Some Laws and Problems of Classical Probability and How Cardano Anticipated Them

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In the history of probability, the sixteen–century physician and mathematician Gerolamo Cardano (1501–1575) was among the first to attempt a systematic study of the calculus of probabilities. Like those of his contemporaries, Cardano's studies were primarily driven by games of chance. Concerning his gambling for twenty–five years, he famously said in his autobiography entitled *The Book of My Life*:

> … and I do not mean to say only from time to time during those years, but I am ashamed to say it, everyday.

Cardano's works on probability were published posthumously in the famous 15–page *Liber de Ludo Aleae* (*The Book on Games of Chance*) consisting of 32 small chapters. Today, the field of probability of widely believed to have been fathered by Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665), whose famous correspondence took place almost a century after Cardano's works. It could appear to many that Cardano has not got the recognition that he perhaps deserves for his contributions to the field of probability, for in the *Liber de Ludo Aleae* and elsewhere he touched on many rules and problems that were later to become classics.

Cardano's contributions have led some to consider Cardano as the real father of probability. Thus we read in mathematician Oystein Ore's biography of Cardano, titled *Cardano: The Gambling Scholar*:

> … I have gained the conviction that this pioneer work on probability is so extensive and in certain questions so successful that it would seem much more just to date the beginnings of probability theory from Cardano's treatise rather than the customary reckoning from Pascal's discussions with his gambling friend de Méré and the ensuing correspondence with Fermat, all of which took place at least a century after Cardano began composing his *De Ludo Aleae*.

The mathematical historian David Burton seemed to share the same opinion, for he said:

> For the first time, we find a transition from empiricism to the theoretical concept of a fair die. In making it, Cardan [Cardano] probably became the real father of modern probability theory.

However, in spite of Cardano's several contributions to the field, in none of the problems did Cardano reach the level of mathematical sophistication and maturity that was later to be evidenced in the hands of his successors. Many of his investigations were either too rudimentary or just erroneous. On the other hand, Pascal's and Fermat's work were rigorous and provided the first impetus for a systematic study of the mathematical theory of probability. In the words of A.W.F. Edwards:

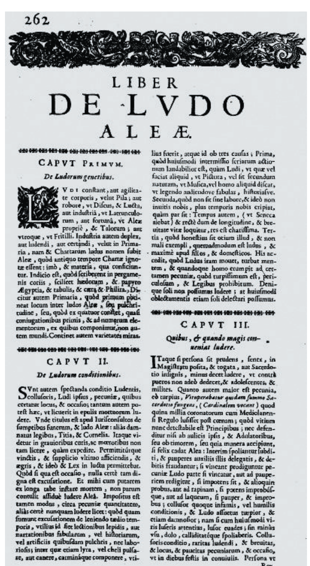
> … in spite of our increased awareness of the earlier work of Cardano (Ore, 1953) and Galileo (David, 1962) it is clear that before Pascal and Fermat no more had been achieved than the enumeration of the fundamental probability set in various games with dice or cards.

Moreover, from the *De Ludo Aleae* (the full English version of this book can be found in Ore's biography of Cardano) it is clear that Cardano is unable to dis-

GEROLAMO CARDANO (1501–1575)

Cardano's early years were marked by illness and mistreatment. He was encouraged to study mathematics and astrology by his father and, in 1526, obtained his doctorate in medicine. Eight years later, he became a mathematics teacher, while still practicing medicine. Cardano's first book in mathematics was the *Practica arithmetice*. In his greatest math work, *Ars Magna* (*The Great Art*), Cardano gave the general solution of a "reduced" cubic equation (i.e., a cubic equation with no second-degree term), and also provided methods to convert the general cubic equation to the reduced one. These results had been communicated to him previously by the mathematician Niccolò Tartaglia of Brescia (1499–1557) after swearing that he would never disclose the results. A bitter dispute thereby ensued between Cardano and Tartaglia, and is nicely documented in Hellman's *Great Feuds in Mathematics*. Cardano's passion for gambling motivated him to write the *Liber de ludo aleae*, which he completed in his old age and was published posthumously.

Figure 1. Gerolamo Cardano (1501–1575) (taken from *http://commons.wikimedia.org/wiki/File:Gerolamo_Cardano.jpg)*



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LIBER
DE LVDO
ALEÆ.

CAPVT PRIMVM.

De Ludorum generibus.

CAPVT II.

De Ludorum conditionibus.

CAPVT III.

Quibus, & quando magis conueniat ludere.

Figure 2. First page of the *Liber de Ludo Aleae*, taken from the Opera Omnia (Vol. I)

associate the unscientific concept of luck from the mathematical concept of chance. He identifies luck with some supernatural force, which he calls the "authority of the Prince." In Chapter 20, titled "On Luck in Play," Cardano states:

> In these matters, luck seems to play a very great role, so that some meet with unexpected success while others fail in what they might expect ...
>
> If anyone should throw with an outcome tending more in one direction than it should and less in another, or else it is always just equal to what it should be, then, in the case of a fair game there will be a reason and a basis for it, and it is not the play of chance; but if there are diverse results at every placing of the wagers, then some other factor is present to a greater or less extent; there is no rational knowledge of luck to be found in this, though it is necessarily luck.

Cardano thus believes that there is some external force that is responsible for the fluctuations of outcomes from their expectations. He fails to recognize such fluctuations are germane to chance and not because of the workings of supernatural forces. In their book *The Empire of Chance: How Probability Changed Science and Everyday Life,* Gigerenzer et al. thus wrote:

> … He [Cardano] thus relinquished his claim to founding the mathematical theory of probability. Classical probability arrived when luck was banished; it required a climate of determinism so thorough as to embrace even variable events as expressions of stable underlying probabilities, at least in the long run.

We now outline some of the rules and problems that Cardano touched on and which were later to be more fully investigated by his more sophisticated successors.

Definition of Classical (Mathematical) Probability

In Chapter 14 of the *De Ludo Aleae*, Cardano gives what some would consider the first definition of classical (or mathematical) probability:

> So there is one general rule, namely, that we should consider the whole circuit, and the number of those casts which represents in how many ways the favorable result can occur, and compare that number to the rest of the circuit, and according to that proportion should the mutual wagers be laid so that one may contend on equal terms.

Cardano thus calls the "circuit" what is known as the sample space today (i.e., the set of all possible outcomes when an experiment is performed). If the sample space is made up of r outcomes which are favorable to an event, and s outcomes which are unfavorable, and if all outcomes are equally likely, then Cardano correctly defines the odds of the event by r/s. This corresponds to a probability of $r/(r + s)$. Compare Cardano's definition to:

- The definition given by Gottfried Wilhelm Leibniz (1646–1716) in 1710:

  > If a situation can lead to different advantageous results ruling out each other, the estimation of the expectation will be the sum of the possible advantages for the set of all these results, divided into the total number of results.

- Jacob Bernoulli's (1654–1705) statement from his 1713 probability opus *Ars Conjectandi:*

  > . . . if the integral and absolute certainty, which we designate by letter α or by unity 1, will be thought to consist, for example, of five probabilities, as though of five parts, three of which favor the existence or realization of some events, with the other ones, however, being against it, we will say that this event has 3/5α, or 3/5, of certainty.

- Abraham de Moivre's (1667–1754) definition from the 1711 *De Mensura Sortis*:

  > If p is the number of chances by which a certain event may happen, and q is the number of chances by which it may fail, the happenings as much as the failings have their degree of probability; but if all the chances by which the event may happen or fail were equally easy, the probability of happening will be to the probability of failing as p to q.

- The definition given in 1774 by Pierre–Simon Laplace (1749–1827), with whom the formal definition of classical probability is usually associated. In his first probability paper, Laplace states:

  > The probability of an event is the ratio of the number of cases favorable to it, to the number of possible cases, when there is nothing to make us believe that one case should occur rather than any other, so that these cases are, for us, equally possible.

However, although the first four definitions (starting from Cardano's) all anteceded Laplace's, it is with the latter that the classical definition was fully appreciated and began to be formally used.

Multiplication Rule for the Independence of Events

One of Cardano's other important contributions to the theory of probability is "Cardano's formula." Suppose an experiment consists of t equally likely outcomes of which r are favorable to an event. Then the odds in favor of the event in one trial of the experiment is $r/(t–r)$. Cardano's formula then states that, in n independent and identical trials of the experiment, the event will occur n times with odds $r^n/(t^n–r^n)$. This is the same as saying that, if an event has probability p $(=r/t)$ of occurring in one trial of an experiment, then the probability that it will occur in all of n independent and identical trials of the experiment is p^n. While this is an elementary result nowadays, Cardano had some difficulty establishing it. At first he thought it was the odds that ought to be multiplied. Cardano calculated the odds against obtaining at least one 1 appearing in a toss of three dice as 125 to 91. Cardano then proceeded to obtain the odds against obtaining at least one 1 in two tosses of three dice as $(125/91)^2 \approx 2$. Thus, on the last paragraph of Chapter 14 of the *De Ludo Aleae*, Cardano writes:

> Thus, if it is necessary for someone that he should throw an ace twice, then you know that the throws favorable for it are 91 in number, and the remainder is 125; so we multiply each of these numbers by itself and get 8,281 and 15,625, and the odds are about 2 to 1. Thus, if he should wager double, he will contend under an unfair condition, although in the opinion of some the condition of the one offering double stakes would be better.

However, in the very next chapter, titled "On an Error Which Is Made About This," Cardano realizes that it is not the odds that must be multiplied. He comes to understand this by considering an event with odds 1:1 in one trial of an experiment. His multiplication rule for the odds would still give an odds of $(1/1)^3$ = 1:1 for three trials of the experiment, which is clearly wrong. Cardano thus writes:

> But this reasoning seems to be false, even in the case of equality, as, for example, the chance of getting one of any three chosen faces in one cast of one die is equal to the chance of getting one of the other three, but according to this reasoning there would be an even chance of getting a chosen face each time in two casts, and thus in three, and four, which is most absurd. For if a player with two dice can with equal chances throw an even and an odd number, it does not follow that he can with equal fortune throw an even number in each of three successive casts.

Cardano thus correctly calls his initial reasoning "most absurd," and then gives the following correct reasoning:

> Therefore, in comparisons where the probability is one–half, as of even faces with odd, we shall multiply the number of casts by itself and subtract one from the product, and the proportion which the remainder bears to unity will be the proportion of the wagers to be staked. Thus, in 2 successive casts we shall multiply 2 by itself, which will be 4; we shall subtract 1; the remainder is 3; therefore a player will rightly wager 3 against 1; for if he is striving for odd and throws even, that is, if after an even he throws either even or odd, he is beaten, or if after an odd, an even. Thus he loses three times and wins once.

Cardano thus realizes that it is the probability, not the odds, that ought to be multiplied. However, in the very next sentence following his above correct reasoning, he makes a mistake again when considering three consecutive casts for an event with odds 1:1. Cardano wrongly states that the odds against the event happening in three casts is $1/(3^2 – 1)=1/8$, instead of $1/(2^3 – 1)=1/7$. Nevertheless, further in the book, Cardano does give the correct general rule:

> Thus, in the case of one die, let the ace and the deuce be favorable to us; we shall multiply 6, the number of faces, into itself: the result is 36; and

two multiplied into itself will be 4; therefore the odds are 4 to 32 , or, when inverted, 8 to 1.

If three throws are necessary, we shall multiply 3 times; thus, 6 multiplied into itself and then again into itself gives 216; and 2 multiplied into itself and again into 2, gives 8; take away 8 from 216: the result will be 208; and so the odds are 208 to 8, or 26 to 1. And if four throws are necessary, the numbers will be found by the same reasoning, as you see in the table; and if one of them be subtracted from the other, the odds are found to be 80 to 1.

In the above, Cardano has considered an event with probability 1/3, and correctly gives the odds against the event happening twice as $(3^2 - 1)/1=8$, happening thrice as $(3^3 - 1)/1=26$, and so on. Cardano thus finally reaches the following correct rule: if the odds in favor of an event happening in one trial of an experiment is $r/(t–r)$ then in n independent and identical trials of the experiment, the odds against the event happening n times is $(t^n–r^n)/r^n$.

Law of Large Numbers

The law of large numbers is attributed to the Jacob Bernoulli (1654–1705) and states that, in n independent tosses of a coin with probability of heads p such that the total number of heads is S_n, the proportion of heads converges in probability to p as n becomes large. In other words, consider an experiment which consists of tossing a coin n times, yielding a sequence of heads and tails. Suppose the experiment is repeated a large number of times, resulting in a large number of sequences of heads and tails. Let us consider the proportion of heads for each sequence. Bernoulli's law implies that, for a given large n, the fraction of sequences (or experiments) for which the proportions of heads is arbitrarily close to p is high, and increases with n. That is, the more tosses we perform for a given experiment, the greater the probability that the proportion of heads in the corresponding sequence will be very close to the true p. This fact is very useful in practice. For example, it gives us confidence in using the frequency interpretation of probability.

Bernoulli had every right to be proud of his law, which he first enunciated it in the fourth part of the *Ars Conjectandi* and which he called his "golden theorem":

> This is therefore the problem that I now wish to publish here, having considered it closely for a period of twenty years, and it is a problem of which the novelty as well as the high utility together with its grave difficulty exceed in value all the remaining chapters of my doctrine. Before I treat of this "Golden Theorem" I will show that a few objections, which certain learned men have raised against my propositions, are not valid.

Bernoulli's law was the first limit theorem in probability. It was also the first attempt to apply the calculus of probability outside the realm of games of chance. In the latter, the calculation of odds through classical (or mathematical) reasoning had been quite successful because the outcomes were equally likely. However, this was also a limitation of the classical method. Bernoulli's law provided an empirical framework that enabled the estimation of probabilities even in the case of outcomes which were not equally likely. Probability was no longer only a mathematically abstract concept. Rather, now it was a quantity that could be estimated with increasing confidence as the sample size became larger.

About 150 years before Bernoulli's times, Cardano had anticipated the law of large numbers, although he never explicitly stated it. In *Cardano: The Gambling Scholar*, Ore has written:

> It is clear ... that he [Cardano] is aware of the so-called law of large numbers in its most rudimentary form. Cardano's mathematics belongs to the period antedating the expression by means of formulas, so that he is not able to express the law explicitly in this way, but he uses it as follows: when the probability for an event is p then by a large number n of repetitions the number of times it will occur does not lie far from the value $m = np$.

Throwing of Three Dice

More than a century after Cardano's times, the Grand Duke of Tuscany asked the renowned physicist and mathematician Galileo Galilei (1564–1642) the following question: "Suppose three dice are thrown and the three numbers obtained added. The total scores of nine, ten, eleven and twelve can all be obtained in six different combinations. Why then is a total score of ten or eleven more likely than a total score of nine or twelve?"

To solve this problem, consider Table 1, which shows each of the six possible combinations for the scores of nine to twelve. Also shown is the number of ways (i.e., permutations) in which each combination can occur.

For example, reading the first entry under the column 12, we have a 6–5–1. This means that, to get a total score of 12, one could get a 6, 5, 1 in any order. Next to the 6–5–1 is the number 6. This is the number of different orders in which one can obtain a 6, 5, 1. Hence we see that the scores of nine to twelve can all be obtained using six combinations for each. However, because different combinations can be realized in a different number of ways, the total number of ways for the scores 9, 10, 11 and 12 are 25, 27, 27, and 25 respectively. Hence scores of ten or eleven are more likely than scores of nine or twelve.

The throwing of three dice was part of the game of passadieci, which involved adding up the three numbers and getting at least eleven points to win. Galileo gave the solution in his 1620 probability paper *Sopra le Scoperte dei Dadi*. In his paper, Galileo states:

> But because the numbers in the combinations in three–dice throws are only 16, that is, 3.4.5, etc. up to 18, among which one must divide the

said 216 throws, it is necessary that to some of these numbers many throws must belong; and if we can find how many belong to each, we shall have prepared the way to find out what we want to know, and it will be enough to make such an investigation from 3 to 10, because what pertains to one of these numbers, will also pertain to that which is the one immediately greater.

However, unbeknownst to Galileo, the same problem had actually already been successfully solved by Cardano almost a century earlier. The problem appeared in Chapter 13 of Cardano's *Liber de Ludo Aleae*. Consider Figure 3 which shows Cardano's solution to the problem of throwing three dice, as it appears in Chapter 13 of the *Liber de Ludo Aleae*. Cardano writes:

> In the case of two dice, the points 12 and 11 can be obtained respectively as (6,6) and as (6,5). The point 10 consists of (5,5) and of (6,4), but the latter can occur in two ways, so that the whole number of ways of obtaining 10 will be 1/12 of the circuit and 1/6 of equality. Again, in the case of 9, there are (5,4) and (6,3), so that it will be 1/9 of the circuit and 2/9 of equality. The 8 point consists of (4,4), (3,5), and (6,2). All 5 possibilities are thus about 1/7 of the circuit and 2/7 of equality. The point 7 consists of (6,1), (5,2), and (4,3). Therefore the number of ways of getting 7 is 6 in all, 1/3 of equality and 1/6 of the circuit. The point 6 is like 8, 5 like 9, 4 like 10, 3 like 11, and 2 like 12.

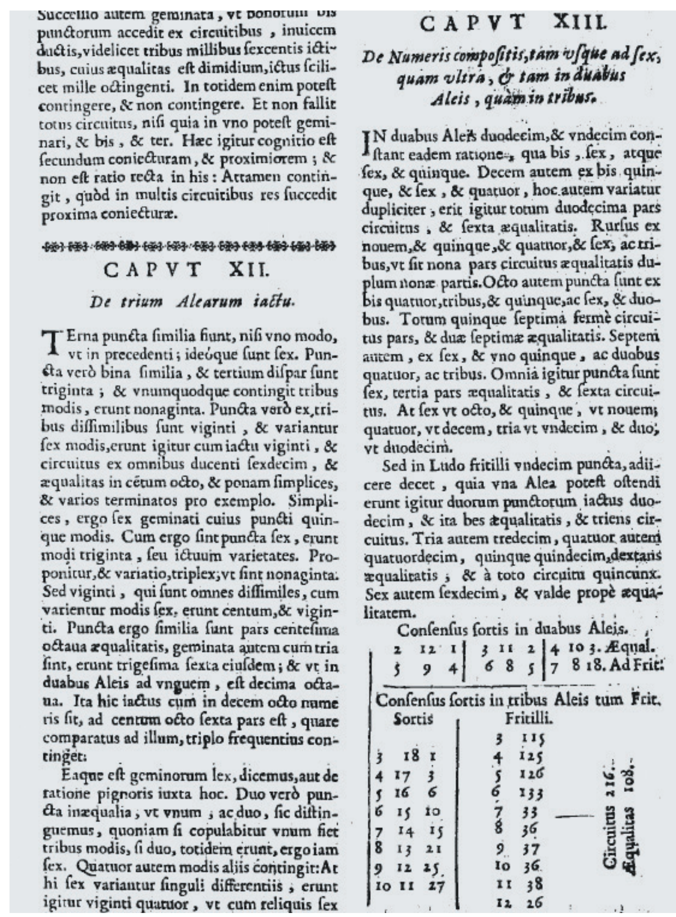
Problem of Points

It is fairly common knowledge that the gambler Antoine Gombaud (1607–1684), better known as the Chevalier de Méré, had been winning consistently by betting even money that a six would come up at least once in four rolls with a single die. However, he had now been losing, when in 1654 he met his friend, the amateur mathematician Pierre de Carcavi (1600–1684). This was almost a century after Cardano's death. De Méré had thought the odds were favorable on betting that he could throw at least one sonnez (i.e. double–six) with twenty–four throws of a pair of dice. However, his own experiences indicated that twenty–five throws were required. Unable to resolve the issue, the two men consulted their mutual friend, the great mathematician, physicist, and philosopher Blaise Pascal (1623–1662). Pascal himself had previously been interested in the games of chance. Pascal must have been intrigued by this problem and, through the intermediary of Carcavi, contacted the eminent mathematician, Pierre de Fermat (1601–1665), who was a lawyer in Toulouse. Pascal knew Fermat through the latter's friendship with Pascal's father, who had died three years earlier. The ensuing correspondence, albeit short, between Pascal and Fermat is widely believed to be the starting point of the systematic development of the theory of probability.

Cardano had also considered the *Problem of Dice* in Chapter 11 of his book, and had reached an incor-

Table 1. Combinations and Number of Ways Scores of Nine to Twelve Can Be Obtained When Three Dice Are Thrown

Score	12		11		10		9	
	6-5-1	6	6-4-1	6	6-3-1	6	6-2-1	6
	6-4-2	6	6-3-2	6	6-2-2	3	5-3-1	6
	6-3-3	3	5-5-1	3	5-4-1	6	5-2-2	3
	5-5-2	3	5-4-2	6	5-3-2	6	4-4-1	3
	5-4-3	6	5-3-3	3	4-4-2	3	4-3-2	6
	4-4-4	1	4-4-3	3	4-3-3	3	3-3-3	1
Total No. of Ways	25		27		27		25	



Succeſſio autem geminata, vt bonorum bis punctorum accedit ex circuitibus, inuicem ductis, videlicet tribus millibus ſexcentis ictibus, cuius æqualitas eſt dimidium, ictus ſcilicet mille octingenti. In totidem enim poteſt contingere, & non contingere. Et non fallit totus circuitus, niſi quia in vno poteſt geminari, & bis, & ter. Hæc igitur cognitio eſt ſecundum coniecturam, & proximiorem; & non eſt ratio recta in his: Attamen contingit, quòd in multis circuitibus res ſuccedit proxima coniecturæ.

CAPVT XII.

De trium Alearum iactu.

TErna puncta ſimilia fiunt, niſi vno modo, vt in precedenti; ideóque ſunt ſex. Puncta verò bina ſimilia, & tertium diſpar ſunt triginta; & vnumquodque contingit tribus modis, erunt nonaginta. Puncta verò ex tribus diſſimilibus ſunt viginti, & variantur ſex modis, erunt igitur cum iactu viginti, & circuitus ex omnibus ducenti ſexdecim, & æqualitas in cētum octo, & ponam ſimplices, & varios terminatos pro exemplo. Simplices, ergo ſex geminati cuius puncti quinque modis. Cum ergo ſint puncta ſex, erunt modi triginta, ſeu ictuum varietates. Proponitur, & variatio, triplex; vt ſint nonaginta. Sed viginti, qui ſunt omnes diſſimiles, cum varientur modis ſex, erunt centum, & viginti. Puncta ergo ſimilia ſunt pars centeſima octaua æqualitatis, geminata autem cum tria ſint, erunt trigeſima ſexta eiuſdem; & vt in duabus Aleis ad vnguem, eſt decima octaua. Ita hic iactus cum in decem octo numeris ſit, ad centum octo ſexta pars eſt, quare comparatus ad illum, triplo frequentius continget:

Eaque eſt geminorum lex, dicemus, aut de ratione pignoris iuxta hoc. Duo verò puncta inæqualia, vt vnum, ac duo, ſic diſtinguemus, quoniam ſi copulabitur vnum fiet tribus modis, ſi duo, totidem erunt, ergo iam ſex. Quatuor autem modis aliis contingit: At hi ſex variantur ſinguli differentiis, erunt igitur viginti quatuor, vt cum reliquis ſex

CAPVT XIII.

De Numeris compoſitis, tam vſque ad ſex, quam vltra, & tam in duabus Aleis, quàm in tribus.

IN duabus Aleis duodecim, & vndecim conſtant eadem ratione, qua bis, ſex, atque ſex, & quinque. Decem autem ex bis quinque, & ſex, & quatuor, hoc autem variatur dupliciter, erit igitur totum duodecima pars circuitus, & ſexta æqualitatis. Rurſus ex nouem, & quinque, & quatuor, & ſex, ac tribus, vt ſit nona pars circuitus æqualitatis duplum nonæ partis. Octo autem puncta ſunt ex bis quatuor, tribus, & quinque, ac ſex, & duobus. Totum quinque ſeptima fermè circuitus pars, & duæ ſeptimæ æqualitatis. Septem autem, ex ſex, & vno quinque, ac duobus quatuor, ac tribus. Omnia igitur puncta ſunt ſex, tertia pars æqualitatis, & ſexta circuitus. At ſex vt octo, & quinque, vt nouem; quatuor, vt decem, tria vt vndecim, & duo, vt duodecim.

Sed in Ludo fritilli vndecim puncta, adiicere decet, quia vna Alea poteſt oſtendi erunt igitur duorum punctorum iactus duodecim, & ita bes æqualitatis, & triens circuitus. Tria autem tredecim, quatuor autem quatuordecim, quinque quindecim, dextans æqualitatis; & à toto circuitu quincunx. Sex autem ſexdecim, & valde propè æqualitatem.

Conſenſus ſortis in duabus Aleis.

2	12	1	3	11	2	4	10	3.	Æqual.
5	9	4	6	8	5	7	8	18.	Ad Frit.

Conſenſus ſortis in tribus Aleis tum Frit.

Sortis			Fritilli.	
			3	115
3	18	1	4	125
4	17	3	5	126
5	16	6	6	133
6	15	10	7	33
7	14	15	8	36
8	13	21	9	37
9	12	25	10	36
10	11	27	11	38
			12	26

Circuitus 216.
Æqualitas 108.

Figure 3. Cardano's solution to the problem considered by Galileo, as it appears in Chapter 13 of the *Liber de Ludo Aleae*. The bottom left column on the right page has the two last rows reading 9, 12, 25 and 10, 11, 27. These correspond, respectively, to a total of 25 ways of obtaining a total of 9 or 12 with three dice, and a total of 27 ways of obtaining a total of 10 or 11 with three dice.

HOW MANY THROWS OF TWO DICE DO WE NEED FOR MORE THAN AN EVEN CHANCE OF AT LEAST ONE DOUBLE-SIX?

With one throw of two dice, the probability of a double-six is (1/6) x (1/6)=1/36, and the probability of a no double-six is 35/36. In n throws of two dice, the probability of no double-sixes at all is $(35/36)^n$. Thus the probability of at least one double-six in n throws of two dice is $1-(35/36)^n$. Solving $1-(35/36)^n > 1/2$, we obtain $n > 24.6$. Hence, we have more than an even chance of a double-six with 25 throws of two dice.

rect answer of 18 throws of two dice for a probability of at least half for at least one double–six. Cardano reaches this answer by making use of an erroneous principle, which Ore calls a "reasoning on the mean" (ROTM). According to the ROTM, if an event has a probability p in one trial of an experiment, then in n trials the event will occur np times on average, which is then wrongly taken to represent the probability that the event will occur in n trials. In our case, we have $p = 1/36$ so that, with $n = 18$ throws, the event "at least a six" is wrongly taken to occur an average $np = 18(1/36) = 1/2$ of the time.

The Problem of Points was another problem de Méré asked Pascal in 1654 and was the question that really launched the theory of probability in the hands of Pascal and Fermat. It goes as follows: "Two players A and B play a fair game such that the player who wins a total of 6 rounds first wins a prize. Suppose the game unexpectedly stops when A has won a total of 5 rounds and B has won a total of 3 rounds. How should the prize be divided between A and B ?"

The problem had already been known hundreds of years before the times of these mathematicians. It had appeared in Italian manuscripts as early as 1380. However, it first came in print in Fra Luca Pacioli's 1494 *Summa de Arithmetica, Geometrica, Proportioni, et Proportionalita*.

To solve the Problem of Points, we need determine how likely A and B are to win the prize if they had continued the game, based on the number of rounds they have already won. The relative probabilities of A and B winning thus determine the division of the prize. Player A is one round short, and player B three rounds short, of winning the prize. The maximum number of hypothetical remaining rounds is (1+3)–1=3, each of which could be equally won by A or B. The sample space for the game is $S = \{A_1, B_1A_2, B_1B_2A_3, B_1B_2B_3\}$. Here, B_1A_2, for example, denotes the event that B would win the first remaining round and A would win the second (and then the game would have to stop since A is only one round short). However, the four sample points in S are not equally likely. Event A_1 occurs if any one of the following four equally likely events occurs: $A_1A_2A_3$, $A_1A_2A_3$, $A_1B_2A_3$, and $A_1B_2B_3$. Event B_1A_2 occurs if any one of the following two equally likely events occurs: $B_1A_2A_3$ and $B_1A_2B_3$. In terms of equally likely sample points, the sample space is thus $S=\{A_1A_2A_3, A_1A_2B_3, A_1B_2A_3, A_1B_2B_3, B_1A_2A_3, B_1A_2B_3, B_1B_2A_3, B_1B_2B_3,$. There are in all eight equally likely outcomes, only one of which ($B_1B_2B_3$) results in B hypothetically winning the game. A thus has a probability 7/8 of winning. The prize should therefore be divided between A and B in the ratio 7:1.

The above solution is essentially the one Fermat gave in his correspondence with Pascal. The latter solved the problem differently, using two alternative methods. The first involved recursive equations and the second the Arithmetic Triangle.

However, Cardano had also considered the Problem of Points in the *Practica arithmetice*. His major insight was that the division of stakes should depend on how many rounds each player had yet to win, not on how many rounds they had already won. Many mathematicians had previously believed the prize should be divided in the same ratio as the total number of games the players had already won. Thus, in the *Summa de Arithmetica*, Pacioli gave the incorrect answer of 5:3 for the division ratio.

However, in spite of Cardano's major insight into the Problem of Points, he was unable to give the correct division ratio. In the *Practica arithmetice*, Cardano uses the concept of the "progression" of a number to solve the Problem of Points, where he defines progression(n) = 1+2+…+n. He then claims that, if players A and B are a and b rounds short of winning, respectively, then the division ratio between A and B should be in accordance with the progressions of b and a, which gives $b(b+1):a(a+1)$. In our case, a =1, b =3, giving an incorrect division ratio of 6:1.

The Arithmetic Triangle

Pascal was able to relate the Problem of Points to the Arithmetic Triangle. Consider players A and B who are a and b rounds short of winning a prize, so that the maximum number of remaining hypothetical rounds is $a+b–1$. Pascal correctly identifies the value of $a+b–1$ with each row of the triangle, such that the corresponding entries, counting from the left, give the number of ways A can win 0, 1, 2, …, rounds. Now player A wins if she wins any of the remaining a, a+1,…, $a+b–1$ rounds. Pascal shows that the number of ways this can happen is given by the sum of the first b entries in the arithmetic table for row $a+b–1$. Similarly, player B wins if she wins any of the remaining b, b+1,…, $a+b–1$ rounds. The number of ways this can happen is given by the sum of the last a entries in the arithmetic table for row $a+b–1$. Pascal is thus able to give the general division rule for a fair game between A and B from the entries of his Arithmetic Triangle:

(sum of the first b entries for row $a+b–1$):(sum of the last a entries for row $a+b–1$)

Although Pascal solves only the case when A and B were $b–1$ and b rounds short in his correspondence with Fermat, he is able to prove the general division rule above in his 1665 *Traité du Triangle Arithmétique*. Applying this simple rule to our question in the problem, we have a=1, b =3, $a+b-1$=3, and a division of stakes of (1+3+3):1=7:1 between *A* and *B*, as required.

In the 1570 *Opus novum de proportionibus*, Cardano also made use of the Arithmetic Triangle well before Pascal. In a 1950 paper, the mathematical historian Boyer reports:

> …in 1570 Cardan[o] published his *Opus novum de proportionibus*, and in this work the Pascal triangle appears in both forms and with varying applications. In connection with the problem of the determination of roots of numbers, Cardan[o] used the familiar earlier form, citing Stifel as the putative discoverer.

Furthermore, Cardano was also aware of what later came to be known as Pascal's identity:

$$\binom{n-1}{r-1}+\binom{n-1}{r}\equiv\binom{n}{r},$$

where n and r are positive integers such that $r \le n$. More importantly, in his paper, Cardano showed that he was aware of the following recursive formula

$$\binom{n}{r}=\frac{n-r+1}{r}\binom{n}{r-1},$$

Boyer writes:

> Had Cardan[o] applied his rule to the expansion of binomials, he would have anticipated the binomial theorem for positive integral powers.

THE ARITHMETIC TRIANGLE

The Arithmetic Triangle was a major component in the development of the calculus of probabilities. This triangle has ones along its two outermost diagonals, and each inner entry is the sum of the two nearest top entries.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

⋮ ⋮ ⋮ ⋮

Moreover, for a given row n, the i^{th} entry counting from left is denoted by $\binom{n}{i}$ and has a special property. It gives the number of ways of choosing i objects out of n identical objects. In the binomial expansion of $(x+y)^n$, $\binom{n}{i}$ is equal to the coefficient of x^i and is thus called a binomial coefficient. The Arithmetic Triangle was known well before Pascal and is called Yang Hui's Triangle in China in honor of the Chinese mathematician Yang Hui (1238–1298) who used it in 1261. Others have called it Halayudha's Triangle since the Indian writer Halayudha used it in the 10th century. The triangle was first called Pascal's triangle by Montmort in his *Essay d'Analyse sur les Jeux de Hazard*.

The St. Petersburg's Paradox

This problem has been undoubtedly one of the most discussed in the history of probability and statistics, and goes as follows: "A player plays a coin–tossing game in a casino. The casino agrees to pay the player 1 dollar if heads appears on the initial throw, 2 dollars if head appears first on the second throw, and in general 2^{n-1} dollars if heads first appears on the n^{th} throw. How much should the player give the casino as an initial down–payment if the game is to be fair (i.e. the expected profit of the casino or player is zero)?"

To "solve" this problem, note that he player wins on the n^{th} throw if all previous (n–1) throws are tails and the n^{th} thrown is a head. This occurs with probability $(1/2)^{n-1}(1/2)=1/2^n$ and the player is then paid 2^{n-1} dollars by the casino. The casino is therefore expected to pay the player the amount $[(1/2)\times1] + [(1/2^2)\times2] + [(1/2^3)\times2^2] + \ldots = 1/2 + 1/2 + 1/2 + \ldots = \infty$. Thus it seems that no matter how large an amount the player initially pays the casino, he or she will always emerge with a profit. Theoretically, this means that only if the player initially pays the casino an infinitely large sum will the game be fair. However, there is a problem in this "solution," for although it seems to imply that the player is expected to be paid an infinite amount of money, in fact she will almost surely be paid much less than that. To see why, we can calculate how likely it is that she will win within the first seven tosses. The probabilities of winning within the first three, four, and five tosses can be shown to be, respectively, .875, .938, and .969. Thus it is very likely that the player will have won within the five tosses. Given that this actually occurs, the player will paid 2^4 = 16 dollars by the casino. So paying the casino 16 dollars would seem fair for all intents and purposes. And yet our solution appears to imply that the player should pay the casino an infinite amount to make the game fair. On one hand, the theory seems to predict the player is expected to emerge with a huge profit however large a sum he or she pays the casino. On the other hand, even paying the casino 1000 dollars seems unfair since most certainly the player will end up being paid at most 16 dollars.

One way of getting around this conundrum is to use the notion of mean utility, which was the approach taken by Daniel Bernoulli (1700–1782). Bernoulli suggested that, if a player has an initial fortune which changes by a small amount, then the corresponding

BERNOULLI'S CONCEPT OF MEAN UTILITY

Suppose a player has an initial fortune x_0, and has a probability p_j (for j= 1,…,K) of winning an amount a_j, where $\sum_j p_j = 1$. If at any stage the player's fortune changes from x to $(x + dx)$, then Bernoulli suggested that the change in the player's utility U for that fortune is $dU \propto dx/x$. From, $dU \propto dx/x$ we get $U=k\ln x+C$, where k and C are both constants. Bernoulli then denotes the mean utility of the player's final fortune by

$$\sum_{j=1}^{K} p_j \ln\left(\frac{x_0 + a_j}{x_0}\right),$$

Now, suppose the player has an initial fortune x_0 and pays an initial amount ξ for the game. Then, for a fair game, Bernoulli sets the player's mean utility for his or her final fortune to be zero, obtaining:

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \ln(x_0 + 2^j - \xi) = \ln x_0.$$

This equation can be solved for ξ when the value of x_0 is specified. For example, if the player's initial fortune is x_0 = 10 dollars, then he or she should pay an upfront amount of $\xi \approx 3$ dollars. On the other hand if x_0 = 1000 dollars, then $\xi \approx$ 6 dollars.

change in the player's utility for that fortune should be inversely proportional to the player's initial fortune. This means that, for a given change in the player's fortune, the more money the player has the less will the change in utility be. Using the concept of mean utility, we can show that, if the player's initial fortune is 10 dollars, then he or she should pay an upfront amount of approximately 3 dollars. On the other hand if the player's initial fortune is 1000 dollars, then approximately 6 dollars should be paid. These are much more reasonable upfront payments than that suggested in our solution. Note that this version of the St. Petersburg problem does not restrict the casino's fortune, but instead conditions on the player's initial fortune.

The St. Petersburg Problem was first proposed by Nicholas Bernoulli (1687–1759) in a letter to Pierre Rémond de Montmort (1678–1719) in 1713. It was published in Montmort's second edition of the *Essay d'Analyse sur les Jeux de Hazard*:

> *Fourth Problem*. A promises to give one ecu to B, if with an ordinary die he scores a six, two ecus if he scores a six on the second throw, three ecus if he scores this number on the third throw, four ecus if he gets it on the fourth throw, & so on. We ask what the expectation of *B* is.
>
> *Fifth Problem*. We ask the same thing if A promises B ecus according to the sequence 1, 2, 4, 8, 16, etc., or 1, 3, 9, 27, etc., or 1, 4, 9, 16, 25, etc., or 1, 8, 27, 64, etc., instead of 1, 2, 3, 4, 5, etc. like before.

It was then taken up again by Daniel Bernoulli (1700–1782) and published in a similar form we showed above in the St. Petersburg Academy Proceedings in 1738. Almost every prominent mathematician of the time discussed the problem, which from then on became known as the St. Petersburg Paradox.

An even earlier version of the St. Petersburg Problem appears in Cardano's *Practice Arithmetice* and is as follows:

> A rich and a poor man play for equal stakes. If the poor man wins, on the following day the stakes are doubled and the process continues; if the rich man wins once, the play is ended once and for all.

Cardano states that the rich man is at a great disadvantage in this game. Let's see if that is the case. Suppose that the game is played by tossing a fair coin, the poor man's initial stake is one unit, and the rich man's initial capital is c units. Then the maximum number of possible games occurs when the rich man keeps on losing until he is ruined. Suppose this maximum is n. Then the total amount lost by the rich man in n games is $1 + 2 + 4 + ... +2^{n-1} = 2^n-1$, where n is the largest integer such that $2^n - 1 \le c$. Now, it can be shown that both the rich and poor men have the same expected profit of zero. However, the rich man's can win the series if he wins on the m^{th} toss, which happens with probability $2-m$, so that his overall probability of winning is the sum $(1/2)+(1/2)^2+\ldots+(1/2)^n = 1-2^{-n}$. On the other hand, the poor man's overall probability of winning the series is 2^{-n}. Thus, contrary to Cardano's assertion, the rich man is ever more likely to ruin the poor man as the number of tosses increases.

Conclusion

Cardano touched on many problems and rules of probability that were later to take a prominent position in the literature. However, he was unable to make any real "breakthrough." This has more to do more with the fact a proper symbolic mathematical language had not been developed in the 16th century than anything else. Nevertheless, once this was done, Cardano's successors were able to more appropriately deal with his initial investigations. C

Further Reading

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