index for each combination of test and sine. It is moderately easy to change the expression for the importance index into a form that is closer to being calculable. Consider the S_{a^*}(test) term in the numerator of the definition (10) of the importance

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[7] Nothing in the development of the power function, however, demanded assumption 3. In fact, according to that development, there might be many channels other than a^* that are sensitive to f_a. And these others could have different bandwidths. The only restriction on the bandwidths is that all the channels sensitive to f_a must be equally sensitive to the test stimulus (assumption 5). Thus, the product of each channel's bandwidth times its peak sensitivity must be approximately constant [see equations (4) and (8)].

index $I(f_a,$ test). By the third property (4) of the linear channels stipulated in assumption 2 applied to channel a^*,

$$S_{a^*}(\text{test}) = W_{a^*} \cdot S_{a^*}(f_{a^*}) \cdot F_{\text{test}}(f_{a^*}).$$

By assumption 6 (close enough channel frequencies), f_a can be substituted for f_{a^*} in the above expression for S_{a^*}(test). Then, in the definition (10) of the importance index, S_{a^*}(test) can be replaced by the above equivalent number of channels, M_a^E, no longer appears replaced with the quantity it equals by definition (9). These substitutions yield

$$I(f_a, \text{test}) = \left(\frac{W_{a^*} \cdot S(f_a) \cdot F_{\text{test}}(f_a)}{S(\text{test})} \right)^k.$$

In this new expression for the importance index the equivalent number of channels, M_a^E, no longer appears explicitly. It appears implicitly, instead, as part of the psychophysical contrast sensitivity $S(f_a)$.

Substituting this new expression for the importance index into the power law in (11) gives

$$\frac{S_{\text{test}}(f_a)}{S(f_a)} = \alpha \left(\frac{W_{a^*} \cdot S(f_a) \cdot F_{\text{test}}(f_a)}{S(\text{test})} \right)^{\beta k}. \quad (12)$$

This new form of the power law is quite manageable as it contains only two variables, α and W_{a^*} which are not either observable in the data or calculable from knowledge of the experiment. $S_{\text{test}}(f_a)$, $S(\text{test})$, $S(f_a)$, and k are observable in the data; $F_{\text{test}}(f_a)$ is calculable from knowledge of what the stimulus is, and β is known from the steepness parameter k and the pair of points used in the extrapolation. The value of α depends on the minimal set. The value changes by less than a factor of two, however, for the minimal sets investigated. Further, since the minimal set is assumed to be the same for all frequencies, (assumption 7), the value of α will not depend on the frequency of sine-wave used. The bandwidth parameter (W_{a^*}) of the most sensitive channel for frequency f_a might well depend on frequency, however. It will be the major free parameter in comparing the model's predictions of $S_{\text{test}}(f_a)$ to data.

Notice that we will be working with only one channel bandwidth (W_{a^*}) for each frequency. This practice is consistent with assumption 3 that there is only one bandwidth of channel at each frequency.[7]

Predictions for test-plus-test experiments

Shapley and Tolhurst and Kulikowski and King-Smith also did some test-plus-test experiments in which the thresholds for combinations of two aperiodic test stimuli were measured. It is not easy to deal with this kind of experiment in general using the model described above. But calculations of results for special cases can easily be done.

COMPARISON OF MODEL AND DATA

Shape of the test-plus-sine contrast interrelationship functions

Although Shapley and Tolhurst (1973) and Kulikowski and King-Smith (1973) published few complete test-plus-sine contrast interrelationship functions, their published data strongly suggest that the empiri-
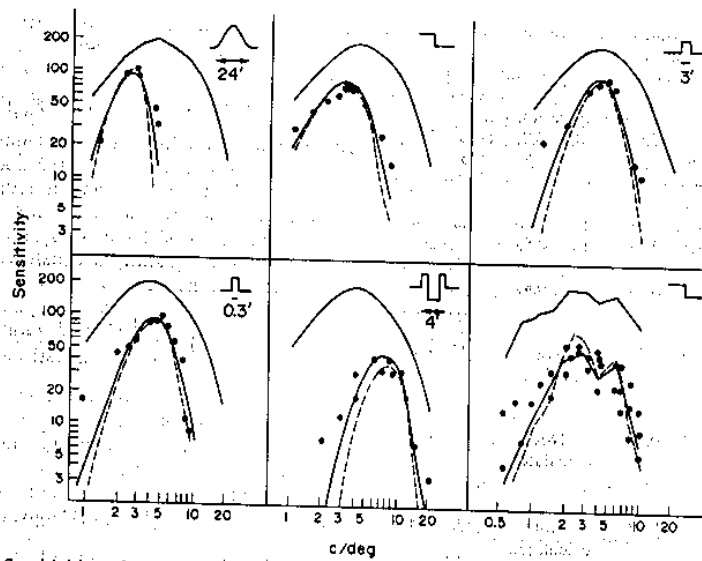
Fig. 3. Sensitivities of test-stimulus detectors [S_{test} (f)] plotted as a function of spatial frequency. Solid points show the data collected by Shapley and Tolhurst (1973), lower right panel, and by Kulikowski and King-Smith (1973), other panels. The sensitivities from Shapley and Tolhurst's study are plotted at 1/3 their actual value. For each panel, the test stimulus used is shown as an inset in the upper right of the panel. The lower two curves are predictions of the probability summation among channels model for $S_{test}(f)$. The predictions from all cases of the model (rectangular, concave, etc.) are the same. The upper curve in each panel is the psychophysical contrast sensitivity function, $S(f)$, shown for comparison.

cal functions are not as curved as the functions predicted by the rectangular case of the probability summation model (solid curves, Fig. 2). The small degree of curvature in the functions predicted by some other cases, however (e.g. the dotted line in Fig. 2) is certainly compatible with existing data.

The rectangular case of the probability summation model, including the subcase where only one channel responds to a sine, can be rejected, therefore, on the basis of the shape of the contrast interrelationship functions. Other cases of the probability summation model are consistent with this data.

Sensitivity of test stimulus detectors, $S_{test}(f)$

A second way of evaluating the model's fit to data is illustrated in Fig. 3. The points in the figure are the sensitivities of test stimulus detectors, the $S_{test}(f)$ data, from Shapley and Tolhurst's experiment (lower right panel) and Kulikowski and King-Smith's experiment (other panels). The small insets illustrate the particular test stimuli used. There is a good deal of difference between the results for an edge collected by Kulikowski and King-Smith (upper middle panel) and those collected by Shapley and Tolhurst (lower right panel). This difference will reappear later as a difference in the parameters of the model. The upper curve is the ordinary psychophysical contrast sensitivity function [the function drawn through the sensitivities of the observer to different frequencies of sine, i.e. the $S(f)$ data]. The lower solid and dashed curves are predictions of $S_{test}(f)$ from the probability summation model derived using the power law in (12) (details below).

Overall, agreement between the data and the model's predictions is extremely good. There may be small but consistent discrepancies at extreme frequencies, that is, the predicted functions may be somewhat narrower than the observed ones. These discrepancies may result from the inadequacy of assumption 5 (the nearby-channels assumption) in frequency ranges where the test-stimulus spectrum and/or the psychophysical contrast sensitivity function is changing quickly. Also, experimental error is larger for these extreme frequencies because the ratio of $S_{test}(f)$ to $S(f)$ is low. In any case, the overall agreement is much more impressive than the discrepancies. Both the frequency most effective in reducing the test-stimulus threshold [i.e. the frequency at the peak of the $S_{test}(f)$ function] and the range of somewhat effective frequencies are almost perfectly predicted by the probability summation model.

To make the predictions shown in Fig. 3 required, at most, one free parameter for each aperiodic stimulus plus one free parameter for each frequency of sine. To fit the same data in the way done by the original investigators (assuming broadband channels and no probability summation) required, in effect, one parameter for each data point (one for each combination of test stimulus and frequency of sine) which, in the case of Kulikowski and King-Smith's study, is approximately five times as many as used in Fig. 3.

Details of making the predictions in Fig. 3. Since the thresholds for the test-stimulus alone [the $S(test)$ data] were not known for all test stimuli and since, as will be discussed further below, these thresholds are more closely related to channel bandwidth than are the $S_{test}(f)$ data, it was decided to fit the $S_{test}(f)$ data without using the $S(test)$ data.

Instead, the following adjustment was made in the power law (12). The bandwidth W_a at frequency f_a was expressed as a constant (called the relative bandwidth w_a) times the

Table 2. A possible decomposition of parameters used in predictions in Fig. 3. Available S(test) data

	$\alpha = 1.1 \quad \beta = 0.33$			$\alpha = 1.0 \quad \beta = 0.47$			Data
	$\dfrac{W}{S(\text{test})}$	W	$S(\text{test})$	$\dfrac{W}{S(\text{test})}$	W	$S(\text{test})$	$S(\text{test})$
Kulikowski and King-Smith							
Blurry bar	0.011	1.4	127	0.013	2	150	Unknown
Edge	0.016	1.4	88	0.019	2	105	75
Wide bar	0.017	1.4	84	0.020	2	99	100
Narrow bar	0.152	1.4	9	0.183	2	11	10
Triphasic blip	0.054	1.4	26	0.125	2	16	Unknown
Shapley and Tolhurst							
Edge	0.0012	0.375	310	0.0016	0.50	310	310

bandwidth at 1.0 c/deg (called W). Then equation (12) can be rewritten as follows in order to separate (a) the known quantities from (b) the parameter depending on sine-wave frequency from (c) the parameter depending on the test stimulus:

$$S_{\text{test}}(f_a) = S(f_a) \cdot [F_{\text{test}}(F_a) \cdot S(f_a)]^{\beta k}$$
$$\times [w_{a^*}]^{\beta k} \times \alpha \left(\frac{W}{S(\text{test})}\right)^{k\beta}.$$

Remember that the steepness parameter k can be known from independent data since it is determined by the steepness of the psychometric function. For the predictions shown in Fig. 3, k was set equal to 5. A value for k of 4 would also be consistent with the available psychometric functions (Sachs et al., 1971). The predictions of $S_{\text{test}}(f)$ using a k of 4 are slightly better than those shown in Fig. 3. Also, β is known once k and the extrapolation pair have been specified. In fact, the solid curve predictions in Fig. 3 result from assuming β equals 0.33 [the extrapolation pair (0, 0.5)] and the dashed curves from β equals 0.47 [the extrapolation pair (-0.5, $+0.5$)]. The psychophysical contrast sensitivity function $S(f)$ is independent data that was collected by Shapley and Tolhurst and Kulikowski and King-Smith and so is known. And the spectrum $F_{\text{test}}(f)$ is determined by the test stimulus. Hence, for each frequency f_a, the leftmost factor in the right side of the above expression is known.

The middle factor, however, contains the relative bandwidth parameter w_a that must be chosen for each frequency f_a. The following choices were made. For Kulikowski and King-Smith's experiment, the relative bandwidth parameter w_a was assumed to be constant (equal to 1.0) at all frequencies. For Shapley and Tolhurst's experiment, w_a was assumed to be proportional to frequency. This difference between relative bandwidths in Shapley and Tolhurst's and in Kulikowski and King-Smith's experiment will be discussed later at length. The right-most factor above is the parameter that was left free to vary separately for each test-stimulus and its actual value can be recovered from Table 2.

If the probability summation model is ultimately to be accepted, the most conservative approach to the counting

[8] In fact, by making an assumption about the density of channels along the spatial frequency axis, one can reduce the number of parameters still further by requiring only one parameter for all test stimuli instead of the one parameter for each test stimulus (see footnote 4). This is how the fits in Fig. 3 were actually done. In the process, one gets an estimate related to channel density. The density estimate is extremely unstable however, that is, a large range of values produces satisfactory predictions. Therefore, the details of this procedure are not presented here.

of free parameters is overestimation rather than underestimation of the number. The larger the number of free parameters, the more difficult it is to accept the model. As mentioned before, the number of free parameters involved in making the predictions of Fig. 3 was, at most, one free parameter for each frequency (relative bandwidth) plus one free parameter for each test stimulus.

If the probability summation model is ultimately to be rejected, however, the most conservative approach is underestimation of the number of free parameters. The smaller the number of free parameters, the harder it is to reject the model. The actual freedom involved in making parameter choices was much less than the above accounting acknowledges. First, relative bandwidths were not chosen independently for different frequencies. On the basis of previous data, it was decided that relative bandwidth would either be constant across frequencies or increase in direction proportional to frequency. Second, the parameter depending on test stimulus[8] (the right-most factor in the above equation) must be decomposable into reasonable values of α, S(test), and bandwidth at 1 c/deg (W). Such a decomposition is carried out in Table 2 and the available S(test) data are presented for comparison to show that the values are reasonable. Table 2 assumes the minimal set is rectangular. If other minimal sets are used, the estimate of bandwidth at 1 c/deg (W) becomes smaller by a factor of 1.5 or less.

Independence of $S_{\text{test}}(f)$ predictions from absolute bandwidth and minimal set

It can easily be shown that the predictions of $S_{\text{test}}(f)$ are independent of the absolute value of bandwidth. Consider what would happen if the bandwidth of every channel in the model were suddenly doubled, that is, if W were doubled. The observer's sensitivity to broadband test stimuli [S(test)] would also double since, by (4), each channel's sensitivity would double and, by (7), therefore, so would the observer's sensitivity. Thus, the right-most factor in the above equation (the parameter depending on test stimulus) would remain unchanged, and so would the predicted $S_{\text{test}}(f)$ for all frequencies and test stimuli. Thus, the $S_{\text{test}}(f)$ data by themselves can tell us nothing about the absolute value of bandwidth, only about the relative values at different frequencies. The comparison of the $S_{\text{test}}(f)$ data with the $S(f)$ data, however, does give information about absolute bandwidth, and is described in the next section.

Sensitivity to the test stimuli and bandwidth estimates

The method described in the last section for evaluating the probability summation model had the

advantage of directly comparing predictions to the data for the sensitivities of test-stimulus detectors (Fig. 3). The method described in this section uses both that data and the data for the sensitivities of the observer to the test stimuli [S(test)]. This method allows one to see clearly exactly what would be required to reject the probability summation model and also allows one to obtain directly an estimate of bandwidth at each frequency.

As discussed earlier, the power law in the form of equation (12) is an almost parameter-free prediction of the probability summation model: only α, which depends on channel shape, and the bandwidth W_{a^*} are free to vary. It will be useful to rearrange (12) so that the bandwidth W_{a^*} is isolated on the left side:

$$W_{a^*} = \left(\frac{1}{\alpha}\right)^{[(1)/k\beta]} \cdot S_{tot}(f_a)^{[(1)/k\beta]}$$
$$\times \left(\frac{1}{S(f_a)}\right)^{[(1+k\beta)/k\beta]} \cdot S(\text{test}) \cdot \frac{1}{F_{test}(f_a)}. \quad (13)$$

Suppose the contrast sensitivity function $S(f)$ and also the complete data for one test stimulus, i.e. $S(\text{test})$ and the function $S_{tot}(f)$, are known. Suppose also, as will be supposed for the rest of this discussion, that the steepness parameter k and the extrapolation pair are known. Then, choose a particular minimal set so that α is determined. It is now possible to compute from equation (13) the bandwidth parameter W_{a^*} for each frequency of sine f_a.

Another way of looking at this fact is that, in the above situation, in which there is data from only one test stimulus, it is impossible to reject the probability summation model. For each frequency, a value of the bandwidth can be found which will make the three kinds of data [$S(f)$, $S(\text{test})$, $S_{tot}(f)$] compatible with each other.

In order to reject the probability summation model, therefore, one must have data from at least two test stimuli. Then independent computations of W_{a^*} for a single frequency f_a can be made using the data from different test stimuli. The computations from different test stimuli should agree if the model is correct, since bandwidth is supposed to be a fixed characteristic of the channel having a given best frequency. If they do not agree, the model can be rejected. (Changing the assumed minimal set and thereby changing the value of α changes the calculated bandwidths by the same amount for all frequencies and all test stimuli. The choice of α is, therefore, irrelevant in this test of the model.)

Figure 4 shows the bandwidth parameters calculated for different frequencies and for different test stimuli from the data of Shapley and Tolhurst (left panels) and Kulikowski and King-Smith (right panels, different test stimuli shown by different insets[9]). The top panels ($\beta = 0.47$) come from the extrapolation pair (−0.5, 0.5) and the bottom panels ($\beta = 0.33$) from the pair (0, 0.5). The value of the steepness par-
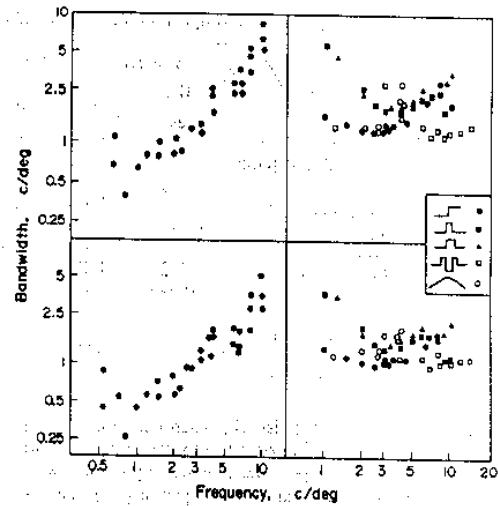
[9] Points for the two extreme frequencies used with the triphasic blip have been omitted from Fig. 4. The data they derive from are inherently extremely variable, since the measured ratio of $S_{tot}(f)$ to $S(f)$ was very low for these two points.

Fig. 4. Bandwidth parameters (as a function of frequency) estimated from the $S_{tot}(f)$ data, the $S(\text{test})$ data and the probability summation among channels model. The bandwidth parameters estimated from Shapley and Tolhurst's data for an edge stimulus are shown in the left half of the figure. Those estimated from Kulikowski and King-Smith's data for five different test stimuli are shown in the right half. The difference betweeen the estimates in the top half of the figure and those in the bottom half are explained in the text. All the estimates were made using the rectangular case of the model. If the bandwidth parameter estimates had been made using some other case shown in Fig. 1. they would all have been smaller by the same factor of less than 1.5. If the bandwidth estimates had been made assuming the existence of probability summation across space, they would have been somewhat larger. The open symbols are from the two test stimuli for which $S(\text{test})$ was not known; thus the vertical position of each of the two kinds of open symbols was free to vary.

ameter k was 5. The solid symbols are from test stimuli for which all relevant data were known, and the open symbols for test stimuli for which $S(\text{test})$ was not given. When $S(\text{test})$ was not known, the values could be calculated only up to one multiplicative constant, so the vertical position of each set of open symbols was arbitrarily chosen.

The bandwidth estimates shown in Fig. 4 were computed assuming a rectangular minimal set because that minimal set produces the largest bandwidth estimates. The effects of assuming other minimal sets are easy to state. Changing the minimal set increases α by a factor of less than 2 (Table 1). Increasing α decreases the estimated bandwidth for all frequencies and all test stimuli by the same amount [equation (13)]. If α increases by a factor of 2, the bandwidth parameter decreases by a factor of 1.5 when $k\beta = 5 \times 0.33$ (bottom panels Fig. 4) or a factor of 1.34 when $k\beta = 5 \times 0.47$ (top panels). Thus the absolute values of bandwidths shown in Fig. 4 may be overestimates by a factor of less than 1.5.

The agreement in Fig. 4 (right half) between the bandwidths calculated from the five stimuli used by Kulikowski and King-Smith is very good, particularly for frequencies greater than 2 c/deg. What discrepancies there are may be due to experimental error or to minor inadequacies in the model (e.g. the nearby-

channels assumptions may be violated at extreme frequencies). There is a decided difference between the form of the function relating bandwidth to frequency for the Kulikowski and King-Smith data (almost constant with frequency) and for the Shapley and Tolhurst data (increasing dramatically with frequency). This discrepancy cannot, however, be taken as evidence against the probability summation model. The discrepancy might be true bandwidth differences due to the use of different observers or different mean luminances. It might also be an artifactual result of Kulikowski and King-Smith's small field size. Field size is, in fact, an important variable and its possible consequences are described in the Discussion section.

Not only is there agreement among the bandwidth estimates calculated from different test stimuli (Fig. 4) but there is also rather good agreement between these estimates and previous estimates of channel bandwidth from sine-plus-sine experiments (Sachs et al., 1971; Lange, Sigel and Stecher, 1973; Pantle, 1973; Quick and Reichert, 1975; Kulikowski and King-Smith, 1973; King-Smith and Kulikowski, 1975).

King-Smith and Kulikowski (1975) recently collected data for the sensitivities of test-stimulus detectors, $S_{test}(f)$, using more different test stimuli. The various test stimuli used in this recent study did not contain significantly overlapping ranges of spatial frequency. As is therefore expected on the basis of the probability summation model, the frequency ranges where the measured $S_{test}(f)$ functions were definitely greater than zero did not overlap significantly either. Thus, these data also are consistent with the probability summation model. However, since the frequency ranges did not overlap and the S(test) data were not available, neither the shape of the function relating bandwidth estimate to frequency nor the absolute values of the bandwidth could be compared across test stimuli to test the probability summation model more rigorously.

There was, however, one interesting aspect of the bandwidth estimates computed from King-Smith and Kulikowski (1975). For the couple of test stimuli containing spatial frequencies between 1 and 8 c/deg, the bandwidth estimates increased in proportion to frequency as did those from Shapley and Tolhurst's study but not those from Kulikowski and King-Smith's (1973) earlier study. The field size used in this later King-Smith and Kulikowski study (7.5°) was large, more like the size used by Shapley and Tolhurst (5.5°) than that by Kulikowski and King-Smith (2.5°).

Threshold for the test stimuli alone. It is important to notice that the agreement in Fig. 4 among the absolute values of W's for different test stimuli involves two kinds of data: $S_{test}(f)$, the test-detector sensitivities; and S(test), the psychophysical sensitivity for the test stimulus alone. Thus, for the test stimuli for which S(test) was given (the solid symbols in Fig. 4), the measured values of S(test) have been shown to be consistent with the probability summation model.

Test-plus-test data

Kulikowski and King-Smith (1973) and Shapley and Tolhurst (1973) showed some experimental results for combinations of edges with lines and lines

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with lines. These results also can probably be handled by the probability summation model.

The experimental results were what the original investigators expected on the basis of their test-plus-sine experiments assuming the existence of broadband detectors and no probability summation. To use one concrete example, they found that the effectiveness of a line in reducing the threshold for an edge is equal to the sum of the effectivenesses of the sinusoidal components of the line. It would be more work than seems worthwhile to calculate the complete predictions for these test-plus-test experiments using the probability summation model (assumptions 1–7). It is easy to argue, however, on the basis of some sample calculations, that the test-plus-test data are probably consistent with the probability summation model. It is easy to show, for example, that the effectiveness of a compound stimulus which is the sum of a large number of *equally effective* sinusoidal components is approximately equal to the sum of the effectiveness of the components (for a wide range of numbers of channels and kinds of broadband stimuli). If the same additivity of effectiveness holds with *unequally effective* sinusoidal components, as seems likely, the test-plus-test results would be successfully predicted by the probability summation model.

DISCUSSION

The possibility of probability summation across space

There may be probability summation among receptive fields located at different spatial positions within a single channel as well as among receptive fields in different channels (King-Smith and Kulikowski, 1975; Stromeyer and Klein, 1975; Graham, 1977). Such probability summation across space would occur if the channels were like those described at the end of assumption 2 except that there was uncorrelated variability in the responses of receptive fields located at different positions.

Test-plus-sine results. How could the general probability summation model above (assumptions 1–7) handle the possibility of probability summation across space? The only modification would need to be in assumption 2 which postulates completely linear channels. A number of computer calculations were done (see footnote 4) to investigate how well the three properties of assumption 2 do in describing a version of the model including probability summation across space. In fact, the three properties describe it well. The computer calculations suggest that the predictions computed from assumptions 1–7 (Fig. 3) are very close to the predictions that would be obtained from a version of the model that explicitly and accurately included probability summation across space. The version with probability summation across space, therefore, can accurately account for the test-plus-sine data (including thresholds for the test stimuli alone). If the version including probability summation across space is actually correct, however, the bandwidths W estimated in Fig. 4 and Table 2 are underestimates of the channel bandwidth.[10] How great an underestimate is extremely difficult to say because the bandwidth estimates and the estimates of the area over which probability summation occurs seem to be heavily interdependent.

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[10] "Channel bandwidth" used in the context of probability summation across space still refers to the bandwidth of the frequency response of the receptive fields making up the channel.

Sine-plus-sine results. If probability summation across space is the correct assumption, the bandwidth estimates originally made from sine-plus-sine experiments (Sachs *et al.*, 1971, for example) are also underestimates of the true channel bandwidth (King-Smith and Kulikowski, 1975; Stromeyer and Klein, 1975). An explanation of why is given in Graham (1977). Exactly how much of an underestimate depends on various properties of the channels, e.g. the density of channels, and is not easy to calculate in any case. One sample calculation is given in Graham and Rogowitz (1976) based on some previous theoretical work by Stromeyer and Klein (1975). Preliminary calculations suggest that the same bandwidth can explain both the test-plus-sine and the sine-plus-sine experiments under the assumption of probability summation across space.

FM grating results. Another kind of stimulus, a frequency-modulated (FM) grating, was used by Stromeyer and Klein to investigate the spatial pooling properties and bandwidths of channels. Graham and Rogowitz's (1976) reanalysis of their study showed that the results are probably consistent with either the assumption of probability summation across space or the assumption of no probability summation across space (given probability summation across channels). Again, the bandwidth necessary on the assumption of probability summation across space is larger (and by about the same amount as for the sine-plus-sine case).

King-Smith and Kulikowski (1975). In contrast to their earlier conclusion that both narrowband and broadband channels would be necessary to explain test-plus-sine and sine-plus-sine results, King-Smith and Kulikowski have recently suggested (1975) that a single bandwidth would be sufficient if there is probability summation both among channels and across space (i.e. in their terms, if there is probability summation among sub-units[11]). Although they only report one sample calculation to support their suggestion, the calculations reported here (Fig. 3 and Discussion) show that a single bandwidth is indeed sufficient if there is probability summation among channels and across space.

In fact, the calculations reported here show that a single bandwidth is sufficient even if there is no probability summation across space as long as there is probability summation across channels. King-Smith and Kulikowski (1975) disagree with this latter conclusion. They try to show that probability summation across space is necessary by computing some sample predictions (p. 250 and Appendix). Their method of computing predictions involves *a priori* specification of parameters like bandwidth and

number of channels rather than the derivation of a relationship (like the power law here) that allows estimation of the best-fitting parameter values. From these calculations they conclude that, without probability summation across space, the test-plus-sine data cannot be explained on the basis of probability summation among channels of a single bandwidth. This conclusion is in direction contradiction to the conclusions here. This discrepancy is perhaps due to the *a priori* specification of parameters (most probably the specification of bandwidth as a function of frequency) that is required by their method of doing theoretical calculations.

Although the results from the test-plus-sine experiments cannot be taken as support for the existence of probability summation across space, King-Smith and Kulikowski (1975) do present other evidence which seems to be better support.

Conclusions about bandwidth and probability summation across space. A large amount of data seems to be consistent with a model in which there is probability summation among relatively narrowband channels. These data include results from test-plus-sine, sine-plus-sine, and FM grating experiments. The data appear to be consistent with either of two versions of the model, one assuming probability summation across space and one not. The bandwidth necessary on the assumption of *no* probability summation across space is calculable (Fig. 4 here; Sachs *et al.*, 1971, for example). And the bandwidths necessary for the different experiments are in satisfactory agreement (but see Discussion below on bandwidths at different frequencies). The bandwidth necessary on the assumption of probability summation across space is definitely larger but more difficult to calculate exactly. It is therefore not yet certain, although it seems likely, that the bandwidths for the different experiments agree under this assumption also.

Further, whether probability summation across space is the correct assumption or not has yet to be firmly established although existing evidence (King-Smith and Kulikowski, 1975; Mostafavi and Sakrison, 1976; Stromeyer and Klein, 1975; Graham and Rogowitz, 1976) does favor probability summation across space over the simple kind of channel described at the end of assumption 2. There are, of course, other possible assumptions about pooling across spatial extent which might be even better.

Bandwidths of channels at different spatial frequencies

If the probability summation across channels model is correct, test-plus-sine experiments provide a relatively simple way of estimating the bandwidth of channels having different best spatial frequencies (Fig. 4). Unfortunately, the estimates derived here from the two experimental studies depend on spatial frequency in two different ways. The estimates in the larger field size where bandwidth estimate increases with frequency (Shapley and Tolhurst, 1973) may be closer to correct than those in the small field size where bandwidth estimate is independent of frequency (Kulikowski and King-Smith, 1973) as will be elaborated below.

Effect of field size on bandwidth estimates. The estimate of the bandwidth parameters [Fig. 4, equation (13)] is based on several kinds of data. One of these,

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[11] King-Smith and Kulikowski (1975) use the word "sub-unit" as the word "receptive field" is used here, and use the word "detector" or "detection process" to refer not to a single receptive field but to whatever it is that detects a particular stimulus. Thus, in their 1975 paper, a "detector" can be a cluster of "sub-units" among which there is probability summation. In Kulikowski and King-Smith (1973), however, the possibility that a detector is actually made up of sub-units is not mentioned nor is the possibility of probability summation. Rather, the detectors themselves are described as being the elementary units, as being equivalent to receptive fields.

the contrast sensitivity function $S(f)$, is known to depend on field size (e.g. Campbell and Robson, 1968). The other two kinds probably do not since both the sensitivity of the visual system to the test stimulus, $S(test)$, and the sensitivity function of the test-stimulus detector, $S_{tes}(f)$, involve a stimulus located at the center of the field.

The most obvious way in which the contrast sensitivity function depends on field size is that the low-frequency decline is more dramatic for smaller field sizes (Hoekstra, van der Goot, van der Brink and Bilsen, 1974; Savoy and McCann, 1975). The contrast sensitivity in a small field relative to that in a large field gets progressively *smaller* as the spatial frequency gets *lower*. Therefore, since the estimate of W depends inversely on contrast sensitivity [see equation (13)], the estimate of W in a small field relative to that in a large field should get progressively *larger* as the frequency gets *lower*. For frequencies below 7 or 8 c/deg, this is exactly the difference between the estimates of W from Kulikowski and King-Smith's (1973) results collected in a small field (2.5° dia) and those from Shapley and Tolhurst's (1973) results collected in a large field (5.5° dia).

Not only do the measured contrast sensitivity functions show this effect of field size, but there is a good theoretical reason why they might. Suppose that the true bandwidth of channels did increase in proportion to the center frequency of the channels. Then the receptive fields associated with a spatial frequency channel will get broader as the center frequency of the channel gets lower. And a grating in a small enough field will not completely cover any of the receptive fields of the lowest frequency channels. Thus the lowest frequency channels will be less sensitive to gratings in a small field than to gratings that are infinite in extent. That is, the contrast sensitivity function will be lower at low spatial frequencies for small fields than for large fields. And the estimates of W for low spatial frequencies will be correspondingly higher.

It remains to be explained why, for the highest spatial frequencies used (8–10 c/deg), the estimates of the bandwidth parameter from Kulikowski and King-Smith's (1973) data are actually smaller than the estimates from Shapley and Tolhurst's data. It is not clear that the difference in field size can explain this difference in bandwidth estimates.

Bandwidths at different frequencies. It is tempting to conclude at this point that channel bandwidth does increase linearly with increasing frequency. This conclusion would be consistent with the bandwidth estimates from the Shapley and Tolhurst and the later King-Smith and Kulikowski (1975) test-plus-sine experiments done in a relatively large field. Inconsistency with the earlier Kulikowski and King-Smith (1973) experiment would be mostly explained away as the consequence of their small field size. Thresholds for sine-wave gratings as field size is varied also seem to be consistent with linearly increasing bandwidths; the measured sensitivity to a grating as a function of number of cycles is the same function for all nominal frequencies (Hoekstra *et al.*, 1974; Savoy and McCann, 1975; Estevez and Cavonius, 1976). Further, results of adaptation and masking experiments are usually interpreted as showing bandwidth linearly increasing with frequency.

Quick and Reichert (1975), however, used sine-plus-sine experiments in a relatively large field to measure the bandwidth at 5, 15 and 21 c/deg. Their results suggest that the bandwidths are equal rather than increasing with frequency. The question of how bandwidths at different frequencies compare is still not definitely answered.

Two other complications in the interpretation of test-plus-sine experiments

Retinal inhomogeneity. The existence of retinal inhomogeneity is well-known, and it is possible that the receptive fields sensitive to low spatial frequencies may tend to be further from the central fovea than the receptive fields sensitive to higher spatial frequencies (Doorn, Koenderink and Bouman 1972; Rubenstein and Limb, 1975). That is, different spatial frequency channels may actually be located at different retinal positions. Since sine-wave stimuli cover a large area, a different-channels-at-different-positions version of the model could certainly account for the known sine-plus-sine results. It might also be able to account for the test-plus-sine results in a manner analogous to that described in this paper, but whether it could or not depends heavily on the precise bandwidths and the exact characteristics of the inhomogeneity.

On the other hand, some recent evidence suggests that, for all spatial frequencies, the central fovea may be more sensitive than any other retinal position under conditions like those usually used in the sine-plus-sine and test-plus-sine experiments (Hilz and Cavonius, 1974; Robson, 1975; Wilson and Giese, 1976). If this is true, the inhomogeneity will not be important in interpreting either sine-plus-sine or test-plus-sine experiments because both sine waves and aperiodic test stimuli are being detected at the central fovea.

Summation in detection or facilitation in masking. The test-plus-sine experiments were analyzed here as detection experiments. The task for the observer was assumed to be the detection of the test-plus-sine combination stimulus, i.e. the discrimination of the combination from a blank field. In the procedure used in the experiments, however, the sine was on continuously, and the observer adjusted the contrast in the test stimulus until he was satisfied. The task for the observer, therefore, might actually have been the discrimination of the combination from the sine, a task that is identical to that in a masking experiment where the sine is the masking stimulus. Further, results like those of Shapley and Tolhurst's and Kulikowski and King-Smith's might well occur (for positive contrasts at least) in a straight masking experiment. The reduction in the test-stimulus threshold due to the presence of some sine-wave (in positive phase) can be seen as an instance of facilitation, and facilitation is known to occur in experiments using sub-threshold sine waves as masking stimuli (Nachmias and Sansbury, 1974; Stromeyer and Klein, 1974).

However, to analyze Shapley and Tolhurst's and Kulikowski and King-Smith's experiment as masking experiments using a probability summation model would require making reasonable assumptions about sine-wave masking, and the necessary information about sine-wave masking is not available. In any case,

the original investigators analyzed the experiments as detection experiments. Since these investigators were also the observers, the correct interpretation of the data is certainly as detection data if either (a) the observer can choose whether to make a detection-type or a masking-type discrimination or if (b) the observer knows which type of discrimination he makes.

SUMMARY

A large amount of available data on the detectability of visual patterns is consistent with a model in which there is probability summation (or equivalent non-linear pooling of outputs) among multiple, relatively narrowband channels. Under this interpretation, the test-plus-sine data shown in any single panel of Fig. 3 do not represent the sensitivity function of a single linear broadband detector as they were originally thought to (Shapley and Tolhurst, 1973; Kulikowski and King-Smith, 1973). They represent the conglomerate action of many relatively narrowband channels with probability summation among the channels.

The fit of the probability summation model's predictions (lower solid and dotted curves in Fig. 3) to the actual data (points) is very good and required many fewer parameters than the explanation in terms of broadband detectors.

The model of probability summation among channels can also account for the thresholds of the aperiodic test stimuli. In the process of accounting simultaneously for these thresholds and for the test-plus-sine data of Fig. 3, one obtains an estimate of the bandwidth of the individual channels at different frequencies. These estimates are shown in Fig. 4 as a function of the best frequency of the channel. The difference between the ways in which the estimates from the two experiments depend on spatial frequency may be an artifact due to the small field size used by Kulikowski and King-Smith (1973). In any case, the estimates are in satisfactory agreement with previous estimates from sine-plus-sine experiments.

These data remain consistent with the model of probability summation among channels even if probability summation across space is also assumed. The bandwidth necessary to explain the data is, however, larger when probability summation across space is assumed than when it is not.

If a model involving multiple relatively narrowband channels is correct, it would be interesting to know something about the responses elicited in different channels by a pure sinusoidal grating. Does a pure sine excite only one channel as has sometimes been assumed for simplicity's sake (e.g. Sachs et al., 1971)? The theoretical predictions described here provide a partial answer to this question. If only one channel responded to a sine, or if several channels responded equally well, a good deal of curvature would be predicted in the contrast interrelationship functions (the functions plotting contrast in test versus contrast in sine when the test-plus-sine combination is just at threshold). This much curvature (solid curves of Fig. 2) appears incompatible with the small amount of published data. Thus, more than one channel probably responds to a sine and the channels responding do not all respond equally.

That a model or theory is consistent with all available data does not make it correct. Probability summation among channels did occur in earlier sine-plus-sine experiments (Sachs et al., 1971). If probability summation did not occur in the test-plus-sine experiments, however, the original investigators' interpretation of these experiments in terms of broadband channels might be correct. Or, of course, some completely different theory might be the appropriate explanation for this whole class of experiments. On the other hand, the ability of the probability summation among multiple channels model to predict the data from test-plus-sine experiments is impressive. With few assumptions or free parameters not supported by independent data, the model successfully accounts for the detectability of a wide variety of patterns, both periodic and aperiodic.

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APPENDIX 1

Contrast interrelationship functions predicted by the probability summation model

General case. Although it involves a number of steps, the derivation of the contrast interrelationship function predicted by the probability summation model requires only algebraic manipulations (see footnote 4). Using assumptions one (probability summation among channels), two (linear channels, properties one and two), and four (Quick's psychometric function), one can prove that

$$ c(\text{test}) \, S(\text{test}) = \left[\cfrac{1}{\displaystyle\sum_{i=1}^{N} \left(\cfrac{S_i(\text{test})}{S(\text{test})} + \rho \cfrac{S_i(\text{sine})}{S(\text{sine})} \right)^k} \right]^{1/k} \quad (A1) $$

where ρ is the ratio of relative sine contrast to relative test contrast, i.e.

$$ \rho = \frac{c(\text{sine}) \, S(\text{sine})}{c(\text{test}) \, S(\text{test})}. \quad (A2) $$

Under assumption 5. Using the nearby-channels assumption allows further simplification of (A1). Some new terminology will be useful. Let M_a be the number of channels having non-zero sensitivity to frequency f_a. For convenience, let the channels be numbered so that the M_a channels sensitive to f_a are first M_a. By the nearby-channels assumption, the sensitivity of each of these first M_a channels to the test stimulus will equal the sensitivity of channel a^* that is,

$$ S_i(\text{test}) = S_{a^*}(\text{test}) \text{ for } i = 1, M_a. $$

Let the relative sensitivities of different channels to a sine-wave frequency f_a be expressed by constants g_{ja}. These constants give the sensitivity of channel j relative to that of the most sensitive channel (a^*), or

$$ g_{ja} = \frac{S_j(f_a)}{S_{a^*}(f_a)}. $$

For channel a^*, the constant g_{a^*a} equals one. For the first M_a channels, g_{ja} will be greater than zero and less than or equal to one. For the other channels, g_{ja} will equal zero.

Then equation (A1) can be modified to

$$ c(\text{test}) \, S(\text{test}) $$

$$ = \left[\cfrac{1}{\displaystyle\sum_{i=1}^{M_a} \left(\cfrac{S_{a^*}(\text{test})}{S_a(\text{test})} + \rho \cfrac{g_{ia}}{(M_a^k)^{1/k}} \right)^k + 1 - M_a \left(\cfrac{S_{a^*}(\text{test})}{S(\text{test})} \right)^k} \right]^{1/k} \quad (A3) $$

This contrast interrelationship function (A3) for the nearby-channels case depends in general on the particular values in the set of non-zero relative sine sensitivities, the set $G_a = \{g_{ia} | i = 1, M_a\}$. But suppose they have been specified (and so, as a result, have been the values M_a and M_a^k). Suppose also that a value of k has been settled on. Then there is only one number left that must be known before (A3) can be used to generate a contrast interrelationship function. That number is $S_{a^*}(\text{test})/S(\text{test})$. Once that number has been specified, sweeping through values of ρ will generate the whole function. Rather than specifying $S_{a^*}(\text{test})/S(\text{test})$, however, the importance index could be specified just as well. (From the importance index one can always calculate $S_{a^*}(\text{test})/S(\text{test})$ since the other variables in the importance index are known.) In short, once the set of relative sine sensitivities G_a and the steepness of the psychometric function have been specified, the family of possible contrast interrelationship functions becomes a one-parameter family, where the parameter is the importance index.

Minimal sets. It is not true, however, that every possible set G_a of relative sine sensitivities leads to a different one-parameter family of contrast interrelationship functions. If, for example, one doubles the number of channels responsive to a sine (i.e. doubles M_a) by duplicating the existing set of sine sensitivities (so that each value of g_{ia} in the original model's set appears twice in the expanded model's set), than the predicted contrast interrelationship function for any given value of importance index remains the same.

There is nothing magic about the number 2. Expanding the set G_a by replicating it any (integral) number of times will leave untouched the contrast interrelationship function for any particular value of importance index. This can be proved using straightforward, if tedious, algebraic manipulations.

Because the model's predictions remain the same for unions of the set G_a with sets identical to itself an indefinite number of times, it is a sufficient description of the model to know the *smallest* set which, when replicated some number of times, can generate G_a. This set will be called the *minimal set* of relative sensitivities to the sine f_a and will be denoted by $\hat{G}_a$.

In terms of minimal sets, one can easily state the dependency of the contrast interrelationship function. The contrast interrelationship function is completely determined by the minimal set $\hat{G}_a$, the steepness parameter k, and the importance index $I(f_a, \text{test})$.

The rectangular case. In the special case of a rectangular minimal set $(\hat{G}_a = \{1\})$, the contrast interrelationship function (A3) reduces to

$$ c(\text{test}) \, S(\text{test}) $$

$$ = \left\{ \frac{1}{\left(I(f_a, \text{test})^{1/k} + \rho \right)^k + 1 - I(f_a, \text{test})} \right\}^{1/k} $$

APPENDIX 2: GLOSSARY

See discussion of general conventions at beginning of the section in the main text called Assumptions of the Model.

Symbols in approximate alphabetical order

α — equals multiplicative constant in power law

a^* — the channel, of all channels, which is most sensitive to a given frequency f_a

β — equals exponent in power law

$c(stim)$ — contrast of stimulus

f_a or f_i — a sine-wave frequency. With subscript a refers to an arbitrary frequency. With all other subscripts, refers to best frequency of channel indicated by subscript

$F_{test}(f)$ — measure of amount of frequency f in test stimulus. Equals four times Fourier transform at frequency f of test stimulus when test stimulus has peak-trough luminance of one unit

g_{ja} — relative sensitivity of channel j to frequency f_a, equals $S_j(f_a)/S_a(f_a)$ (Appendix 1)

G_a — the set of relative sine sensitivities, $\{g_{ja}|j=1, M_a\}$ (Appendix 1)

$\hat{G}_a$, $\hat{G}$ — the minimal set of relative sine sensitivities

$I(f_a, test)$ — the importance index, a measure of the importance of channels sensitive to frequency f_a in the detection of the test stimulus. See equation (10) in text

k — parameter determining steepness in Quick's psychometric function

$L(x)$ — luminance as function of position

M_a — number of channels having non-zero sensitivity to frequency f_a

M_a^E — equivalent number of channels sensitive to frequency f_a. See equation (9) in text

N — total number of channels

$P(stim)$ — probability that observer detects stimulus

$P_i(stim)$ — probability that channel i detects stimulus

$R_i(stim)$ — response of channel i, equals one when $P_i(stim) = 1/2$

$S(stim)$ — reciprocal of contrast for which $P(stim) = 0.5$, called psychophysical sensitivity

$S_i(stim)$ — reciprocal of contrast for which $P_i(stim) = 0.5$, called sensitivity of channel i

$S_{test}(f)$ — sensitivity of test-stimulus detector to frequency f, i.e. the effectiveness of a sine of frequency f in lowering the threshold for the test stimulus; an aperiodic, and, therefore, broadband test stimulus

w_i — relative bandwidth, equals W_i/W

W_i — the bandwidth parameter of channel i

W — the bandwidth parameter at 1 c/deg.

Words in approximate alphabetical order

additive phase: phase in which sine is added to test stimulus in phase with the component in test stimulus of same frequency

best frequency: the frequency, of all frequencies, to which a channel is most sensitive

contrast: one-half the difference between the maximum and minimum luminances divided by the mean luminance

contrast interrelationship function: $c(test)$ $S(test)$ as a function of $c(sine)$ $S(sine)$

contrast sensitivity function: $S(f)$ as a function of f

equivalent number of channels: see M_a^E

extrapolation pair: the two values of relative contrast in the sine-wave for which is known the relative contrast in the test necessary for the test-plus-sine to be at threshold. $S_{test}(f)$ is calculated by linear extrapolation from the two points on the contrast interrelationship function determined by these two values

importance index: see $I(f_a, test)$

mean luminance: luminance averaged over the spatial extent of the pattern

relative contrast: contrast of a stimulus multiplied by the psychophysical sensitivity for the stimulus, $c(stim)S(stim)$

spatial frequency: number of cycles per unit of distance, usually given in cycles per degree of visual angle (c/deg)

subtractive phase: phase in which sine is added to test stimulus 180° out of phase with the component in test stimulus of same frequency.