Limiting Distribution of the Estimator of a Covariance Matrix

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For reasons that I need not go into here, suppose that in the covariance matrix

$$\Sigma = (\sigma_{ij}), \quad i, j = 1, 2, \ldots, m,$$

we are interested in testing the hypothesis $H_0 : \sigma_{ij} = 0, \quad i \neq j$.

Let $\{u_{t\cdot} : t = 1, 2, \ldots T\}$ be a sequence of independent identically distributed random **row** m-element vectors with mean 0 and covariance matrix Σ . An estimator of the covariance matrix is

$$\hat{\Sigma} = \frac{1}{T} U'U, \quad U = (u_{t\cdot}). \tag{1}$$

It is apparent from Eq.(1) that $\hat{\Sigma}$ is an **unbiased** estimator. Consider its limiting distribution, i.e. consider the behavior of

$$\sqrt{T}\mathrm{vec}(\hat{\Sigma} - \Sigma) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \{[I \otimes u'_{t\cdot}]u'_{t\cdot} - \sigma\} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (u'_{t\cdot} \otimes u'_{t\cdot} - \sigma), \quad \sigma = \mathrm{vec}(\Sigma) \tag{2}$$

and note, for future reference, that $\sigma_{\cdot i}$ is the i^{th} column of Σ . The summands in the equation above are i.i.d random (column) vectors of dimension m^2 with mean zero and some covariance matrix, say Φ . Thus, they obey a **central limit theorem** and the remaining problem is to determine this covariance matrix. But

$$\Phi = E[(u'_{t\cdot} \otimes u'_{t\cdot} - \sigma)(u_{t\cdot} \otimes u_{t\cdot} - \sigma') = E(u'_{t\cdot}u_{t\cdot} \otimes u'_{t\cdot}u_{t\cdot}) - \sigma\sigma'. \tag{3}$$

We need to find the expectation of the matrix in the rightmost member, which is a Kronecker product with (i,j) block

$$u_{ti}u_{tj}[u'_{t\cdot}u_{t\cdot}] = u_{ti}u_{tj}(u_{tk}u_{ts}), \quad k, s =, 2, 3, \ldots, m. \tag{4}$$

The typical, (k,s), element of this block is $u_{ti}u_{tj}u_{tk}u_{ts}$. Let us begin with the diagonal blocks, Φ^*_{ii} Since the distribution is symmetric all odd moments vanish and we have that

$$\Phi^*_{ii} = \mathrm{diag}(\sigma_{ii}\sigma_{11}, \sigma_{ii}\sigma_{22}, \ldots, \sigma_{ii}\sigma_{mm}), \tag{5}$$

except that the i^{th} diagonal element is $Eu_{ti}^4 = \mu_{4i}$. For the off diagonal blocks we need to evaluate (for $i \neq j$)

$$\Phi_{ij}^* = E[u_{ti}u_{tj}u_{tk}u_{ts}] = [e_{\cdot i}e_{\cdot j}' + e_{\cdot j}e_{\cdot i}']\sigma_{ii}\sigma_{jj}. \tag{6}$$

It may be verified that

$$\Phi^* = (\Phi_{ij}^*), \quad i, j = 1, 2, \ldots, m \tag{7}$$

is an $m^2 \times m^2$ matrix of **rank** $m(m+1)/2$. This is so because **by construction** all columns except for those containing the **fourth moment** form identical pairs. Thus there are $m(m-1)/2$ distinct pairs and the collection obtained by taking any one element from each pair constitutes a linearly independent set. The covariance matrix of the limiting distribution is, thus,

$$\Phi = \Phi^* - \sigma\sigma', \tag{8}$$

which is also a positive semi-definite matrix of rank $m(m+1)/2$.

We are now interested in a test of the hypothesis $H_0 : \sigma_{ij} = 0, \ i \neq j$. We may proceed as follows: Under the null, Φ^* does not change, but the other component of Φ becomes

$$\sigma\sigma' = (\sigma_{ii}\sigma_{jj}e_{\cdot i}e_{\cdot j}'), \quad i, j = 1, 2, \ldots, m, \tag{9}$$

so that the covariance matrix under the null, say Φ_0 , is given by

$$\Phi_0 = [\Phi_{ij}^* - (\sigma_{ii}\sigma_{jj}e_{\cdot i}e_{\cdot j}'], i, j = 1, 2 \ldots, m, \tag{10}$$

where $e_{\cdot i}$ is an m-element **column** vector all of whose elements are zero save the i^{th} , which is one.

Note that under the null, Eq. (10) implies that the off diagonal blocks are given by

$$\Phi_{ij} = [(\sigma_{ii}\sigma_{jj}e_{\cdot j}e_{\cdot i}'], i, j = 1, 2 \ldots, m, \tag{11}$$

and the matrix Φ_0 is of rank $m(m+1)/2$, by the same argument given after Eq. (7).

To define the test statistic, say ζ , let R be a selection matrix that takes out of σ the distinct elements $\sigma_{ij}, i \neq j$. It is of dimension $\sum_{i=1}^{m-1}(m-i) \times m^2$, or $m(m-1)/2 \times m^2$, and of rank $m(m-1)/2$. This matrix may be defined as

$$R\hat{\sigma} = (\hat{\sigma}_{21}, \hat{\sigma}_{31}, \ldots, \hat{\sigma}_{m1}, \hat{\sigma}_{32}, \ldots, \hat{\sigma}_{m2}, \ldots, \ldots \hat{\sigma}_{mm-1})'. \tag{12}$$

The test statistic is then

$$\zeta = T\hat{\sigma}' R' (R\Phi_0 R')^{-1} R\hat{\sigma} \xrightarrow{\mathrm{d}} \chi^2_d, \quad d = \frac{m(m-1)}{2}. \tag{13}$$

That the matrix above is invertible is guaranteed by the rank of the matrix Φ_0 which is $m(m+1)/2$.