Phys C2601, Physics III: Classical and Quantum Waves Homework Assignment 2: Solutions

October 6, 2009

1

(a) **4 points**

The horizontal displacement of the target is $x - \xi$

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -k(x-\xi) - b\frac{dx}{dt} \\ k = m\omega_0^2 &= \frac{mg}{l} \\ \gamma &= b/m \end{aligned}\right\} \Rightarrow \boxed{\tfrac{d^2x}{dt^2} + \tfrac{g}{l}x + \gamma\tfrac{dx}{dt} = \tfrac{g}{l}\xi}$$

Try $x = Re[Ae^{i(\omega t-\delta)}]$, $\xi = \xi_0 e^{i\omega t}$,

$$Ae^{i(\omega t-\delta)}(i\omega)^2 + \gamma(i\omega)Ae^{i(\omega t-\delta)} + \frac{g}{l}Ae^{i(\omega t-\delta)} = \frac{g}{l}\xi_0 e^{i\omega t}$$

$$\Rightarrow \begin{cases} \omega^2 - \omega_0^2 + \frac{\omega_0^2}{A}\xi_0\cos\delta = 0 \\ \gamma\omega = \frac{\omega_0^2}{A}\xi_0\sin\delta \end{cases}$$

$$\Rightarrow \frac{A^2\gamma^2\omega^2}{\xi_0^2\omega_0^4} + \frac{A^2(\omega^2-\omega_0^2)^2}{\xi_0^2\omega_0^4} = 1$$

Thus,

$$\boxed{x = A\cos(\omega t - \delta) \text{ with } A = \frac{\xi_0\omega_0^2}{\sqrt{\gamma^2\omega^2 + (\omega^2-\omega_0^2)^2}} \text{ and } \tan\delta = \frac{\gamma\omega}{\omega_0^2-\omega^2}}$$

(b) **3 points**

Since at free vibration, $A = A_0 e^{-\frac{\gamma}{2}t}$. $\frac{1}{e} = e^{-\frac{50\pi\gamma}{\omega_0}} \Rightarrow \frac{\omega_0}{\gamma} = 50\pi$. At resonance, $\omega = \omega_0$, then

$$\boxed{A = \frac{\xi_0\omega_0}{\gamma} = 157mm.}$$

(c) **3 points**

Solve $\frac{50\pi\xi_0}{2} = \frac{\xi_0\omega_0^2}{\sqrt{\gamma^2\omega^2+(\omega^2-\omega_0^2)^2}}$, with $\xi_0 = 0.001\text{m}$, $\omega_0^2 = 9.8s^{-2}$, $\gamma^2 = (\frac{\omega_0}{50\pi})^2$.

$$\Rightarrow \omega^4 + (\gamma^2 - 2\omega_0^2)\omega^2 + (1 - \frac{1}{25^2\pi^2})\omega_0^4 = 0$$
$$\Rightarrow \boxed{\omega = 3.1477s^{-1}, 3.1132s^{-1}}$$

2

(a) **4 points**

Rate of doing work is $\vec{F}_{ex} \cdot \vec{v}$, since $\vec{F}_{ex} = b\vec{v}$, the instantaneous rate of doing work against this force is then $P = bv^2$.

(b) **3 points**

$$\begin{aligned}\bar{P} &= \frac{1}{T}\int_0^T bv^2 dt = \int_0^T b\dot{x}^2 dt \\ &= \frac{1}{T}\int_0^{T=\frac{2\pi}{\omega}} bA^2\omega^2 \sin^2(\omega t - \delta) dt \\ &= \boxed{bA^2\omega^2/2}\end{aligned}$$

(c) **3 points**

$$\left.\begin{aligned}\bar{P} &= bA^2\omega^2/2 \\ A(\omega) &= \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \\ \gamma &= b/m, Q = \omega_0/\gamma \\ k &= m\omega_0^2\end{aligned}\right\} \Rightarrow \boxed{\bar{P}(\omega) = \frac{F_0^2\omega_0}{2kQ}\frac{1}{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+\frac{1}{Q^2}}}$$

3

(a) **4 points**

The equations of motion for A and B could be written as following:

$$\begin{cases} m\frac{d^2x_A}{dt^2} = -kx_A - 2k(x_A - x_B) \\ m\frac{d^2x_B}{dt^2} = -kx_B - 2k(x_B - x_A) \end{cases}$$

Define the center of mass coordinate $X_{\text{cm}} = \frac{x_A+x_B}{2}$ and the difference coordinate $x = x_A - x_B$, so from the equation of motion, we could get:

$$\begin{cases} m\frac{d^2X_{\text{cm}}}{dt^2} = -kx_{\text{cm}} \\ m\frac{d^2x}{dt^2} = -5kx \end{cases}$$

$$\Rightarrow \boxed{\omega_1 = \omega_0 = \sqrt{\frac{k}{m}}, \omega_2 = \sqrt{5}\omega_0 = \sqrt{\frac{5k}{m}}}$$

(b) **3 points**

With initial condition at t=0, $X_{\text{cm}} = x = 0$ and $\frac{dX_{\text{cm}}}{dt} = v_0/2$, $\frac{dx}{dt} = v_0$, try solution

$$\begin{cases} X_{\text{cm}} = A_1\cos(\omega_1 t - \delta_1) \\ x = A_2\cos(\omega_2 t - \delta_2) \end{cases}$$

$$\Rightarrow \begin{cases} X_{\text{cm}} = \dfrac{v_0}{2\omega_1}\cos(\omega_1 t - \dfrac{\pi}{2}) \\ x = \dfrac{v_0}{\omega_2}\cos(\omega_2 t - \dfrac{\pi}{2}) \end{cases} \Rightarrow \boxed{\frac{A_1}{A_2} = \frac{\sqrt{5}}{2}}$$

(c) **3 points**

$$x_B = (2X_{\text{cm}} - x)/2 = \boxed{\frac{v_0}{2\omega_0}\sin\omega_0 t - \frac{v_0}{2\sqrt{5}\omega_0}\sin\sqrt{5}\omega_0 t}$$

4

(a) **7 points**

define $\omega_i = \sqrt{k/m_i}$, i=1,2,3

$$\frac{d^2\vec{x}}{dt^2} = \frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\omega_1^2 & \omega_1^2 & 0 \\ \omega_2^2 & -2\omega_2^2 & \omega_2^2 \\ 0 & \omega_3^2 & -\omega_3^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

Try $\vec{x} = \vec{A} \exp i\omega t$

$$\begin{pmatrix} \omega_1^2 & -\omega_1^2 & 0 \\ -\omega_2^2 & 2\omega_2^2 & -\omega_2^2 \\ 0 & -\omega_3^2 & \omega_3^2 \end{pmatrix} \vec{A} = \omega^2 \vec{A},$$

Find eigen value.

$$\begin{vmatrix} \omega_1^2 - \lambda & -\omega_1^2 & 0 \\ -\omega_2^2 & 2\omega_2^2 - \lambda & -\omega_2^2 \\ 0 & -\omega_3^2 & \omega_3^2 - \lambda \end{vmatrix} = 0$$

$$-\lambda^3 + (2\omega_2^2 + \omega_3^2 + \omega_1^2)\lambda^2 - (\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\omega_1^2)\lambda = 0$$

since $\omega_1 = \omega_3$, thus $\boxed{\omega = 0, \omega_1, \sqrt{\omega_1^2 + 2\omega_2^2}}$

(b) **3 points**

$$ratio = \frac{\omega_1}{\sqrt{\omega_1^2 + 2\omega_2^2}} = \sqrt{\frac{3}{11}}$$

5

(a) **3 points**

$$M_1\ddot{x}_1 = -kx_1 + M_2 g \sin\theta = -kx_1 + M_2 \frac{g}{l}(x_2 - x_1)$$

$$M_2\ddot{x}_2 = -M_2 g \sin\theta = -M_2 \frac{g}{l}(x_2 - x_1)$$

(b) **4 points**

Since $M_1 = M_2 = M$, we could write the equation of motion as following:

$$\frac{d^2\vec{x}}{dt^2} = \frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k}{M} + \frac{g}{l} & -\frac{g}{l} \\ -\frac{g}{l} & \frac{g}{l} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

Find eigen value,

$$\begin{vmatrix} \frac{k}{M} + \frac{g}{l} - \lambda & -\frac{g}{l} \\ -\frac{g}{l} & \frac{g}{l} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \boxed{\omega^2 = \frac{g}{l} + \frac{k}{2M} \pm \sqrt{\frac{g^2}{l^2} + \frac{k^2}{4M^2}}}$$

(c) **3 points**

If $g/l \gg k/M$, $\omega^2 \approx \frac{g}{l} + \frac{k}{2M} \pm \frac{g}{l}$, thus $\boxed{\omega \approx \sqrt{\frac{2g}{l}} \;\; or \; \sqrt{\frac{k}{2M}}}$

6

(a) **5 points**

$$\det[\mathrm{A} - \lambda \mathrm{I}] = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm 1$$

$$\lambda = 1 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \Rightarrow \boxed{\text{A=B, } \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\lambda = -1 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \Rightarrow \boxed{\text{A=-B, } \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

(b) **5 points**

$$\det[\mathrm{A} - \lambda \mathrm{I}] = \begin{vmatrix} 1-\lambda & i \\ -i & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\sqrt{2}$$

$$\lambda = \sqrt{2} \Rightarrow \begin{pmatrix} 1-\sqrt{2} & i \\ -i & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \Rightarrow \boxed{A = (1+\sqrt{2})iB, \; \vec{x} = \begin{pmatrix} i \\ \sqrt{2}-1 \end{pmatrix}}$$

$$\lambda = -\sqrt{2} \Rightarrow \begin{pmatrix} 1+\sqrt{2} & i \\ -i & -1+\sqrt{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \Rightarrow \boxed{A = (1-\sqrt{2})iB, \; \vec{x} = \begin{pmatrix} -i \\ \sqrt{2}+1 \end{pmatrix}}$$