Moments of Truncated (Normal) Distributions

Phoebus J. Dhrymes
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Introduction

The purpose of this note is to obtain the moments of truncated distributions as they occur in the dummy endogenous variable models. This can easily be done from first principles, so that it is accessible to readers of varying skills and mathematical sophistication. In fact only calculus is required to read these notes.

In what follows, f, F will denote the pdf and cdf, respectively, of a normal distribution **with mean** μ and **variance** σ^2. Similarly, ϕ, Φ will denote, respectively, the pdf and cdf of the unit normal.

Relation between f and ϕ.

$$f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(u-\mu)^2}, \quad \phi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2}.$$

Thus, we have the relation

$$\Pr(u \leq k) = \int_{-\infty}^{k} f(u)du = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{k-\mu}{\sigma}} e^{-\frac{1}{2}\xi^2} d\xi = \Phi\left(\frac{k-\mu}{\sigma}\right),$$

which is obtained by making the change in variable $\xi = (u-\mu)/\sigma$. A simple implication of the preceding is that for $k_1 < k_2$

$$\Pr(k_1 \leq u < k_2) = \Phi\left(\frac{k_2-\mu}{\sigma}\right) - \Phi\left(\frac{k_1-\mu}{\sigma}\right).$$

Truncated Distributions

I have not found a clean distinction between truncated and censored random variables, but my usage in this context is: a **truncated** random variable, or a truncated distribution, refers to a r.v. whose "natural" range is $(-\infty, \infty)$, but is **constrained to assume values only in** (k_1, k_2) where the k_i are non-random and one of them may well be $\pm\infty$. A **censored** r.v. is one whose value is **only observed through the operation of some random process**, as for example in the threshold models. Notice, e.g., in the "Tobit" model the left hand variable assumes values only in $(0,\infty)$ but it is not a truncated r.v. because whether it assumes the value zero or a value greater than zero is determined by a random process related to the threshold.

Moments of Truncated Normal Distributions

Suppose we wish to obtain an expression for $E(u^m|u \leq k)$, for some fixed k . In dealing with such issues it is convenient to deal with the truncated distribution, i.e. with that part that deals only with Borel sets of the form $(a,b) \subseteq (-\infty, k]$. To convert that part to a **proper distribution** we need to normalize so that its integral over this range is unity. Let f be the distribution in question; then the normalized truncated distribution is given by

$$\frac{f(u)}{F(k)} = \frac{\phi(\xi)}{\Phi\left(\frac{k-\mu}{\sigma}\right)}, \quad \xi \in (-\infty, (k-\mu)/\sigma]. \tag{1}$$

In the literature of dummy endogenous variables we are often called upon to deal with the mean and variance or, more generally, the moments of truncated distributions. We now deal with such issues. First, because the term $(k-\mu)/\sigma$ occurs repeatedly we shall replace it by

$$h = \frac{k-\mu}{\sigma}, \quad \text{so that we can write, e.g.} \quad \Pr(k_1 \leq u < k_2) = \Phi(h_2) - \Phi(h_1).$$

Now consider the m^{th} partial or truncated moment of the r.v. u ; it is given by

$$E(u^m|u \leq k) = \frac{1}{\Phi(h)} \int_{-\infty}^{k} u^m f(u) du$$

It would be desirable to find a routine way of evaluating such integrals. To this end, in the integral above, make the change in variable

$$\xi = \frac{u-\mu}{\sigma}$$

and note that the expression in question is transformed to

$$E(u^m|u \leq k) = \frac{1}{\Phi(h)} \int_{-\infty}^{k} u^m f(u) du = \frac{1}{\Phi(h)} \int_{-\infty}^{h} (\mu + \sigma\xi)^m \phi(\xi) d\xi. \tag{2}$$

In the expression of Eq. (2), expand and note the definition of I_r

$$(\mu + \sigma\xi)^m = \sum_{r=0}^{m} \binom{m}{r} \mu^{m-r} \sigma^r \xi^r, \quad I_r = \frac{1}{\Phi(h)} \int_{-\infty}^{h} \xi^r \phi(\xi) d\xi, \tag{3}$$

to obtain

$$E(u^m|u \leq k) = \sum_{r=0}^{m} \binom{m}{r} \mu^{m-r} \sigma^r I_r. \tag{4}$$

We shall now obtain a **recursive** representation of I_r ; to this end observe that

$$I_r = \frac{1}{\Phi(h)} \int_{-\infty}^{h} \xi^r \phi(\xi) d\xi = -h^{r-1} \frac{\phi(h)}{\Phi(h)} + (r-1) I_{r-2}, \tag{5}$$

where we have made use of integration by parts with $s = \xi^{r-1}$, $dv = \int_{-\infty}^{h} \xi \phi(\xi) d\xi$. To make this recursive representation fully useful, we need to produce the "initial" conditions,

$$I_0 = \frac{1}{\Phi(h)} \int_{-\infty}^{h} \xi^0 \phi(\xi) d\xi = 1, \quad I_1 = \frac{1}{\Phi(h)} \int_{-\infty}^{h} \xi \phi(\xi) d\xi = -\frac{\phi(h)}{\Phi(h)}. \tag{6}$$

It will be notationally convenient to put

$$\rho(h) = \frac{\phi(h)}{\Phi(h)}, \tag{7}$$

to facilitate the derivation of the first few moments of the truncated distribution, from Eq. (4), as follows:

$$E(u^0|u \leq k) = I_0 = 1 \tag{8}$$

$$E(u||u \leq k) = \mu I_0 + \sigma I_1 = \mu - \sigma \rho(h)$$

$$E(u^2|u \leq k) = \mu^2 I_0 + 2\mu\sigma I_1 + \sigma^2 I_2 = \mu^2 - 2\mu\sigma\rho(h) + \sigma^2[1 - h\rho(h)] \tag{9}$$

where we have used the recursion of Eq. (5) to obtain

$$I_2 = -h\rho(h) + I_0 = -h\rho(h) + 1.$$

Since $\mathrm{Var}(u|u \leq k) = E(u^2|u \leq k) - [E(u|u \leq k)]^2$, we easily find from the derivations above

$$\mathrm{Var}(u|u \leq k) = \mu^2 - 2\mu\sigma\rho(h) + \sigma^2[1 - h\rho(h)] - [\mu^2 - 2\mu\sigma\rho(h) + \sigma^2\rho(h)^2 = \sigma^2[1 - h\rho(h) - \rho(h)^2]. \tag{10}$$

Higher moments, central or otherwise, can easily be obtained by the developments above.