Matching to Share Risk

Pierre-Andre Chiappori
Department of Economics
Columbia University

Philip J. Reny*
Department of Economics
University of Chicago

December 2005

Abstract

We consider a matching model in which individuals belonging to two populations ('males' and 'females') can match to share their exogenous income risk. Within each population, individuals can be ranked by risk aversion in the Arrow-Pratt sense. The model permits non transferable utility, a context in which few general results have previously been derived. We show that in this framework (i) a stable matching always exists, (ii) it is essentially unique, and (iii) it is negatively assortative: for any two couples, the more risk averse male is matched with the less risk averse female. We discuss the implications of these results for the empirical analysis of risk-sharing.

1. Introduction

The hypothesis that groups share income-risk efficiently has been studied both theoretically and empirically. A basic theoretical implication of efficient risk-sharing, due to Karl Borch (1962), is the so-called *mutuality principle*.[1] According to this principle, individual consumption should depend only on variations in total group income or on ex-ante individual shocks such as those that might affect an individual's income distribution. In particular, individual consumption should *not* depend on the realization of individual idiosyncratic shocks to income.

Important empirical tests of efficient risk-sharing have exploited the mutuality principle in several ways. For example, one may, following Robert Townsend (1994), regress individual consumption on both aggregate consumption (or aggregate income) at the group level and various individual-specific variables (individual income, proxies for idiosyncratic shocks, etc.). Efficient risk-sharing is rejected if, controlling for aggregate

---

*Reny gratefully acknowledges financial support from the National Science Foundation (SES-9905599).

[1] See also Robert Wilson (1968).

shocks, individual-specific shocks are found to matter. An alternative approach, initiated by John Cochrane (1991), uses an Euler equation formulation relating the marginal utility of consumption at date t to the expected marginal utility of consumption at date $t + 1$ conditional on information available at date t. According to the mutuality principle, the ratio of marginal utility of consumption between periods t and $t + 1$ should be independent of idiosyncratic shocks at date t. In particular, if preferences exhibit constant relative risk aversion, growth in individual log consumption should be orthogonal to idiosyncratic shocks, a property that can readily be tested provided (even short) panel data on consumption are available.

The question that is central to the present paper is this. If risk-sharing is limited to groups whose sizes are smaller than the total size of the population, then what will be the composition of these groups?[2] Will group members be homogeneous with the same risk preferences, or will they be heterogeneous with distinct risk preferences?

The answer has important implications for the empirical risk-sharing literature. For example, if one wishes to apply Townsend's estimation strategy to rural villages under the assumption that households have CRRA (instead of CARA) preferences, then one may have to assume that all households in each village have *identical* risk preferences.[3] Similarly, the Euler equation approach requires that group members' characteristics (preferences, risk tolerance, discount factor, ...) are cross-sectionally uncorrelated with the distribution of income shocks. In practice, most of the empirical risk-sharing literature assumes that group members have identical risk preferences.[4]

A second, although related, potential difficulty with the empirical literature is that one is often forced to employ household consumption data rather than individual consumption data. Group members are then taken to be households, and risk is assumed to be shared between households in the group. This is invariably modeled by assuming that there is a single utility function representing the household's preferences. In reality, of course, the household consists of several individuals, with possibly different

---

[2] There are a variety of reasons that risk-sharing among the entire population might be infeasible. One is the cost of monitoring in the presence of private information about income: smaller groups may be much more effective in monitoring the incomes of their members. Smaller groups may also help reduce the number of contingent markets to a feasible level (see David Cass, Graciela Chichilnisky and Ho-Mou Wu 1996). Finally, Garance Genicot and Debraj Ray (2003) show that the presence of limits to commitment tend to bound the maximum size of risk sharing groups.

[3] For example, when individuals have distinct risk aversion parameters, the relationship between individual and aggregate consumption is highly non linear (it actually admits no closed-form solution); a linear regression is thus misspecified. More importantly, the consumption of any group member depends on the person's Pareto weight, which in turn varies with the ex ante distribution of the person's income. A spurious rejection will then occur if the idiosyncratic variable picks up this effect. Simple differences of log consumption may eliminate the problem, but only when preferences are identical (see for instance Esther Duflo and Christofer Udry 2004).

[4] See for instance Sumru Altug and Robert Miller (1990), Fumio Hayashi, Joseph Altonji and Laurence Kotlikoff (1996), Agnes Lund and Marcel Fafchamps (2003) and Duflo and Udry (2004).

risk preferences. These individuals may be sharing risk themselves and it is their joint income that is used in risk-sharing with the group. This raises a theoretical question, namely whether it is possible to correctly model the household's decision at the group risk-sharing level by representing the household's preferences by a single utility function over its joint income. The answer is negative. A group's attitude toward risk generally depends not only on individual preferences, but also on the members' Pareto weights. Since weights are typically correlated with the characteristics of the members' income distributions, the resulting 'household utility' varies with individual distributions. This generates nontrivial econometric difficulties which, to the best of our knowledge, have not been taken into account in the empirical literature. There is, however, precisely one instance in which these difficulties do not arise, namely when all household members have preferences in the same ISHARA class.[5] In this case (and in this case alone), a household utility representation exists, it is itself in the ISHARA class, and does not depend on the individual Pareto weights.[6] Unfortunately, when this positive result is combined with the very common assumption in the empirical literature that household utility is CRRA, one is again led to the implicit assumption that individuals within the household are identical.

It is thus fair to say that the assumption of identical risk preferences is pervasive (either explicitly or implicitly) in the empirical risk-sharing literature. An obvious problem, however, is that risk preferences are known to vary widely across the population.[7] Consequently, the identical risk-preference assumption must rest upon the idea that risk-sharing groups and households are not formed randomly, otherwise they would almost certainly be composed of members with distinct risk preferences.

Thus, we are led again to our main question. Should one expect the composition of risk-sharing groups to be homogeneous or heterogeneous? The answer, of course, lies within a model in which group formation for the purposes of risk-sharing is *endogenous*.

Our very stylized model is as follows. There are two populations, a population of males and a population of females. Within each population, individual incomes are drawn from some joint distribution; we assume only that each joint distribution is exchangeable (so that males are ex ante identical from any female's viewpoint and vice versa), and that male and female incomes are independent. Any male and any female

---

[5] ISHARA stands for Identical Shape Harmonic Absolute Risk Aversion. A collection of utility functions, $u_1, u_2, ...$, are in the same ISHARA class if the index of absolute risk aversion for each u_i is a harmonic function (i.e., $\frac{1}{a_i+b_i x}$) of income, x, and the shape parameters are identical (i.e., $b_i = b$ for all i). Then the group's preferences can be described by a HARA function with an index of absolute risk aversion equal to $\frac{1}{(\sum_i a_i)+bx}$.

[6] See Robert Wilson (1968), and Maurizio Mazzocco (2004a) for a complete proof.

[7] See for instance Robert B. Barsky et. al. (1997), Luigi Guiso and Monica Paiella (2001), Alma Cohen and Liran Einav (2005), Pierre-André Chiappori and Bernard Salanié (2006) and Chiappori, Townsend and Sam Schuhlhofer-Wohl (2006).

can form a group for the purposes of risk-sharing. No other risk-sharing groups are feasible.[8] Thus, we are considering here a one-to-one matching model in which the sole motivation for matching is to share risk.

To address the question of whether risk-sharing groups (couples, under our assumptions) will consist of homogeneous or heterogeneous individuals, we assume that individuals within each population can be ranked according to their risk aversion. That is, for any two males, one is more risk averse than the other in the sense of Arrow-Pratt, and similarly for any two females.[9] Consequently, risk-sharing groups cannot be 'homogeneous' in our model unless, among matched individuals, the most risk averse male is matched with the most risk averse female, the second most risk averse male is matched with the second most risk averse female, etc. In short, homogeneity of risk-sharing groups requires positive assortative matching.[10]

The key behavioral assumption in our model is that competitive forces will determine which males share risk with which females. More specifically, we shall focus attention on matches (i.e., pairwise assignments of males to females with possibly some singles) that are *stable* in the sense that there are income sharing rules for matched couples such that no individual can improve his payoff by not engaging in risk-sharing, and no male and female can improve their payoffs regardless of the sharing rule they might adopt.

Our two main results are as follows. First, at least one stable match always exists. Second, every stable match is negative assortative in the sense that among the set of matched couples, the most risk averse man is matched with the least risk averse woman, the second most risk averse man is matched with the second least risk averse woman, and so on. Consequently, in contrast to the empirical literature's assumption that members of risk-sharing groups are identical with respect to risk preferences, we find that stability considerations lead instead to the conclusion that group members have distinct, and in some sense even opposite, risk preferences.

While we have confined our attention to a particularly, and deliberately, simplified situation involving pairwise matching, we are confident that a version of our result will hold up more generally. Competitive forces will lead risk-sharing groups, including households, to be composed of individuals whose risk preferences are not identical, and perhaps substantially different.

Finally, from a technical perspective, it is worth mentioning that the model we consider involves matching under non-transferable utility. Indeed, while compensating

---

[8] Such pairwise groups are in fact optimal if the incomes of all of the males are perfectly correlated as are the incomes of all of the females.

[9] But note that an arbitray male need not be more or less risk averse than an arbitrary female.

[10] Note however, that because we do not assume, although we permit, males and females to be comparable in the Arrow-Pratt sense, positive assortative matching does not literally imply that matched individuals are identical. On the other hand, without positive assortative matching, homogeneity has no chance.

transfers within couples are possible (and indeed essential) in our context, the more standard transferable utility property does not hold in general; the non linearities implicit in risk aversion generally imply that the expected utility cost to one spouse of transferring an expected utile to his/her partner is not constant along the Pareto frontier.[11] The non linearity of the Pareto frontier raises interesting problems (e.g., the existence and characterization of stable matches) that have rarely been studied in the literature.[12]

2. The Model

Consider a one-to-one matching model in which n_M males match with with n_F females. Males and females are strictly risk averse, and $u_i : [0, \infty) \to \mathbb{R}$ denotes the von Neumann-Morgenstern utility function of income of male $i \in \{1, ..., n_M\}$, and $v_j : [0, \infty) \to \mathbb{R}$ denotes that of female $j \in \{1, ..., n_F\}$. For every male i, $u_i(\cdot)$ is bounded and continuous on its domain, $u_i'(x) > 0$, $u_i''(x) < 0$ for all $x \in (0, \infty)$, $u_i(0) = 0$, and $\lim_{x \downarrow 0} u_i'(x) = +\infty$. Female utility functions satisfy the same conditions.

We assume that agents in each population can be ranked according to their risk aversion in the Arrow-Pratt sense. That is, for any two males, one of them is more risk averse than the other, and similarly for any two females. We may therefore index each set of agents in order of increasing risk aversion, i.e., so that male i is more risk averse than male j if and only if $i > j$, and similarly for females.[13]

Each individual is endowed with an exogenous non negative and non degenerate random income denoted by $\tilde{x}_i$ for male i and $\tilde{z}_j$ for female j.[14] The n_M random variables $\tilde{x}_1, ..., \tilde{x}_{n_M}$ are assumed to be exchangeable and independent of the n_F exchangeable random variables $\tilde{z}_1, ..., \tilde{z}_{n_F}$.[15] Note that incomes within each population may be correlated and the marginal income distributions of any two members of the same population are identical.

Let $\tilde{y} = \tilde{x}_i + \tilde{z}_j$ be the non degenerate random total income of any male and female.

---

[11] Interestingly, ISHARA is necessary and sufficient for expected utility to be transferable within the couple; see Schuhlhofer-Wohl (2004).

[12] An exception is the recent work by Patrick Legros and Andy Newman (2002) which provides sufficient conditions for assortative matching in a context of imperfectly transferable utilities.

[13] The assumption that individual i is more risk averse than j only means that i's Arrow-Pratt measure of risk aversion is greater than j's when their income realizations are *identical.* Because incomes are random and the incomes received through risk sharing will be endogenous, there is no reason to expect, and we do not assume, that i's Arrow-Pratt measure of risk aversion is greater than j's at their respective *ex-post* incomes when these are distinct.

[14] We adopt the convention that tilde's denote random variables and the absence of a tilde denotes a realization of that random variable.

[15] A collection of random variables are exchangeable if their joint c.d.f. is a symmetric function. Note that income distributions may be discrete or continuous.

Exchangeability guarantees that the distribution of $\tilde{y}$ is independent of i and j. Let $Y \subseteq [0, \infty)$ denote the support of $\tilde{y}$.

Individuals who choose to remain single receive their random income. On the other hand, if male i and female j choose to match and share risk, they can enter into a binding agreement, ex-ante, prior to the realization of their incomes, specifying how their income will be shared. We assume that this binding agreement is ex-ante Pareto efficient. Consequently, as shown by Borch (1962), and given our assumptions, the allocation of income between male i and female j must satisfy the mutuality principle: each agent's income share depends only on the couple's total income $\tilde{y}$. In particular, a couple's efficient risk-sharing agreement can be denoted by a function, $x : Y \to [0, \infty)$, of total income, where $x(y)$ denotes the amount of total income y given to the male, and $y - x(y)$ is the amount given to the female. Under this agreement, male i's expected utility is $Eu_i(x(\tilde{y}))$ and female j's expected utility is $Ev_j(\tilde{y} - x(\tilde{y}))$.

Because individuals can choose to remain single, we focus on sharing rules that are *individually rational* in the sense that both the male's and the female's ex-ante payoffs are at least as large under the agreed upon sharing rule as their respective ex-ante payoffs when single.

We wish to determine which pairs of men and women will choose to share risk and, for those that do, we also wish to determine the sharing rules they adopt. To accomplish this we employ a standard stability criterion.

An ordered-pair (i, j) is called a *couple* if i is a male and j is a female. A *match* is a collection of couples in which each individual appears at most once.[16] Individuals appearing in some couple are said to be *matched*, while individuals not appearing in any couple are said to be *unmatched* or *single*.

A match is *stable* if for each couple there is an individually rational and efficient sharing rule such that the resulting collection, $\mathcal{S}$, of sharing rules has the following property. There does not exist a male i, a female j, and a sharing rule $x(\cdot)$ such that male i and female j each strictly prefer the sharing rule $x(\cdot)$ to his/her sharing rule in $\mathcal{S}$ if matched or to his/her random income if unmatched.

Remark 1. *It is inefficient for a man and a woman to each be unmatched. Indeed, one way they could share income is to mimic being single (i.e., male i keeps $\tilde{x}_i$ and female j keeps $\tilde{z}_j$). Such a sharing rule will however never be efficient, since efficiency requires male and female income shares to rise and fall together (by the mutuality principle and Wilson (1968), Theorem 5), while $\tilde{x}_i$ and $\tilde{z}_j$ are non degenerate and independent. Consequently, there exist sharing rules that are preferred by both individuals over being single. It follows that at any stable match there can be single males* (*if and only if*

---

[16] That is, among all couples, each male (resp., female) index appears at most once as a first (resp., second) coordinate.

$n_M > n_F$) or single females (if and only if $n_M < n_F$), but not both. In particular, stability implies that all individuals are matched if $n_M = n_F$.

A modest amount of additional notation will be useful in what follows. First, let $r_j = Ev_j(\tilde{z}_j)$ denote female j's reservation utility, and let $\bar{v}_j = Ev_j(\tilde{y})$ denote female j's expected utility when she is matched with a male and receives all of the joint income $\tilde{y}$. Note that, because male and female incomes are non negative and non degenerate, $0 < r_j < \bar{v}_j$. Second, let $U_{ij}(v)$ denote male i's maximum expected utility when he is matched with female j, where the maximum is taken over all possible sharing rules that ensure female j an expected payoff of at least $v \in [0, \bar{v}_j]$.[17]

We next provide the more basic of our two main results.

3. Existence of a Stable Match

Theorem 3.1. *There exists at least one stable match*

Proof. Without loss of generality, we may assume that $n_M \geq n_F$. Consider the following artificial game between the n_M males, $i = 1, ..., n_M$, and a referee, player $i = 0$. The referee chooses a utility, $v_j \in [r_j, \bar{v}_j]$, for each female $j = 1, ..., n_F$. Male i chooses, for each female j, a probability $p_{ij} \in [0, 1]$, subject to $\sum_j p_{ij} \leq 1$. Let $p_i = (p_{i1}, ..., p_{in_F})$ denote male i's strategy.

Given the referee's strategy $v = (v_1, ..., v_{n_F})$ and the vector of male strategies $p = (p_1, ..., p_{n_M})$, male i's payoff is

$$\pi_i(p, v) = \sum_{j=1}^{n_F} p_{ij} U_{ij}(v_j) + \left(1 - \sum_{j=1}^{n_F} p_{ij}\right) Eu_i(\tilde{x}_i),$$

and the referee's payoff is

$$\pi_0(p, v) = \sum_{j=1}^{n_F} v_j \left[\left(\sum_{i=1}^{n_M} p_{ij}\right) - 1\right].$$

This game has the following interpretation. The referee chooses for each female j a utility level, v_j. Male i chooses, for each female j, a probability $p_{ij} \in [0, 1]$ with which he is matched with j. Hence, male i remains unmatched with probability $1 - \sum_j p_{ij}$. If male i is matched with female j, then male i is permitted to choose any sharing rule giving female j utility at least v_j. Male i's utility will then be $U_{ij}(v_j)$. Male i's

---

[17] The analyses of Borch (1962) and Wilson (1968), together with the assumptions made here, ensure that $U_{ij}(v)$ is always well-defined and continuous on $[0, \bar{v}_j]$.

payoff, $\pi_i(p, v)$, is then his expected utility given his strategy and the strategy of the referee. The referee's payoff can be interpreted as the value of expected excess demand for females, and the referee behaves so as to maximize this.

Because each player's payoff is continuous on the compact convex set of joint strategies and linear in his own strategy, this game possesses a pure strategy Nash equilibrium (p^*, v^*).

We first claim that $\sum_i p^*_{ij} \leq 1$ for each $j = 1, ..., n$. If not, then $\sum_i p^*_{ij} > 1$ for some j. Consequently, for this j, optimal play by the referee requires $v^*_j = \bar{v}_j$. But because $Eu_i(\tilde{x}_i) > 0 = U_{ij}(\bar{v}_j)$, every male strictly prefers being unmatched than matching with j at $\bar{v}_j$. Hence we must have $\sum_i p^*_{ij} = 0$, and this contradiction proves the claim.

Next, we claim that $\sum_i p^*_{ij} = 1$ for each $j = 1, ..., n$. If not, then $\sum_i p^*_{ij'} < 1$ for some j'. Consequently, optimal play by the referee requires $v^*_{j'} = r_{j'}$. But because every male strictly prefers matching with j' at her reservation value to being unmatched (see Remark 1), it must be the case that $\sum_j p^*_{ij} = 1$ for every male i. But this implies $\sum_i \sum_j p^*_{ij} = n_M \geq n_F$, which together with the first claim implies that $\sum_i p^*_{ij} = 1$ for every j. This contradiction establishes the claim.

Hence, letting P denote the $n_M \times n_F$ matrix whose i-th row is p^*_i, the columns add to unity and the rows add to no more than unity. Appending to P $n_M - n_F$ identical columns with i-th entry $(1 - \sum_j p^*_{ij})/(n_M - n_F)$ results in an $n_M \times n_M$ matrix that is doubly stochastic, and hence by Birkhoff's theorem (see, for example, Roger Horn and Charles Johnson 1985) a convex combination of permutation matrices. Observe next that any permutation matrix given positive weight in the convex combination defines a match as follows. Male i is matched to female j if in the permutation matrix there is a one in the i-jth entry, unless j is one of the appended columns in which case male i is unmatched. Moreover, the equilibrium conditions imply that this match has the following properties.

(1) Each female j is matched and her expected utility is $v^*_j \geq r_j$.

(2) If male i is single, then $Eu_i(\tilde{x}_i) = \pi_i(p^*, v^*) \geq U_{ij'}(v^*_{j'})$ for all females j'.

(3) If male i is matched with female j, then $p^*_{ij} > 0$ and so male i's expected utility is $U_{ij}(v^*_j) = \pi_i(p^*, v^*) \geq \max_{j'}[U_{ij'}(v^*_{j'}), Eu_i(\tilde{x}_i)]$.

Property (1) implies that the match is individually rational for the females. Property (2) implies that single males cannot improve their payoffs by mating with a matched female and offering her at least the utility she receives in the current match. Finally, property (3) implies that no matched male can improve his payoff by becoming single (and so we have individual rationality for the males) or by matching with a different female and offering her the utility she receives in the current match. We conclude that the match is stable. ■

Remark 2. *An alternative proof can be obtained through an application of a variant of the Gale-Shapley algorithm applied to a discretized utility space and then considering the limit as the discretization becomes infinitely fine.*

We now consider the most interesting part of the problem, namely the characterization of the stable matches.

4. Stability

Our second main result is the following.

Theorem 4.1. *Any stable match has the following features:*

- *either all of the men or all of the women are matched, and*
- *the matching is negative-assortative in the sense that among matched individuals, the k-th most risk averse male is matched with the k-th least risk averse female for all k.*

In particular, if there are equal numbers of males and females, the unique stable match is negative assortative and all individuals are matched.

Before considering the most general case involving any number of males and females, let us consider first the case in which there are two males and two females. Theorem 4.1 then reduces to the following more fundamental result.

Proposition 4.2. *When there are precisely two males and two females, the negative-assortative match, namely that in which the most risk averse male (female) is matched with the least risk averse female (male), is the unique stable match.*

The proof of Proposition 4.2 relies on the following lemma. Recall from Section 2 that $\bar{v}_j$ is female j's expected payoff from matching with any male and receiving the entire joint income. Recall also that $U_{ij}(v)$ denotes male i's maximum expected payoff when he is matched with female j, where the maximum is taken over all possible sharing rules that ensure female j an expected payoff of at least $v \in [0, \bar{v}_j]$. Finally, recall that male (female) 2 is more risk averse than male (female) 1.

Lemma 4.3. *If* $0 \leq v_1 < \bar{v}_1$ *and* $0 < v_2 \leq \bar{v}_2$ *are such that*

$$U_{11}(v_1) \geq U_{12}(v_2),$$

then

$$U_{21}(v_1) > U_{22}(v_2).$$

Lemma 4.3, whose proof is in the appendix, says the following. Suppose that a male is faced with two options, 1 and 2. Option 1 permits the male to match with the less risk averse female and to choose any sharing rule that gives her an ex-ante expected payoff of at least v_1. Option 2 permits the male to match with the more risk averse female and to choose any sharing rule that gives her an ex-ante expected payoff of at least v_2. The lemma states that if the less risk averse male weakly prefers option 1 to option 2, then the more risk averse male *strictly* prefers option 1 to option 2, so long as option 1 does not require the male to give the entire joint income to the female, and option 2 does not permit the male to keep the entire joint income for himself.

Alternatively, consider the two possible matchings, $\{(1,1),(2,2)\}$ on the one hand and $\{(1,2),(2,1)\}$ on the other, and suppose that for each match, incomes are shared in an efficient and individually rational manner. According to Lemma 4.3, if both females are indifferent between the two matches while male 1 weakly prefers the first, then male 2 strictly prefers the second.[18] We can now prove Proposition 4.2.

Proof of Proposition 4.2. We first argue that $\{(1,1),(2,2)\}$ is not stable. For suppose that it is. Then there are individually rational and efficient sharing rules associated with this match giving each female j utility v_j, say. Individual rationality ensures that $0 < v_j < \bar{v}_j$ for each female j, and efficiency implies that male i's payoff must be $U_{ii}(v_i)$. By stability, male 1 must be unable to strictly improve his payoff by matching with female 2 and ensuring her a payoff of at least v_2 – otherwise he could also make female 2 strictly better off by giving her slightly more income. Hence, $U_{11}(v_1) \geq U_{12}(v_2)$. But then, according to Lemma 4.3, $U_{21}(v_1) > U_{22}(v_2)$. That is, male 2 can strictly improve his payoff by matching with female 1 and choosing a sharing rule that leaves female 1 at least indifferent. But then male 2 can make both himself *and* female 1 strictly better off by giving female 1 slightly more income. We conclude that $\{(1,1),(2,2)\}$ is not stable. Consequently, by Theorem 3.1 and Remark 1, $\{(1,2),(2,1)\}$ is the unique stable match. ■

The proof of Theorem 4.1 is now straightforward.

Proof of Theorem 4.1. The first part of the theorem follows from Remark 1.

Consider next the third part of the theorem in which there are equal numbers of males and females. By Theorem 3.1 there exists at least one stable match, and by the first part of the theorem every individual is matched. It therefore suffices to show that any match different from the negative-assortative one cannot be stable. For any such match, define $\bar{k}$ as the smallest positive integer k such that the k-th most risk averse male (whose index is $n_M - k + 1$) is *not* matched with the k-th least risk averse female (whose index is k). By definition of $\bar{k}$, male $n_M - \bar{k} + 1$ must be matched with a female

---

[18] Note that individual rationality ensures that no individual receives the entire joint income.

(say j) who is *more* risk averse than female $\bar{k}$, and female $\bar{k}$ must be matched with a male (say i) who is *less* risk averse than male $n_M - \bar{k} + 1$. By Proposition 4.2, the submatch $\left(n_M - \bar{k} + 1, j\right), \left(i, \bar{k}\right)$ cannot be stable: the matching $\left(n - \bar{k} + 1, \bar{k}\right), (i, j)$ must be preferred either by male $n - \bar{k} + 1$ and female $\bar{k}$ or by male i and female j. Hence the overall match cannot be stable.

Finally, the second part of the theorem now follows from the third. ■

An ambiguity remains, however, regarding the identity of those who are left single. For example, if there are more males than females, so that some males must be single, is it the most risk averse males? The least risk averse? Or could the single males be located somewhere within the distribution of risk aversion? We now proceed to show, through a simple example, that any of these situations is possible.

An example The example we consider has the following features:

- Both $\tilde{y}_i$ and $\tilde{z}_j$ are uniformly distributed over $[1, 2]$; it follows that their sum $\tilde{y}$ has the following distribution

$$\begin{aligned} P\left(\tilde{y} < a\right) &= 0 \text{ if } a < 2 \\ P\left(\tilde{y} < a\right) &= \frac{1}{2}\left(a-2\right)^2 \text{ if } 2 \leq a < 3 \\ P\left(\tilde{y} < a\right) &= 2a - 5 - \frac{1}{2}\left(a-2\right)^2 \text{ if } 3 \leq a < 4 \\ P\left(\tilde{y} < a\right) &= 1 \text{ if } a \geq 4 \end{aligned}$$

- Individual preferences are CARA.[19] In particular, for a match between a male and a female with respective Arrow-Pratt risk aversion (RA) indices of μ and ϕ, an efficient sharing rule, $x(\cdot)$, is characterized by

$$\begin{aligned} x\left(\tilde{y}\right) &= \frac{\phi \tilde{y}}{\mu + \phi} + k \\ \tilde{y} - x\left(\tilde{y}\right) &= \frac{\mu \tilde{y}}{\mu + \phi} - k \end{aligned}$$

where k is an arbitrary constant.

---

[19] Technically, CARA functions do not satisfy our assumption that the derivative at zero should be infinite. However, in our example individual incomes are always larger than .5 for the specific parameters we consider. Hence one can replace the CARA form $-\exp(-\sigma x)$ by the function

$$u\left(x\right) = \begin{cases} -e^{-\sigma/2}(1 + \sigma - \sigma\sqrt{2x}), & \text{if } x \leq .5 \\ -\exp\left(-\sigma x\right), & \text{if } x > .5 \end{cases},$$

without changing the conclusions. The redefined function satisfies our conditions.

We first compute the maximum level of expected utility a female with RA ϕ can obtain when matched with a male with RA μ. This corresponds to an efficient sharing rule in which the constant k is such that the male is indifferent between being matched with the female and being single. This implies that k must satisfy the following.

$$e^{-\mu k} = \frac{\left(e^{-\mu} - e^{-2\mu}\right)\phi^2\mu}{(\mu+\phi)^2\left(e^{\mu\frac{\phi}{\mu+\phi}} + e^{-\mu\frac{\phi}{\mu+\phi}} - 2\right)} e^{\frac{3\phi\mu}{\mu+\phi}}.$$

Letting $V(\phi,\mu)$ denote the corresponding utility of the female, it can be shown that

$$V(\mu,\phi) = -\frac{(\mu+\phi)^{2\left(1+\frac{\phi}{\mu}\right)}}{\phi^2\mu^2}\left(e^{\mu\frac{\phi}{\mu+\phi}} + e^{-\mu\frac{\phi}{\mu+\phi}} - 2\right)^{1+\frac{\phi}{\mu}}\left(\left(e^{-\mu}-e^{-2\mu}\right)\phi^2\mu\right)^{\frac{-\phi}{\mu}} e^{-3\phi}$$

a graph of which ($\times 10^6$) is provided in Figure 1 for μ between 0 and 10 and ϕ between 6 and 9

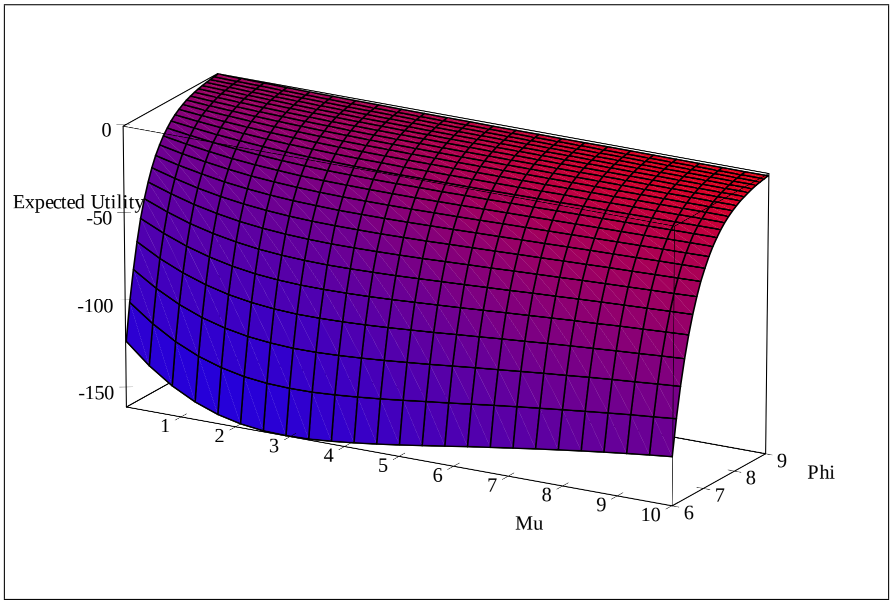

Figure 1 - Female maximum utility as a function of RA coefficients

In particular, Figure 2 provides a section of $V(\phi,\mu)\times 10^6$ through the plane $\phi = 7$

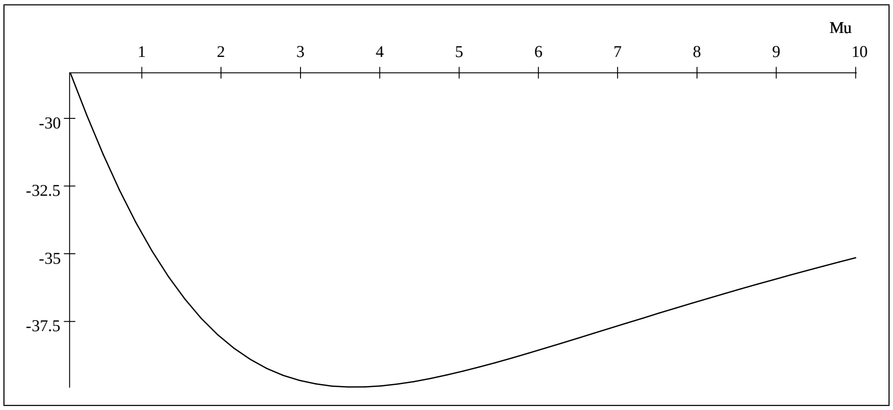

Figure 2: section for $\phi = 7$

The key point to note here is that this section is non monotonic. According to Figure 2, when the female's RA parameter is $\phi = 7$, her maximum utility $V(7, \mu)$, as a function of her mate's risk aversion, μ, decreases and then increases.

Consider now a situation in which three males (with respective RA parameters μ_1, μ_2 and μ_3) are to be matched with two females with identical RA parameter ϕ. We shall make use of the following lemma.

Lemma 4.4. *In any stable match, male i remains single only if*

$$V(\phi, \mu_i) = \min_{i'} V(\phi, \mu_{i'}).$$

Proof. Fix any stable match. By Remark 1, precisely one male, say male 1, is single. Assume, by way of contradiction, that $V(\phi, \mu_i) < V(\phi, \mu_1)$ for some matched male i. Male i's mate cannot receive more than $V(\phi, \mu_i)$. Hence, a match with 1 in which i's mate receives $(V(\phi, \mu_i) + V(\phi, \mu_1))/2$ would be strictly preferred by her and by male 1. Hence, the match is not stable. ■

In particular, when $\phi = 7$, consider any three values of μ on the horizontal axis of Figure 2. The one of them giving the smallest value to the function defines the single male.[20] Clearly then, all negative assortative matches are possible. Referring to Figure 2, we see that for $\mu_1 = 1, \mu_2 = 2, \mu_3 = 3$, the most risk averse agent 3 is single; for $\mu_1 = 4, \mu_2 = 5, \mu_3 = 6$, the least risk averse agent 1 is single; finally, for $\mu_1 = 1, \mu_2 = 4, \mu_3 = 7$, the intermediate agent 2 is single.

The intuition is as follows. When selecting a mate, a female must take into account two conflicting considerations. On the one hand, she may prefer a *less* risk averse partner because he is willing to accept a larger share of the income risk. On the other

---

[20] To apply Theorems 3.1 and 4.1, simply perturb one of the female's ϕ slightly.

hand, she may prefer a *more* risk averse partner because he is willing to accept a smaller amount of joint income so long as it is almost guaranteed. That is, males with 'extreme' measures of risk aversion – extreme in either direction – are attractive. Thus, roughly speaking, we should expect the male with the least extreme risk aversion parameter to remain single. The examples above bear this out.[21]

5. Concluding Remarks

The theoretical nature of our investigation has led us to consider a very simple model. In practice of course, matching is not exclusively based on risk aversion, and other characteristics also play a key role. An interesting extension would permit the agents to choose their income distributions prior to matching. There could then be a trade-off between choosing a distribution that fits one's risk preferences and making oneself attractive to others for the purposes of risk sharing.

Nonetheless, even our simple model has interesting consequences. In particular, risk-sharing leads to negative assortative matching. Relatively risk averse individuals are eager to match with less risk averse partners, who can provide the coverage they need at low cost; conversely, relatively risk neutral individuals exploit their comparative advantage by matching with the risk averse, who are willing to give up a large risk premium in exchange for coverage. To the extent that risk-sharing may play a role in marital decisions, one would expect intrahousehold differences in risk aversion to be large – a conclusion that fits empirical evidence pretty well.[22]

Our findings appear to be especially relevant to the empirical literature on risk sharing, as discussed in the introduction, although a complete analysis of the impact of our results upon this literature, while surely important, is beyond the scope of the present paper. In addition, our results provide support for the 'individualistic' approach of Mazzocco (2005), who shows that Euler equations can be estimated at the *individual* level, even for couples, using labor supply behavior.[23] Finally, our results suggest that risk-aversion heterogeneity, being endogenous, should be taken seriously, which may lead to a more extensive use of long panels.[24]

---

[21] A natural conjecture stemming from this intuition which we have been unable to prove, is that the set of unmatched individuals is an interval.

[22] For instance, Mazzocco (2004a), using HRS data, shows that even when agents are gathered into four wide (and potentially heterogenous) classes of risk aversion, half married men are found to belong to a different category than their spouse. A different investigation, using the Consumer Expenditure Survey, leads to the same conclusion (Mazzocco 2004b).

[23] Mazzocco finds that traditional Euler equations, estimated at the household level, are rejected for couples but not for singles. Moreover, his 'individualistic' generalization, which independently analyzes individuals within the couple, is not rejected.

[24] See Chiappori, Schuhlhofer-Wohl and Townsend (2005).

APPENDIX: Proof of Lemma 4.3

The following well-known result will be helpful.

Proposition A.1. *If a differentiable real-valued function defined on an interval is nonnegative at* x_0 *and its derivative is positive whenever the function vanishes, the function is positive at all* $x > x_0$ *and can be zero at no more than one point.*[25]

Remark. The interval, I say, need not be open and x_0 need not be an interior point of I when the derivative at $x \in I$ is defined by $f'(x) = \lim_{x' \to x, x' \in I} \frac{f(x')-f(x)}{x'-x}$.

Recall the statement of the lemma.

Lemma 4.3. If $0 \leq v_1 < \bar{v}_1$ and $0 < v_2 \leq \bar{v}_2$ are such that

$$U_{11}(v_1) \geq U_{12}(v_2),$$

then

$$U_{21}(v_1) > U_{22}(v_2).$$

Proof of Lemma 4.3. Fix v_1 and v_2 as in the statement of the lemma. If $v_1 = 0$, then we are done because $U_{21}(v_1) = Eu_2(\tilde{y}) > U_{22}(v_2)$, where the second inequality follows because $v_2 > 0$ requires the male to strictly share the joint income with the female with positive probability. Hence, we may assume that both v_1 and v_2 are strictly positive. We may similarly assume that $v_1 < \bar{v}_1$ and $v_2 < \bar{v}_2$.

Let $x_{ij} : Y \to [0, \infty)$ denote the sharing rule employed by male i and female j that maximizes male i's utility subject to female j receiving at least utility v_j. By definition, male i then receives utility $U_{ij}(v_j) = Eu_i(x_{ij}(\tilde{y}))$.

We first extend $x_{ij}(\cdot)$ to all of $[0, \infty)$. As shown in Wilson (1968), there are Pareto weights, $\lambda_i, \lambda_j > 0$ (strict positivity follows because $0 < v_j < \bar{v}_j$) such that $x_{ij}(\cdot)$ solves

$$\max_{x:Y \to [0,\infty)} \lambda_i Eu_i(x(\tilde{y})) + \lambda_j Ev_j(\tilde{y} - x(\tilde{y})),$$

subject to $0 \leq x(y) \leq y$ for all $y \in Y$. Hence, $x_{ij}(0) = 0$ and because $u_i'(0) = v_j'(0) = +\infty$, $x_{ij}(y)$ is the unique solution to

$$\lambda_i u_i'(x) = \lambda_j v_j'(y - x), \tag{A.1}$$

---

[25] A proof of the first part is as follows. If $N = \{x > x_0 : f(x) \leq 0\}$ is nonempty, it contains a smallest member, $\bar{x} > x_0$; otherwise $f(x_0) = 0$ and $f'(x_0) \leq 0$, a contradiction. Consequently, $f(\bar{x}) = 0$ and f assumes a minimum at $\bar{x}$ on the interval $[x_0, \bar{x}]$, implying that $f'(\bar{x}) \leq 0$, a contradiction. Hence, N is empty. The second part follows immediately from the first.

for almost every positive $y \in Y$. Clearly, we can use (A.1) to uniquely extend $x_{ij}(\cdot)$ to all of $[0, \infty)$. Moreover, $x'_{ij}(y)$ exists for all $y > 0$ by the implicit function theorem. Consider now the difference $x_{11}(y) - x_{12}(y)$ as a function of $y \in [0, \infty)$. A first claim is the following:

Claim 1. There exists $\bar{y} \geq 0$, possibly infinite, such that $x_{12}(y) - x_{11}(y)$ is negative for $0 < y < \bar{y}$ and positive for $y > \bar{y}$.

To prove Claim 1, suppose that $x_{12}(\bar{y}) = x_{11}(\bar{y}) = \bar{x}$ for some $\bar{y} > 0$. Because $v_2 > 0$ and $u'_1(0) = v'_2(0) = +\infty$, we know that $0 < \bar{x} < \bar{y}$. Consequently, (A.1) implies that

$$x'_{1j}(\bar{y}) = \frac{\phi_j(\bar{y} - \bar{x})}{\mu_i(\bar{x}) + \phi_j(\bar{y} - \bar{x})} \in (0, 1), \text{ for } j = 1, 2.$$

where $\phi_j(z) = -v''_j(z)/v'_j(z)$ and $\mu_i(x) = -u''_i(x)/u'_i(x)$ are the female and male Arrow-Pratt measures of risk aversion. Since $\phi_2(\bar{y} - \bar{x}) > \phi_1(\bar{y} - \bar{x})$ by assumption, we must have $x'_{12}(\bar{y}) > x'_{11}(\bar{y})$. Hence, whenever $x_{12}(y) - x_{11}(y)$ vanishes on $(0, \infty)$, it's derivative is positive. Claim 1 now follows from Proposition A.1.

By Claim 1, the function $\Delta(y) = u_1(x_{11}(y)) - u_1(x_{12}(y))$ is positive for $0 < y < \bar{y}$ and negative for $y > \bar{y}$. Let $[0, \bar{u}_1)$ denote the range of $u_1(\cdot)$. Then, defining $f : [0, \bar{u}_1) \to \mathbb{R}$ so that $f(u_1(x)) = u_2(x)$ for all $x > 0$, the fact that male 2 is strictly more risk averse than male 1 implies that $f(\cdot)$ is strictly increasing and strictly concave. Because $x_{11}(y)$ is strictly increasing in y and $u_1(\cdot)$ is strictly increasing, the function $g(y) = f'[u_1(x_{11}(y))]$ is positive and strictly decreasing in y. Hence, if $\tilde{y}$ has c.d.f. $H(\cdot)$,

$$\begin{aligned} E[g(\tilde{y})\Delta(\tilde{y})] &= \int_0^{\bar{y}} g(y)\Delta(y)dH(y) + \int_{\bar{y}}^{\infty} g(y)\Delta(y)dH(y) \\ &> \int_0^{\bar{y}} g(\bar{y})\Delta(y)dH(y) + \int_{\bar{y}}^{\infty} g(\bar{y})\Delta(y)dH(y) \\ &= g(\bar{y})E[u_1(x_{11}(\tilde{y})) - u_1(x_{12}(\tilde{y})] \\ &= g(\bar{y})[U_{11}(v_1) - U_{12}(v_2)] \\ &\geq 0, \end{aligned}$$

where the final inequality follows by hypothesis. Hence, substituting the definition of $\Delta(\cdot)$ into the left-hand side above yields

$$E[g(\tilde{y})(u_1(x_{11}(\tilde{y})) - u_1(x_{12}(\tilde{y})))] > 0. \tag{A.2}$$

The concavity of $f(\cdot)$ implies that

$$f[u_1(x_{11}(y))] - f[u_1(x_{12}(y))] \geq f'[u_1(x_{11}(y))](u_1(x_{11}(y)) - u_1(x_{12}(y))). \tag{A.3}$$

Since $u_2 = f \circ u_1$, we may combine (A.2) and (A.3) to conclude that

$$E[u_2(x_{11}(\tilde{y})) - u_2(x_{12}(\tilde{y}))] > 0. \tag{A.4}$$

For $j = 1$ and 2, define the following value functions for $0 \leq \lambda \leq 1$.

$$\pi_j(\lambda) = \max_{x:Y \to [0,\infty)} E\left[(1-\lambda)\, u_1(x(\tilde{y})) + \lambda u_2(x(\tilde{y}))\right] \quad \text{s.t. } E\left[v_j(\tilde{y} - x(\tilde{y}))\right] \geq v_j,$$

and subject to $0 \leq x(y) \leq y$ for almost every $y \in Y$. A key property of the these value functions is the following.

Claim 2. If $\bar{\lambda} \in [0,1]$ is such that $\pi_1\left(\bar{\lambda}\right) = \pi_2\left(\bar{\lambda}\right)$ then $\pi_1'\left(\bar{\lambda}\right) > \pi_2'\left(\bar{\lambda}\right)$.

To prove Claim 2, assume first that $\bar{\lambda} = 0$ and $\pi_1(0) = \pi_2(0)$. Then $\pi_j(0) = E\left[u_1(x_{1j}(\tilde{y}))\right]$ and, by the envelope theorem (see Milgrom and Segal (2002), Theorem 3), $\pi_j'(0) = E\left[u_2(x_{1j}(\tilde{y}))\right] - E\left[u_1(x_{1j}(\tilde{y}))\right]$.[26] Consequently,

$$\begin{aligned} \pi_1'(0) - \pi_2'(0) &= \left(E\left[u_2(x_{11}(\tilde{y}))\right] - E\left[u_1(x_{11}(\tilde{y}))\right]\right) \\ &\quad - \left(E\left[u_2(x_{12}(\tilde{y}))\right] - E\left[u_1(x_{12}(\tilde{y}))\right]\right) \\ &= E\left[u_2(x_{11}(\tilde{y}))\right] - E\left[u_2(x_{12}(\tilde{y}))\right] \\ &> 0, \end{aligned}$$

where the second equality follows because $\pi_1(0) = \pi_2(0)$ and the inequality follows from A.4.

Hence, if $\pi_1(0) = \pi_2(0)$ then $\pi_1'(0) > \pi_2'(0)$. Of course, this conclusion holds for all utility functions satisfying our hypotheses. Consequently, if instead $\bar{\lambda} > 0$, and we define $\hat{u}_1 = \left(1 - \bar{\lambda}\right) u_1 + \bar{\lambda} u_2$, and we define for $j = 1, 2$ the value function $\hat{\pi}_j(\lambda)$ as before, but replacing the utility function u_1 with $\hat{u}_1$, then, because u_2 is a concavification of $\hat{u}_1$ (e.g. consider the Arrow-Pratt measures) we may similarly conclude that if $\hat{\pi}_1(0) = \hat{\pi}_2(0)$ then $\hat{\pi}_1'(0) > \hat{\pi}_2'(0)$. But this is equivalent to the statement that if $\pi_1\left(\bar{\lambda}\right) = \pi_2\left(\bar{\lambda}\right)$ then $\pi_1'\left(\bar{\lambda}\right) > \pi_2'\left(\bar{\lambda}\right)$. This establishes Claim 2.

In view of Proposition A.1, a direct consequence of Claim 2 is that $\pi_1(0) \geq \pi_2(0)$ implies that $\pi_1(1) > \pi_2(1)$. Since $\pi_j(0) = U_{1j}(v_j)$ and $\pi_j(1) = U_{2j}(v_j)$ this completes the proof of Lemma 4.3. ■

---

[26] Milgrom and Segal's (2002) Theorem 3 applies because, under our assumptions, the solution to the optimization problem defining $\pi_j(\lambda)$ is unique and, as $\lambda \to \lambda_0$, the corresponding sequence of such solutions (being a sequence of nondecreasing functions) has a pointwise convergent subsequence whose limit is a solution to the problem for λ_0.

References

S. Altug and R. Miller, "Household Choices in Equilibrium", *Econometrica,* Vol. 58, No. 3 (May, 1990), pp. 543-570.

K. Borch, "Equilibrium in a Reinsurance Market," *Econometrica*, Vol. 30, No. 3 (July, 1962), pp.424-444.

R. Barsky, F. Juster, M. Kimball and M. Shapiro, "Preference Parameters and Behavioral Heterogeneity: An Experimental Approach in the Health and Retirement Study", *Q.J.E.*, Vol. 112, No. 2 (May, 1997), pp. 537-579.

D. Cass, G. Chichilnisky and H.M. Wu, "Individual Risk and Mutual Insurance", *Econometrica*, vol. 64, no. 2 (March, 1996), pp. 333-41.

P.-A. Chiappori and B. Salanié, "Demand for Automobile Insurance: An Econometric Model", Working Paper, Columbia University (2005).

P.-A. Chiappori, R. Townsend and S. Schuhlhofer-Wohl, "Sharing Risk with Heterogenous Risk Aversion", Mimeo, University of Chicago (2005).

J. Cochrane, "A Simple Test of Consumption Insurance", *J. Polit. Econ.*, Vol. 99, No. 5 (October, 1991), pp. 957-976.

A. Cohen and L. Einav, "Estimating Risk Preferences from Deductible Choice", NBER Working Paper No. 11461 (2005).

G. Genicot and D. Ray, "Group Formation in Risk-Sharing Arrangements", *Rev. Econ. Studies*, vol. 70, no. 1 (January, 2003), pp. 87-113.

L. Guiso and M. Paiella, "Risk aversion, wealth and background risk", CEPR Discussion Paper No. 2728 (2001).

F. Hayashi, J. Altonji and L. Kotlikoff, "Risk-Sharing between and within Families", *Econometrica,* Vol. 64, No. 2 (March, 1996), pp. 261-294

R. Horn and C. Johnson, *Matrix Analysis.* Cambridge University Press, Cambridge, 1985 (reprinted 1992).

P. Legros and A. Newman, "Assortative Matching in a Non-transferable World", CEPR Discussion Papers: 3469 (2002).

S. Lund and M. Fafchamps, "Risk Sharing Networks in Rural Philippines", *J. Devel. Econ.*, 71: 261-87 (2003).

M. Mazzocco, “Savings, Risk Sharing and Preferences for Risk”, *Amer. Econ. Rev.*, Vol. 94, No. 4, (September, 2004a): pp. 1169-1182.

M. Mazzocco, “Individual Rather Than Household Euler Equations: Identification and Estimation of Individual Preferences Using Household Data,” Manuscript, Department of Economics, University of Wisconsin (2004b).

M. Mazzocco, “Intertemporal Behavior and Household Structure”, Manuscript, Department of Economics, University of Wisconsin (2005).

P. Milgrom and I. Segal, “Envelope Theorems for Arbitrary Choice Sets,” *Econometrica*, Vol. 70, No. 2 (March, 2002), pp. 583-601.

S. Schuhlhofer-Wohl, “Negative Assortative Matching of Risk-Averse Agents With Transferable Expected Utility”, Mimeo, University of Chicago (2004).

R. Townsend, “Risk and Insurance in Village India”, *Econometrica,* Vol. 62, No. 3 (May, 1994), pp. 539-591.

R. Wilson, “The Theory of Syndicates”, *Econometrica,* Vol. 36, No. 1 (January, 1968), pp. 119-132.