Phys C2601, Physics III: Classical and Quantum Waves Homework Assignment 4: Solutions

October 27, 2009

1 1,AM radio, 10 points

(a)

$$\begin{aligned}
V(t) &= A_{\text{mod}}(t)\cos(2\pi\nu_c t) \\
&= A_0\sin(2\pi\nu_c t)\cos(2\pi\nu_c t) + B_0\cos(2\pi\nu_c t) \\
&= \frac{1}{2}A_0\sin[2\pi(\nu_c+\nu_{\text{mod}})t] + \frac{1}{2}A_0\sin[2\pi(\nu_{\text{mod}}-\nu_c)t] \\
&\quad +B_0\cos(2\pi\nu_c t) \\
&= \frac{1}{2}A_0\sin[2\pi(\nu_c+\nu_{\text{mod}})t] - \frac{1}{2}A_0\sin[2\pi(\nu_c-\nu_{\text{mod}})t] \\
&\quad +B_0\cos(2\pi\nu_c t) \qquad (1)
\end{aligned}$$

Usually ν_{mod} is much smaller than ν_c, therefore $V(t)$ is a superposition of three frequencies: ν_c and side bands $\nu_c+\nu_{\text{mod}}$,$\nu_c-\nu_{\text{mod}}$.

(b)

We want $(\nu_c+\nu_{\text{mod}})-(\nu_c-\nu_{\text{mod}})$ =10kHz. Therefore $\nu_{\text{mod}}=5$kHz. The maximum frequency is ν_c+5kHz and the minimum is ν_c-5kHz. Note that a numerical value of ν_c is not given.

2 French 7-8, 10 points

(a)

The freqency is $(3\pi)/(2\pi) = 1.5$Hz.

(b)

Consider a general wave equation

$$y(x,t) = A\sin(kx - \omega t + \delta) \tag{2}$$

where $|k| = 2\pi/\lambda$ is the wavenumber, $\omega = 2\pi\nu$, and δ is some initial phase factor. Here only the absolute value of k is related to the wavelength; its sign will indicate the direction the wave travels and will be useful later. we know that (note that x has a unit of meter)

$$y(x=0,t) = A\sin(-\omega t + \delta) = 0.2\sin 3\pi t \tag{3}$$

$$y(x=1,t) = A\sin(k - \omega t + \delta) = 0.2\sin(3\pi t + \frac{\pi}{8}) \tag{4}$$

Obviously $A = 0.2$, $\omega = 3\pi$. For two sines to be equal, i.e. $\sin x = \sin y$, it requires either $x = y + 2n\pi$ or $x = \pi - y + 2n\pi = (2n+1)\pi - y$ where n is any integer. Here only the latter is possible. Therefore,

$$\delta = (2m+1)\pi \tag{5}$$

$$k + \delta = (2l+1)\pi - \frac{\pi}{8} \tag{6}$$

where m, l are any integers. Let $n = l - m$ (could also be arbitrary integer), we have:

$$k = (2n - \frac{1}{8})\pi \tag{7}$$

We immediately get the wavelength

$$\lambda = \frac{2\pi}{|k|} = \frac{2}{\left|2n - \frac{1}{8}\right|} = \frac{16}{|16n - 1|} \tag{8}$$

Note that here n is any integer (it could even be negative!).The unit is meter.

(c)

$$v = \lambda \cdot \nu = \frac{3}{2} \cdot \frac{16}{|16n-1|} = \frac{24}{|16n-1|} \tag{9}$$

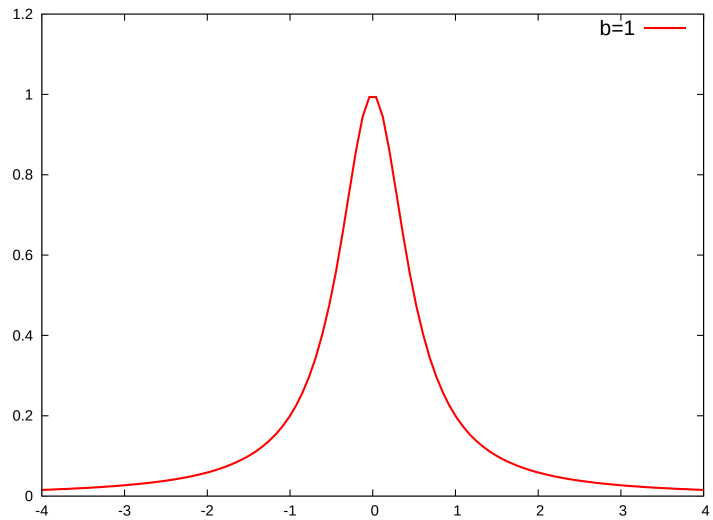

Figure 1: b=1.

(d)

As we can see in Eqn. (7) there is no reason for the wavenumber to have a definite sign. It could be either positive or negative, depending on the choice of the arbitraty integer n. Based on the information given here we cannot determine the direction the wave travels.

3 French 7-13, **10 points**

(a)

See Fig. 1

(b)

We write $y(x,t)$ as a function of $x \pm vt$ and we recognize that the speed of the pulse is $u/2$ and it travels in the $+x$ direction.

(c)

$$v_y(x) = \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} \tag{10}$$

$$= \frac{4b^3 x u}{[b^2 + 4x^2]^2} \tag{11}$$

Note that for $x > 0$, $v_y > 0$ and for $x < 0$, $v_y < 0$. So during a short time Δt the $x < 0$ part goes downward and the $x > 0$ part goes upward.

4 French 7-17, 10 points

(a)

$$\begin{aligned} y &= y_1 + y_2 && (12) \\ &= A[\sin(5x - 10t) + \sin(4x - 9t)] && (13) \end{aligned}$$

We want to find the low frequency modulator, so

$$\begin{aligned} y &= A\left[2\sin\frac{(5x-10t)+(4x-9t)}{2}\cos\frac{(5x-10t)-(4x-9t)}{2}\right] && (14) \\ &= 2A\sin\left(\frac{9}{2}x - \frac{19}{2}t\right)\cos\left(\frac{1}{2}x - \frac{1}{2}t\right) && (15) \end{aligned}$$

(b)

The group velocity is that of the low frequency of the modulator with $k = 1/2\text{m}^{-1}$, $\omega = 1/2\text{sec}^{-1}$. Therefore $v_g = \Delta\omega/\Delta k = 1\text{m/sec}$. Note the units are given.

(c)

There are a sine part and a cosine part in the combined disturbance who each has its own zeroes. When the modulation wave has zero amplitude, for the sine part, $x = \frac{2m}{9}\pi$; for the cosine part, $x = (2n+1)\pi$, m and n are any integer number. So the distance between the closest two zeroes is:

$$\text{minimum of } |\frac{2m}{9}\pi - (2n+1)\pi| = \frac{\pi}{9} \qquad (16)$$

5 French 7-19, 10 points

Due to the negative concavity of the curve, the phase velocities ($= 2\pi\nu/k$), which is the slope of the line connecting a given point and the origin, will be always larger than the group velocities ($= 2\pi\Delta\nu/\Delta k$), which is the slope of the tangent line drawn at that point.

6 Superposition of N components, **10 points**

(a)

Note that $\cos x = \text{Re}[\exp(ix)]$, we have

$$\begin{aligned}
y(t) &= A\sum_{n=0}^{N-1}\cos(w_1 + n\cdot\delta w)t && (17)\\
&= A\cdot\text{Re}\left\{\sum_{n=0}^{N-1}\exp[i(w_1 + n\cdot\delta w)t]\right\} && (18)\\
&= A\cdot\text{Re}\left\{\exp[iw_1 t]\frac{1-\exp[iN\delta wt]}{1-\exp[i\delta wt]}\right\} && (19)\\
&= A\cdot\text{Re}\left\{\exp[iw_1 t]\frac{\exp[-i\frac{\delta w}{2}t]}{\exp[-iN\frac{\delta w}{2}t]}\frac{\exp[-iN\frac{\delta w}{2}t]-\exp[iN\frac{\delta w}{2}t]}{\exp[-i\frac{\delta w}{2}t]-\exp[i\frac{\delta w}{2}t]}\right\} && (20)\\
&= A\cdot\text{Re}\left\{\exp[i(w_1+\frac{N-1}{2}\delta w)t]\frac{\sin[N\frac{\delta w}{2}t]}{\sin[\frac{\delta w}{2}t]}\right\} && (21)\\
&= A\frac{\sin[N\frac{\delta w}{2}t]}{\sin[\frac{\delta w}{2}t]}\cdot\text{Re}\left\{\exp[i(w_1+\frac{N-1}{2}\delta w)t]\right\} && (22)\\
&= A\cos[w_{\text{ave}}t]\frac{\sin[N\frac{\delta w}{2}t]}{\sin[\frac{\delta w}{2}t]} && (23)
\end{aligned}$$

6.1 (b)

See Fig. 2.

Zeroes of $\sin[N\delta wt/2]$:

$$N\frac{\delta w}{2}t = n\pi \tag{24}$$

$$t = \frac{2n\pi}{N\delta w} \tag{25}$$

where n is any integer.

Therefore

$$\Delta t = \frac{2\pi}{N\delta w} \tag{26}$$

$$\Delta w\cdot\Delta t = (N-1)\delta w\cdot\frac{2\pi}{N\delta w} \approx 2\pi. \tag{27}$$

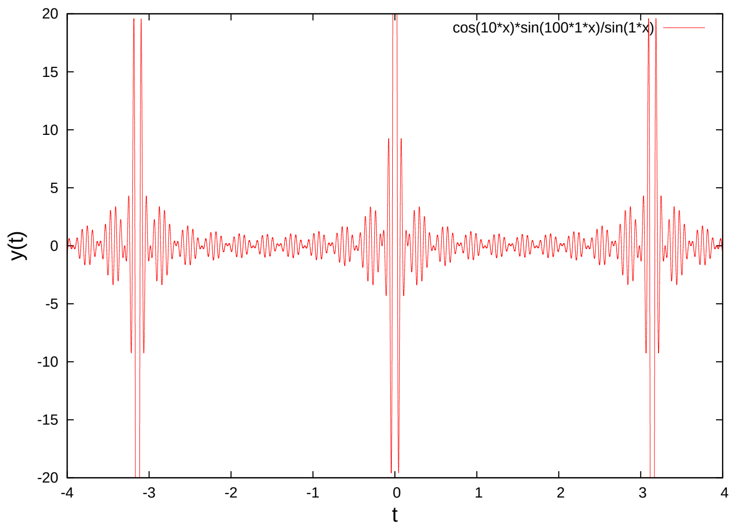

Figure 2: $y(t)$ at $w_{\text{ave}} = 10$, $\delta w = 2$, N=100.