



Classic Problems of Probability

Reviewed by Andrew I. Dale

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Prakash Gorroochurn

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In one of his “Monday Chats” in 1850, Charles-Augustin Sainte-Beuve wrote, “The idea of a *classic* contains in itself something that has a consequence and consistency, that produces cohesion and tradition, that forms itself, that transmits itself and that endures.” While it is perhaps too soon to say whether each of the problems Gorroochurn discusses could be described as a *monumentum aere perennius*, there is no doubt that some of the older ones have displayed the desired quality of endurance, and a number have led to later ones and may even be seen as precursors of modern research topics.

This book, the author notes, “is targeted primarily to readers who have had at least a basic course in probability. Readers in the history of probability might also find it useful” (p. ix).¹ Gorroochurn urges the reader not to focus on the problems and their solutions to the neglect of the discussions, and indeed I, for one, found these discussions the most useful and absorbing parts of the text. Some problems certainly require a more advanced knowledge than that to be gained in a basic course in probability for their full understanding and development (e.g., Borel’s paradox and Jacob Bernoulli’s Law of Large Numbers), but this in no way detracts from one’s enjoyment of the text as a whole.

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¹References to page numbers without a citation are to the book being reviewed.

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Many of the problems discussed here are well known, and the student of probability either will have been or certainly should be exposed to them. They are arranged in time order, ranging from one in Cardano’s *Liber de ludo aleæ* (1539/1663) to Parrondo’s game-theoretic paradox of 1996. We find here well-known curiosities (and sometimes perplexities) such as the de Méré paradoxes, Buffon’s needle problem, Bertrand’s chords, and birthday coincidences, and also those less well known in general, such as Montmort’s matches problem, a question posed by Samuel Pepys to Isaac Newton, Newcomb’s problem, and Simpson’s paradox.

Some of the problems discussed here are described as “paradoxes”. The more one thinks about this word, however, the less clear one becomes about its meaning. What *exactly* is a paradox? Székely [16, p. xii] says “It is important to distinguish paradoxes from fallacies. The first one is a true though surprising theorem while the second one is a false result obtained by reasoning that seems correct.” Quoting one Phillips (probably Sir Richard Phillips, 1767–1840), de Morgan [5, vol. I, p. 3] finds added to the notion of strangeness the notion of absurdity, and he points out that a paradox is “contrary to common opinion.” And if the touch of absurdity is to be preserved, then we must be grateful to the *Oxford English Dictionary* for advising us that, in the early nineteenth century, “paradox” was an alternative name for *Ornithorhynchus anatinus*, the duck-billed platypus.

De Morgan [5, vol. I, p. 31] can perhaps be seen as having anticipated Székely by writing:

The counterpart of *paradox*, the isolated opinion of one or of few, is the general opinion held by all the rest; and the counterpart of false and absurd paradox is what is called the “vulgar error”, the *pseudodox*.

The masterpiece on this last-mentioned topic is, of course, Sir Thomas Browne’s *Pseudodoxia Epidemica*, first published in 1646.

Crudely speaking, one may classify the solutions of the problems discussed by Gorroochurn as the orthodox, the heterodox, and the paradox (using the latter word here as an obsolete adjective), for we have problems such as Cardano's (a simple one involving dice tossing and indisputable reasoning), d'Alembert's coin tossing problem (whose solution hinges heavily, I believe, on the acceptance of an appropriate, if unusual, sample space), and Simpson's paradox (in which, for example, a positive association between treatment and survival among men and among women may, somewhat surprisingly, be reversed or disappear altogether when the populations are combined).

Let's look at a few of Gorroochurn's problems.

De Méré's (first) problem is stated by Gorroochurn (p. 13) as follows:

When a die is thrown four times, the probability of obtaining at least one six is a little more than $1/2$. However, when two dice are thrown 24 times, the probability of getting at least one double-six is a little less than $1/2$. Why are the two probabilities not the same, given the fact that $\Pr[\text{double-six for a pair of dice}] = 1/36 = 1/6 \cdot \Pr[\text{a six for a single die}]$, and you compensate for the factor of $1/6$ by throwing $6 \cdot 4 = 24$ times when using two dice?

The phrase "at least one" is used by many English authors who discuss this problem (e.g., Feller [8]). However, the original version, given in David [4, p. 137], runs as follows (my translation):

If one undertakes to throw a six with one die, there is an advantage in undertaking to do it in 4 throws, as 671 to 625. If one undertakes to throw double-six with two dice there is a disadvantage in undertaking to do it in 24 throws.

The problem was later discussed by Christiaan Huygens in Propositions X and XI of his *De ratiociniis in ludo aleæ* of 1657 (an annotated reprint of this book was given by Jacob Bernoulli in his *Ars Conjectandi* of 1713). These propositions are translated by Arbuthnot [1, vol. II, pp. 273–274] as follows:

To find at how many Times one may undertake to throw 6 with one Die ... To find out how many Times one may undertake to throw 12 with two Dice.

Note that Arbuthnot gives, in the case of both four and five throws, the odds on a six as 671 against 625.

The question is then whether the original should be *interpreted* in the sense of "at least one". Independent of the formulation of the question, however, in all cases the solution to the second

part of this problem is that he who undertakes to do it in 24 throws lacks an even chance of winning, while he who undertakes to do it in 25 throws has a better than even chance of winning.

In November 1693 Samuel Pepys asked Newton for help with the following problem:

- A—has 6 dyes in a box with which he is to fling a 6
- B—has in another box 12 dyes with which he is to fling 2 sixes
- C—has in another box 18 dyes with which he is to fling 3 sixes.
- Q—Whether B and C have not as easy a taske as A at even luck?

Chaundy and Bullard [3] note that Pepys had been asked to solve this problem by John Smith, writing master of Christ's Hospital in London, an institution of which Pepys was a governor. It has been suggested that Smith's interest was perhaps not altogether academic, since he was later dismissed from his post on being found to have charged the boys for the use of pens and paper.

Gorroochurn takes care to note that, while Pepys originally posed the problem in terms of an exact number of sixes, Newton pointed out that "in reading the Question it seemed to me at first to be ill stated," and he therefore changed it to "at least one six." It is interesting to note that a variation on this old problem has recently been applied to size-estimation problems (see Varagnolo et al. [17]).

In Bertrand [2, p. 2] we find the following problem (stated slightly differently by Gorroochurn):

Three chests, identical in appearance, each have two drawers, each drawer containing one coin [*médaille*]. The coins in the first chest are gold, those in the second are silver, while the third chest has one gold and one silver. One chest is chosen. What is the probability that it contains one gold coin and one silver coin in its drawers?

Note that Bertrand does not say, as Gorroochurn does, that the chest is chosen *at random*, though that this is so emerges from the subsequent discussion (the answer is clearly $1/3$). Then a drawer in the selected chest is chosen (again, it transpires, at random) and opened. It might be argued, suggests Bertrand, that no matter what coin is revealed, there are only two possible cases: the closed drawer contains either a coin of the same metal as the one displayed or a different one. Of these two cases only one is favorable to the event that the chest has different coins, and thus the probability that one has chosen the chest with different coins is $1/2$.

Such an argument, Bertrand notes, although it may appear correct, is in fact false: the two possible cases after the chosen drawer has been opened are not equally probable. Using Bayes's Theorem (Problem 14 in Gorroochurn), any first-year student should be able to solve this problem. What is interesting, though, as Gorroochurn notes, is its connection to the Prisoner's Problem and the now notorious Monty Hall problem.

Simpson's paradox, Senn [15] has noted, is closely connected with the Will Rogers phenomenon. (Will Rogers was an American humorist who suggested during the depression of the 1930s that "When the Okies left Oklahoma and went to California, they raised the average intelligence level of both states.") This was perhaps first described in the 1980s by Feinstein et al. who found survival improvement stage by stage in different groups of cancer patients but no evidence of overall improvement.

A basic principle of rational behavior is the sure-thing principle, stated by Savage [14, p. 21] as follows:

If the person would not prefer f to g , either knowing that the event B obtained, or knowing that the event $\sim B$ obtained, then he does not prefer f to g

(here f and g are possible acts). Put otherwise, the sure-thing principle says that one's preferences should not be affected by outcomes that occur regardless of which actions are taken (i.e., that are "sure-things") or that elements common to any pair of outcomes may be ignored. Simpson's paradox, like the intransitivity of some preference orderings in matters of social choice, violates this principle.

Finally, let us have a look at Parrondo's paradox, the solution of which requires some knowledge of game theory and Markov chains. The idea here is that two losing games can produce a winning expectation when they are played in an alternating sequence. Harmer and Abbott [9] have suggested that such games may have important applications in sociology, physics, genetics, and biology. Commenting on the counterintuitive nature of this paradox, Gorroochurn writes (p. 273), "by randomly playing two losing games, the player comes out a winner!" But before one gets too excited about this, one must remember that the idea of a "fair game" is known to be correctly formulated in terms of a martingale, where the "impossibility of gambling systems" holds (see Feller [8]).

It must be almost impossible to write a book with the wealth of detail that Gorroochurn has provided without a few slips. I mention two here: the first is in connection with the photograph of Adrien-Marie Legendre on page 98. This is now known to be of the French politician Louis Legendre,

not the mathematician [6]. The second and more serious problem is on page 133. Here the myth is perpetuated that the person depicted is Thomas Bayes. The attribution apparently first occurred in Terence O'Donnell's book [12]; for remarks casting grave doubt on O'Donnell's attribution, see [10].

For those readers who already have copies of the books by Mosteller [11], Székely [16], and Whitworth [18] (any one of the many editions) on their shelves, this book will form a useful adjunct, perhaps particularly for the excellence of its coverage of modern references. Those who do not have easy access to these works will benefit enormously from *Classic Problems of Probability*.

References

- [1] J. ARBUTHNOT, *Miscellaneous Works of the Late Dr. Arbuthnot. With an Account of the Author's Life*, 2 vols., London, 1770.
- [2] J. BERTRAND, *Calcul des Probabilités*, Paris: Gauthier-Villars et fils, 1889.
- [3] T. W. CHAUNDY and J. E. BULLARD, John Smith's problem, *The Mathematical Gazette* **44**, 1960, 253–260.
- [4] F. N. DAVID, Mr. Newton, Mr. Pepys & Dyse: A historical note, *Annals of Science* **13**, no. 3, 1957, 137–147.
- [5] A. DE MORGAN, *A Budget of Paradoxes*, 2 vols., 2nd ed. (D. E. Smith, ed.), Chicago: Open Court, 1915.
- [6] P. DUREN, Changing faces; The mistaken portrait of Legendre, *Notices of the AMS* **56**, no. 11, 2009, 1440–1443.
- [7] A. R. FEINSTEIN, D. A. SOSIN, and C. K. WELLS, The Will Rogers phenomenon: Improved technologic diagnosis and stage migration as a source of nontherapeutic improvement in cancer prognosis, *Transactions of the Association of American Physicians* **97**, 1984, 19–24.
- [8] W. FELLER, *An Introduction to Probability Theory and Its Applications*. Vol. I, New York: John Wiley and Sons, 1968.
- [9] G. P. HARMER and D. ABBOTT, Parrondo's paradox, *Statistical Science* **14**, no. 2, 1999, 206–213.
- [10] *IMS Bulletin* **17**, no. 3, 1988, 276–278.
- [11] F. MOSTELLER, *Fifty Challenging Problems in Probability with Solutions*, New York: Dover, 1987.
- [12] T. O'DONNELL, *History of Life Assurance in Its Formative Years. Compiled from approved sources by Terence O'Donnell*, Chicago: American Conservation Company, 1936.
- [13] C.-A. SAINTE-BEUVE, Qu'est-ce qu'un classique? *Causeries du Lundi*, Part 3, 3rd ed., Paris: Garnier Frères.
- [14] L. J. SAVAGE, *The Foundations of Statistics*, 2nd ed., New York: John Wiley and Sons, 1972.
- [15] S. SENN, *Dicing with Death: Chance, Risk and Health*, Cambridge: Cambridge University Press, 2003.
- [16] G. J. SZÉKELY, *Paradoxes in Probability and Mathematical Statistics*, Dordrecht: D. Reidel, 1986.
- [17] D. VARAGNOLO, L. SCHENATO, and G. PILLONETTO, A variation of the Newton-Pepys problem and its connections to size-estimation problems, *Statistics & Probability Letters* **83**, 2013, 1472–1478.
- [18] W. A. WHITWORTH, *Choice and Chance, with 1000 Exercises*, 5th ed. Reprinted 1942, Cambridge: Deighton, Bell and Co., 1901.