Phys C2601, Physics III: Classical and Quantum Waves Homework Assignment 3: Solutions

October 6, 2009

1

(a) **5 points**

$$y(x,t) = A \sin kx \cos \omega t$$

thus,

$$\left.\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= -Ak^2 \sin kx \cos \omega t \\ \frac{\partial^2 y}{\partial t^2} &= -A\omega^2 \sin kx \cos \omega t \\ \frac{\partial^2 y}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \end{aligned}\right\} \Rightarrow \boxed{v = \frac{\omega}{k}}$$

(b) **5 points**

$$\begin{aligned} y(x,t) &= A \sin kx \cos \omega t \\ &= \boxed{\frac{A}{2}[\sin(kx+\omega t) - \sin(kx-\omega t)]} \end{aligned}$$

2

(a) **5 points**

Apply the boundary conditions: $y(0,t) = y(L,t) = 0$ to the standing wave solution $y(x,t) = \sum_n A \sin k_n x \cos(\omega_n t - \delta_t)$,

$$\Rightarrow \omega_n = v k_n = \sqrt{\frac{T}{M/L}} \frac{n\pi}{L},$$

$$\Rightarrow \boxed{\nu_n = \frac{\omega_n}{2\pi} = \frac{n}{2}\sqrt{\frac{T}{ML}}}$$

(b) **5 points**

N=3 coupled oscillator.

$$\omega_n = 2\omega_0 \sin \frac{n\pi}{2(N+1)} = 2\sqrt{\frac{12T}{ML}} \sin \frac{n\pi}{8}$$

$$\Rightarrow \boxed{\nu_n = \frac{\omega_n}{2\pi} = \frac{2\sqrt{3}}{\pi}\sqrt{\frac{T}{ML}} \sin \frac{n\pi}{8}}$$

3

(a) **5 points**

$$\boxed{\nu_n = c/\lambda_n = \frac{nc}{2L}}$$

(b) **5 points**

$$n = \frac{2L\nu}{c} = \frac{2L(\nu_0 \pm \Delta\nu)}{c} = 5 \times 10^6 \pm 10 \Rightarrow \boxed{21 \text{ modes}}$$

$$L \leq \frac{c}{2\Delta\nu} \Rightarrow \boxed{L_{max} = 15cm}$$

4

(a) **5 points**

$y(x,t) = A\sin(\frac{n\pi}{L}x)\cos(\omega_n t - \delta_n)$, with $\omega_n = vk_n = \sqrt{\frac{T}{M/L}}\frac{n\pi}{L}$

$$\begin{aligned}
K.E &= \int_0^L \frac{M}{2L}\left(\frac{\partial y}{\partial t}\right)^2 dx \\
&= \frac{M}{2L}A^2\omega_n^2\sin^2(\omega_n t - \delta_n)\int_0^L \sin^2(\frac{n\pi}{L}x)dx \\
&= \frac{M}{4}A^2\omega_n^2\sin^2(\omega_n t - \delta_n)
\end{aligned}$$

$$\Rightarrow E_{\text{tot}} = K.E_{max} = \frac{M}{4}A^2\omega_n^2 = \boxed{\frac{TA^2n^2\pi^2}{4L}}$$

(b) **5 points**

$$\begin{cases}
y_1(x,t) = A_1\sin(\frac{\pi}{L}x)\cos(\omega_1 t) \\
y_3(x,t) = A_3\sin(\frac{3\pi}{L}x)\cos(\omega_3 t - \frac{\pi}{4}) \\
y(x,t) = y_1(x,t) + y_3(x,t)
\end{cases}$$

$$\begin{aligned}
K.E &= \int_0^L \frac{M}{2L}\left(\frac{\partial y}{\partial t}\right)^2 dx \\
&= \frac{M}{2L}\left[\int_0^L \left(\frac{\partial y_1}{\partial t}\right)^2 dx + \int_0^L \left(\frac{\partial y_3}{\partial t}\right)^2 dx + 2\int_0^L \left(\frac{\partial y_3}{\partial t}\right)\left(\frac{\partial y_1}{\partial t}\right) dx\right] \\
&= \frac{M}{2L}\left[\int_0^L \left(\frac{\partial y_1}{\partial t}\right)^2 dx + \int_0^L \left(\frac{\partial y_3}{\partial t}\right)^2 dx + 0\right] \\
&= \boxed{\frac{T\pi^2}{4L}(A_1^2 + 9A_3^2)}
\end{aligned}$$

5

(a) **5 points**

The work done against the tension gives the string its initial potential energy and at this moment, this string has zero K.E.

$$
\begin{aligned}
E_{\text{tot}} = W &= \int_0^h 2T\frac{y}{\sqrt{y^2+L^2/4}}dy \\
&\simeq \int_0^h 2T\frac{y}{L/2}dy \\
&= \boxed{\frac{2Th^2}{L}}
\end{aligned}
$$

(b) **5 points**

$y(x,t) = \sum_n A_n \sin(\frac{\pi}{L}nx)\cos\omega_n t$, with $A_n = \frac{2}{L}\int_0^L y(x,t=0)\sin(\frac{\pi}{L}nx)$,

$$
y(x,t=0) = \begin{cases} 2hx/L & 0 \le \text{x} \le \text{L/2} \\ 2h - 2hx/L & \text{L/2} \le \text{x} \le \text{L} \end{cases}
$$

$$
\begin{aligned}
\Rightarrow A_n &= \frac{2}{L}\left[\int_0^{L/2}\frac{2h}{L}x\sin\frac{\pi n}{L}xdx + \int_{L/2}^{L}(2h - \frac{2h}{L}x)\sin\frac{\pi n}{L}xdx\right] \\
&= \frac{8}{\pi^2 n^2}\sin\frac{n\pi}{2}
\end{aligned}
$$

Then $A_1 = \frac{8h}{\pi^2}$, $A_2 = 0$, $A_3 = -\frac{h}{\pi^2}$, $A_4 = 0$,... thus every $T = 2\pi/\omega_1 = \frac{2\pi}{\sqrt{\frac{T}{M/L}}\frac{n\pi}{L}} = \boxed{2\sqrt{\frac{ML}{T}}}$

6

(a) **5 points**

$y(x,t) = \sum_n A_n \sin(\frac{\pi}{L}nx)\cos(\omega_n t + \delta_n)$ with $\boxed{\omega_n = \sqrt{\frac{T}{M/L}}\frac{n\pi}{L}}$

$$
\left.\frac{\partial y}{\partial t}\right|_{t=0} = -\sum_n \omega_n A_n \sin(\frac{\pi}{L}nx)\sin(\omega_n t + \delta_n) = 0 \Rightarrow \delta_n = 0
$$

$$
\begin{aligned}
\Rightarrow A_n &= \frac{2}{L}\int_0^L Ax(L-x)\sin\frac{\pi n}{L}xdx \\
&= \frac{4AL^2}{n^3\pi^3}(1 - \cos n\pi)
\end{aligned}
$$

$$\boxed{y(x,t) = \sum_n A_n \sin(\tfrac{\pi}{L}nx)\cos\omega_n t \quad \text{with } A_n = \tfrac{4AL^2}{n^3\pi^3}(1-\cos n\pi)}$$

(b) **5 points**

$y(x,t) = \sum_n A_n \sin(\frac{\pi}{L}nx)\cos(\omega_n t + \delta_n)$, with $\boxed{\omega_n = \sqrt{\frac{T}{M/L}}\frac{n\pi}{L}}$

$$y(x,t)|_{t=0} = \sum_n A_n \sin(\frac{\pi}{L}nx)\cos(\omega_n t + \delta_n) = 0 \Rightarrow \delta_n = \pi/2$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = -\sum_n \omega_n A_n \sin(\frac{\pi}{L}nx)\sin(\omega_n t + \pi/2) = Bx(L-x)$$

$$\Rightarrow -\omega_n A_n = \frac{2}{L}\int_0^L Bx(L-x)\sin\frac{\pi n}{L}xdx$$

$$= \frac{4BL^2}{n^3\pi^3}(1-\cos n\pi)$$

$$\boxed{y(x,t) = \sum_n A_n \sin(\tfrac{\pi}{L}nx)\sin\omega_n t \quad \text{with } A_n = \tfrac{4BL^2}{n^3\pi^3\omega_n}(1-\cos n\pi)}$$