Necessary and sufficient conditions for transferable utility

P.A. Chiappori  E. Gugl
Columbia University  University of Victoria

May 13, 2015

1 Introduction

Many results in microeconomics and game theory require that preferences exhibit the Transferable Utility (TU) property, whereby, for a well chosen cardinalization of utilities, the set of Pareto efficient allocations is a straight line with slope -1 (or, for more than two agents, an hyperplane orthogonal to the unit vector) for all price and income bundles, and generally for all economic environments. For instance, in collective models of the household (Chiappori 1988, 1992), TU implies that household (aggregate) demand does not depend on Pareto weights; this allows to reconcile the unitary model with an explicit representation of individual preferences while addressing issues of intrahousehold redistribution (and inequality).1 Another important property of TU preferences is that Pareto frontiers do not intersect when prices, incomes or other factors change; this property, in turn, is crucial for the so-called Coase theorem to hold.2 Lastly, a host of recent works on the market for marriage refer to a frictionless matching framework a la Becker (1973) and Shapley and Shubik (1971). Under TU, the notion of stability is equivalent to total surplus maximization, a fact that considerably simplifies theoretical and empirical analysis.

Yet, while the TU assumption is made on a regular basis, its exact meaning is not clear. Consider, for example, a two-person household consuming private and public goods under a budget constraint. What do we need to assume on individual preferences to get the TU property? Partial answers have been given to this problem. For instance, Bergstrom and Varian (1985) consider the case of purely private consumptions, and show that preferences must be of

---

*Chiappori gratefully acknowledges financial support from the NSF (Award 1124277). Errors are ours.

1In particular, Becker’s well-known ‘rotten kid’ theorem can be formulated in a TU framework; see for instance Bergstrom (1989) and Browning, Chiappori and Weiss (2015) for a general discussion.

2A typical application is Becker-Coase theorem, which states that divorce laws should have no impact of divorce rates; see Chiappori, Iyigun and Weiss 2015.
the Gorman polar form (more on this later on). Alternatively, Bergstrom and Cornes (1983) analyze a model in which all commodities but one are publicly consumed; they show that, for a given price and income vector, the efficient bundle of public goods does not change as we move from one Pareto efficient allocation to another if and only if preferences are of the generalized quasi linear (GQL) form. One can readily check that GQL preferences imply TU (a property that also holds in the general case with many public and private goods). Conversely, Bergstrom (1989), still considering the case with one private good, shows that if the demand for public goods is the same for all efficient allocations, then TU requires GQL. To the best of our knowledge, however, the fact that, in Bergstrom and Cornes’s context with one private good only, TU implies both GQL and identical demand for public goods has not been proved. In the general case, with an arbitrary number of private and public goods, this statement would actually be incorrect; GQL is sufficient for TU but not necessary, as shown by Gugl (2014) who exhibits utility functions that are not QGL but lead to TU. So far, no necessary and sufficient characterization of preferences leading to TU in the general case has been provided.

The goal of this note is to fill this gap. We provide a general condition that is necessary and sufficient for TU in a general context. Specifically, we refer to the notion of Conditional Indirect Utility introduced by Blundell, Chiappori and Meghir (2005), defined as the maximum utility level an individual can reach by choosing the optimal bundle of private consumption for given values of private prices, total private expenditures and conditional on a given vector of public consumption. We introduce a specific property of individual preferences, the Affine Conditional Indirect Utility (ACIU), which states that for a well chosen cardinal representation the conditional indirect utility is affine in total expenditures; and we show that TU obtains if and only if (i) each individual preferences exhibit the ACIU property, and (ii) the coefficient of total expenditures (which can be a function of private prices and public consumptions) is the same for all individuals. We show that this result generalizes the previous ones, in the sense that it boils down to Gorman polar form in the absence of public consumptions, and to GQL with only one private good; moreover, we provide an example that is neither Gorman nor GQL but satisfies our characterization, and therefore generates TU.

2 The framework

Consider a group of two agents (‘wife’ and ‘husband’) who consume \( n \) private and \( N \) public commodities that can be purchased on a market.\(^3\) Let \( x_m = (x^1_m, \ldots, x^n_m) \), where \( m = h, w \), denote member \( m \)’s private consumption and \( p = (p^1, \ldots, p^n) \) the corresponding price vector. Similarly, \( X = (X^1, \ldots, X^N) \) denotes the household’s public consumption purchased at price \( P = (P^1, \ldots, P^N) \).

\(^3\)The goods are public within the household only; they are privately purchased on the market. One may think of housing or expenditures on children as typical examples. Also, extension to an arbitrary number of group members is straightforward.
Finally, let $y_m$ denote member $m$’s income, and let $y = y_h + y_w$ be the household’s total (pooled) income, so that the household’s budget constraint is:

$$\sum_{i=1}^{n} p^i (x^i_h + x^i_w) + \sum_{j=1}^{N} P^j X^j = y$$

We assume away consumption externalities; therefore $m$’s preferences only depend on the vector $(X, x_m)$. Moreover, each agent’s preferences are assumed reflexive, transitive, complete, continuous, strongly increasing and strictly convex; therefore they can be represented by utility functions $u_m (X, x_m), m = w, h$, that are strictly increasing and strictly quasi-concave.

We say that preferences satisfy the transferable utility (TU) property if there exists a cardinal representation $u_m (X, x_m)$ of $m$’s preferences, $m = w, h$, such that the Pareto frontier takes the form

$$u_w + u_h = K (p, P, y)$$

In words, for a well chosen cardinalization of preferences, the Pareto frontier is a straight line with slope equal to $-1$ for all values of prices and income.

Next, following Blundell, Chiappori and Meghir (2005), we define the conditional indirect utility of person $m (= w, h)$ by:

$$v_m (X, p, \rho) = \max_{x = \rho} u_m (X, x)$$

(1)

In words, $v_m$ is the maximal utility agent $m$ can reach when endowed with a vector $X$ of public consumption and an amount $\rho$ to spend on private goods. The conditional private demand vector $\xi_m (X, p, \rho)$ is defined as the solution to (1): it is the vector of private consumptions that maximizes $m$’s utility for prices $p$, private expenditures $\rho$ and public consumption $X$.

The underlying intuition is that any Pareto efficient decision mechanism can be represented as a two-stage process. In stage one, agents jointly decide on the level $X$ of public consumption, and on the distribution of the remaining resources between agents; she gets $\rho_w = \rho$ and he gets $\rho_h = y - P'X - \rho$. In stage two, agents each choose their optimal vector of private consumption, conditional on the vector $X$ and subject to the budget constraint $p'x_m = \rho_m$.

The conditional indirect utility reflects the outcome of the second stage; the first stage, which characterizes Pareto efficient allocations of household resources between public and private consumptions, can then be written as:

$$\max_{X, \rho} v_w (X, p, \rho)$$

(2)

under the constraint

$$v_h (X, p, y - P'X - \rho) \geq \bar{v}_h$$

(3)

for some given $\bar{v}_h$. Then the TU property requires the following: one can find specific cardinalizations $(v_w, v_h)$ of individual preferences such that, for all
Pareto efficient allocations, the Lagrange multiplier of constraint (3) is equal to 1, irrespective of the price-income bundle.

In what follows, we only consider allocations such that \( p_w > 0 \) and \( p_h > 0 \). Note that Inanda conditions would be sufficient to guarantee that all efficient allocations satisfy this constraint.

### 3 The main result

We can now state our main result. It is based on the following definition:

**Definition 1** The preferences of individual \( m \) satisfy the Affine Conditional Indirect Utility (ACIU) property if they admit a cardinal representation such that the corresponding, conditional indirect utility of \( m \) is an affine function of the sharing rule:

\[
v_m (X, p, \rho) = \alpha_m (X, p) \rho + \beta_m (X, p),
\]

where \( \alpha_m \) (resp. \( \beta_m \)) is \((-1)\)-homogeneous (resp. 0- homogenous) in \( p \).

The result is the following:

**Proposition 2** Preferences satisfy the TU property if and only if each individual preferences satisfy the ACIU property, and moreover

\[
\alpha_w (X, p) = \alpha_h (X, p) = \alpha (X, p)
\]

**Proof.** We first show that (4) and (5) are sufficient. If these properties hold, then the program defined by (2) and (3) becomes:

\[
\max_{X, \rho} \alpha (X, p) ((1 - \mu) \rho + \mu (y - P' X)) + \beta_w (X, p) + \mu \beta_h (X, p)
\]

where \( \mu \) denotes the Lagrange multiplier of the constraint. The maximization with respect to \( \rho \) requires \( \mu = 1 \); then the Pareto frontier can be written as:

\[
u_w + \nu_h = \max_X [(y - P' X) \alpha (X, p) + \beta_w (X, p) + \beta_h (X, p)]
\]

Conversely, assume that the TU property holds, and consider the program (where \( p \) is omitted for brevity):

\[
\max_{X, \rho} v_w (X, \rho) + v_h (X, y - P' X - \rho)
\]

For notational convenience, we present the argument for the case of a single public good \((N = 1)\); the extension to \( N \geq 2 \) is straightforward. The crucial remark is that the solution to this program cannot be unique (since all Pareto efficient allocations solve it). It follows that the first order conditions:

\[
\frac{\partial v_w}{\partial \rho} (X, \rho) - \frac{\partial v_h}{\partial \rho} (X, y - PX - \rho) = 0
\]

\[
\frac{\partial v_w}{\partial X} (X, \rho) - \frac{\partial v_h}{\partial X} (X, y - PX - \rho) - P \frac{\partial v_h}{\partial \rho} (X, y - PX - \rho) = 0
\]
considered as two equations in \((X, \rho)\), have a continuum of solutions (the set of solutions will typically be a one-dimensional manifold). This implies either that one equation (in that case, the first) is degenerate, or that the two equations are redundant (technically, they are not transversal). We consider both cases successively.

1. Assume, first, that
   \[
   \frac{\partial v_w}{\partial \rho} (X, \rho) - \frac{\partial v_h}{\partial \rho} (X, y - PX - \rho) = 0
   \]
   for all values of \((X, \rho, P, y)\). Then consider the change in variables:
   \((X, \rho, P, y) \rightarrow (X, \rho, P, t)\) where \(t = y - PX\)
   then
   \[
   \frac{\partial v_w}{\partial \rho} (X, \rho) - \frac{\partial v_h}{\partial \rho} (X, t - \rho) = 0 \quad (8)
   \]
   for all \(t\). Differentiating in \(t\) gives that \(\partial v_h/\partial \rho\) is independent of \(\rho\), so is \(\partial v_w/\partial \rho\) by (8), which proves the result.

2. Assume, now, that the equations are redundant, and define:
   \[
   F(X, \rho, P, t) = \frac{\partial v_w}{\partial \rho} (X, \rho) - \frac{\partial v_h}{\partial \rho} (X, t - \rho)
   \]
   \[
   G(X, \rho, P, t) = \frac{\partial v_w}{\partial X} (X, \rho) - \frac{\partial v_h}{\partial X} (X, t - \rho) - P \frac{\partial v_h}{\partial \rho} (X, t - \rho)
   \]
   with the same change in variables as before. Since \(\partial F/\partial P = 0\) it must be the case that \(\partial G/\partial P = 0\), therefore \(\partial v_h/\partial \rho = 0\), which is a particular case of 1.

In words, Proposition 2 states that for TU to obtain, preferences must have the following properties. First, if we fix the vector \(X\) of public consumption and consider individual utilities as functions of private consumptions only, the corresponding (conditional) indirect utilities must be (possibly after an increasing transformation) affine in total expenditures for all values of prices and income. Second, the coefficient of total expenditures (which may depend on private prices but also on consumption of public goods) must be the same for all agents. These conditions are necessary and sufficient: preferences of this form satisfy TU, and any preferences that satisfy TU must have cardinal representations of this form.

An important corollary is the following:

**Corollary 3** If preferences satisfy the TU property then, for any \((p, P, y)\), the household’s demand for public goods is the same for all Pareto efficient allocations.
The proof of this Corollary follows directly from Proposition 2, since for any efficient allocation $X$ solves
\[ \max_X \left[ (y - P'X) \alpha(X, p) + \beta_w(X, p) + \beta_h(X, p) \right] \]
It should be noted, however, that Proposition 2 is needed to establish this result. Indeed, under TU any Pareto efficient allocation must solve:
\[ \max_{X, \rho} v_w(X, p, \rho) + v_h(X, p, y - P'X - \rho) \]
TU implies that this maximization program has a continuum of solutions (all efficient allocations, typically a one-dimensional manifold). However, what is not clear is whether they all correspond to the same demand for public goods; in principle, it could be the case that different efficient allocations correspond to different values of the sharing rule $\rho$ and different demands for public goods $X$ (one for each possible choice of $\rho$). The corollary states that such a situation is not possible.

One can easily check that the form characterized by Proposition 2 encompasses several results already available in the literature. For instance:

- In the absence of public goods, then (4) boils down to the Gorman polar form (GPF) discussed by Bergstrom and Varian (1984). Indeed, in that case, the conditional indirect utility coincides with the standard indirect utility, since the sharing rule coincides with income; therefore $m$’s indirect utility and demand for commodity $i$ are:
  \[ v_m(X, p) = \alpha(p) y + \beta_m(p) \text{ and } \]
  \[ x^i_m(p, y) = \alpha^i(p) y + \beta^i_m(p) \]

  Note, however, that Bergstrom and Varian’s result does not extend directly to the case of public goods. The problem, here, is related to Corollary 3: to apply Bergstrom and Varian’s result, one needs to rule out the possibility of different efficient allocations involving different demands for public goods.

- If, on the other hand, there is only one private good ($n = 1$), then we can normalize $p$ to be 1, and we get the Generalized Quasi Linear (GQL) form of Bergstrom and Cornes (1983); indeed, the sharing rule is then the individual consumption of the private good, and $m$’s direct utility becomes
  \[ u_m(X, x) = \alpha_m(X) x + \beta_m(X) \]

  Specifically, Bergstrom and Cornes show the following result: when $n = 1$, a necessary and sufficient condition for all Pareto efficient allocations to generate the same demand for public good is that preferences are GQL; and it is well known that GQL preferences imply TU. Our result also establishes the converse property - namely that, when $n = 1$, TU requires GQL preferences, implying that $X$ must be identical across all efficient allocations.
• The GQL form can readily be extended to \( n \geq 2 \) by considering direct utility functions of the form:

\[
   u_m (X, x_1, \ldots, x_n) = \alpha_m (X, x_2, \ldots, x_n) x_1 + \beta_m (X, x_2, \ldots, x_n)
\]

One can readily check that the corresponding, conditional indirect utility is affine in the sharing rule (this property comes directly from the fact that for a given vector of public consumptions, \( u_m \) as a function of \( x_1, \ldots, x_n \) is quasi-linear). For \( \alpha_h = \alpha_w \), it therefore implies TU. The converse, however, is not true; indeed, the next section provides an example of preferences which are not GQL but still imply TU.

Lastly, the condition can readily be translated in terms of conditional demands:

**Corollary 4** Preferences satisfy the TU property if and only if the conditional private demand vectors \( \xi_m (X, p, \rho), m = w, h \), are affine functions of the sharing rule:

\[
   \xi^i_m (X, p, \rho) = a^i (X, p) \rho_m + b^i_m (X, p), \ m = h, \ w.
\]  

(9)

where

\[
   a^i (X, p) = \frac{\partial \alpha (X, p)}{\partial p_i} \quad \text{and} \quad b^i_m (X, p) = \frac{\partial \beta_m (X, p)}{\partial p_i}
\]

and \( \alpha \) is \((-1)\)-homogeneous in \( p \), \( \beta_m \) is \(0\)-homogeneous in \( p \). In particular, \( a_i \) is \((-1)\)-homogeneous in \( p \) and \( \beta^i_m \) is \(0\)-homogeneous in \( p \)

**Proof.** The envelope theorem applied to (1) gives a conditional version of Roy’s identity:

\[
   \xi^i_m (X, p, \rho) = \frac{\partial v_m (X, p, \rho)}{\partial p_i} \quad \frac{\partial v_m (X, p, \rho)}{\partial \rho}
\]

Plugging (4) into this relationship gives the result. ■

In particular, different Pareto efficient allocations may correspond to different vectors of individual private consumptions for all private commodities, in contrast to a standard property of GQL demands (for which individual demands for all private goods except one are identical over the set of Pareto efficient allocations). However, the aggregate demand for private good \( i \) is:

\[
   \xi^i (X, p, \rho) = a^i (X, p) \left( \sum_{m=w, h} \rho_m \right) + \sum_{m=w, h} b^i_m (X, p)
   = a^i (X, p) (y - P'X) + \sum_{m=w, h} b^i_m (X, p)
\]

which is identical over all Pareto-efficient allocations.
4 Example

We now provide an example of preferences that are neither GPF nor GQL, but still satisfy TU. This example is a direct generalization of Gugl (2014). Consider the following preferences:

\[ u_m(X, x_m) = \frac{a(X)}{\Gamma_m} \prod_{k=1}^{n_1} (x_m^k - e_m^k)^{\gamma_m^k} + \beta_m (X, x_{m+1}^{n_1}, \ldots, x_m^n), \quad m = h, w, \]  

with

\[ \sum_{k=1}^{n_1} \gamma_m^k = 1 \quad \text{and} \quad \Gamma_m = \prod_{k=1}^{n_1} (\gamma_m^k)^{\gamma_m^k}. \]  

The conditional indirect utility of \( m \) is defined by:

\[ v_m(X, p, \rho) = \max_x a(X) \prod_{k=1}^{n_1} \left( x_m^k - e_m^k \right)^{\gamma_m^k} + \beta_m (X, x_{m+1}^{n_1}, \ldots, x_m^n) \]  

under

\[ \sum_{k=1}^{n} p_k x_m^k = \rho \]

The maximand in (11) is separable in \( (x_m^1, \ldots, x_m^{n_1}) \); therefore the program can be solved using two stage budgeting. Define

\[ \varrho = \sum_{k=1}^{n_1} p_k x_m^k, \]

then \( (x_m^1, \ldots, x_m^{n_1}) \) must solve:

\[ \max_{(x^1, \ldots, x^{n_1})} \prod_{k=1}^{n_1} \left( x_k^k - e_m^k \right)^{\gamma_m^k} \]

under (12), which gives

\[ x_m^k = e_m^k + \gamma_m^k \frac{\varrho - \sum_{i=1}^{n_1} p_i e_m^i}{p_k} \]

The first stage is therefore:

\[ \max_{\varrho, x_m^{n_1+1}, \ldots, x_m^n} \left( a(X) \prod_{k=1}^{n_1} (p_k)^{-\gamma_m^k} \right) \left( \varrho - \sum_{i=1}^{n_1} p_i e_m^i \right) + \beta_m (X, x_{m+1}^{n_1+1}, \ldots, x_m^n) \]

under

\[ \varrho + \sum_{k=n_1+1}^{n} p_k x_m^k = \rho \]
The maximand is quasi-linear in $\varrho$; assuming that $\varrho > 0$, this implies that $(x_{m+1}^{n}, ..., x_{m}^{n})$ only depends on $(p, X)$, so that:

$$
\beta_m (X, x_{m+1}^{n}, ..., x_{m}^{n}) = B_m (p, X) \quad \text{and}
$$

$$
\varrho = \rho - \sum_{k=n+1}^{n} p_k x_k^m = \rho - C_m (p, X)
$$

Finally, we get that:

$$
v_m (X, p, \rho) = a (X) \prod_{k=1}^{n+1} (p_k)^{-\gamma_k} \rho
$$

$$
+ B_m (p, X) - a (X) \prod_{k=1}^{n+1} (p_k)^{-\gamma_k} \left( C_m (p, X) \sum_{i=1}^{n+1} p_i e_i^m \right)
$$

which has the ACIU property for all prices if and only if $\gamma_w^k = \gamma_h^k$ for $k = 1, ..., n_1$. The corresponding Pareto frontier is therefore defined by:

$$
u_w + u_f = K (p, P, y)
$$

where

$$
K (p, P, y) = \max_X \left[ a (X) \prod_{k=1}^{n+1} (p_k)^{-\gamma_k} (y - P^X) \right. \\
\left. + \sum_{m=w, h} \left( B_m (p, X) - a (X) \prod_{k=1}^{n+1} (p_k)^{-\gamma_k} \left( C_m (p, X) \sum_{i=1}^{n+1} p_i e_i^m \right) \right) \right]
$$

5 Extension: domestic production

Our results can readily be extended to allow for domestic production. To see why, assume that some of the public goods are internally produced, using inputs that can be purchased on the market at some exogenous price; note that these can include time spent by household members, provided that the later participate in the labor market (so that their time can be valued at some exogenous market wage). Formally, assume that

$$
X^s = F^s \left( q^{s,1}, ..., q^{s,n} \right), \ s = 1, ..., S
$$

and the new budget constraint is:

$$
\sum_{i=1}^{n} p^i (x_h^i + x_w^i) + \sum_{(s,i)=(1,1)}^{(n,S)} p^i q^{s,i} + \sum_{j=S+1}^{N} P^j X^j = y
$$

Now, modify the two-stage process previously described as follows: in stage one, agents now decide on the purchase of public consumption $(X^j, \ j = S + 1, N)$, on the input purchases $(q^{s,i}, \ s = 1, S, i = 1, n)$, and on the distribution of the remaining resources between agents, whereas stage two is unchanged.
Note that domestic productions enter individual utilities only through the corresponding outputs, \((X^1, ..., X^S)\); therefore we can keep the same definition of the conditional indirect utilities, namely (1).

We now proceed to show that Proposition 2 applies to that case as well. A first remark is that the second stage consumptions solve the program (1) for \(\rho = \rho_m, m = w, h\). Indeed, if it was not the case, one could increase \(m\)'s utility without changing neither public nor \(k\)'s private consumptions (where \(k \neq m\)), hence without decreasing \(k\)'s utility, a contradiction.

The first stage program can therefore be written as:

\[
\max_{(q^{1,1}, ..., q^{S,n}, X)} v_w (X, p, \rho_w) + \mu v_h (X, p, \rho_h) \\
\rho_w + \rho_w + \sum_{(s,i) \in (1,1)} p^i q^{s,i} + \sum_{j=S+1}^{N} P^j X^j = y \\
X^s = F^s (q^{1,1}, ..., q^{S,n}), \ s = 1, ..., S
\]

Assume, now, that the collective indirect utilities are of the form (4), with \(\alpha_w (X, p) = \alpha_h (X, p)\). Then, assuming that \(\rho_m > 0, m = w, h\), first order conditions give \(\alpha_w (X, p) = \lambda = \mu \alpha_h (X, p)\) where \(\lambda\) is the Lagrange multiplier of the budget constraint. This implies \(\mu = 1\), which in turn implies the TU property. The converse statement can be shown using the same argument as in Proposition 2.

References


