Phys C2601, Physics III: Classical and Quantum Waves Homework Assignment 5: Solutions

October 24, 2009

1 French 7-20

(a)

Denote the area of the cross section by A. The kinetic energy is given by

$$E_k = \frac{1}{2}\rho A l \left(\frac{dy}{dt}\right)^2 \tag{1}$$

The potential energy is given by

$$U = (\rho A y) g y \tag{2}$$

The total energy is given by

$$E_{\text{tot}} = E_k + U = \frac{1}{2}\rho A l \left(\frac{dy}{dt}\right)^2 + (\rho A g) y^2 \tag{3}$$

Compare this to the general form:

$$E_{\text{tot}} = \frac{1}{2} m v^2 + \frac{1}{2} K x^2 \tag{4}$$

we realize that $m = \rho A l$, $K = 2\rho A g$. The period of oscillation is given by

$$T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{l}{2g}} = \pi\sqrt{\frac{2l}{g}} \tag{5}$$

And the angular frequency is

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{2g}{l}} \tag{6}$$

(b)

We shall consider both the phase velocity and the group velocity. Roughly we expect they have similar order of magnitude. Using $\lambda \sim 2l$ and the wave number $k = 2\pi/\lambda$ (not to be confused with K in the previous subsection), we have the phase velocity:

$$v_p = \frac{\omega}{k} = \sqrt{\frac{2g}{l}}\frac{\lambda}{2\pi} \sim \sqrt{4g}\frac{\sqrt{\lambda}}{2\pi} = \frac{(g\lambda)^{1/2}}{\pi} \tag{7}$$

The group velocity:

$$v_g = \frac{d\omega}{dk} \sim \frac{d\left(\sqrt{\frac{4g}{\lambda}}\right)}{d\left(\frac{2\pi}{\lambda}\right)} = \frac{-\frac{1}{2}\sqrt{4g}\lambda^{-\frac{3}{2}}d\lambda}{-2\pi\lambda^{-2}d\lambda} = \frac{\sqrt{g\lambda}}{2\pi} = \frac{1}{2}v_p \tag{8}$$

(c)

$$v = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8m \cdot s^{-2} \cdot 500m}{2\pi}} = 27.93m/s \tag{9}$$

2 French 7-23

We assume the wave is described by $y(x,t)$. At the point $x = 0$, we know its speed, which is $\partial y(x=0,t)/\partial t$. This information is shown in Fig. 1(a). Integrating over t, we get $y(x=0,t)$, which is shown in Fig. 1(b).

It is obvious that the wave goes to the $+x$ direction with a speed v. Therefore we shall note $y(x,t) = f(x - vt)$. Defining $\alpha = x - vt$, then we know $f(\alpha)$ from the information of $y(0,t)$. These informations are shown in Fig. 1(c) and (d). We further denote the mass density by μ and the tension of the string by T. Note that $v = \sqrt{T/\mu}$.

The kinetic energy density is given by

$$\frac{dE_k}{dx} = \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 = \frac{\mu}{2}\left(\frac{\partial f(x-vt)}{\partial t}\right)^2 = \frac{\mu}{2}v^2\left(f'(\alpha)\right)^2 = \frac{T}{2}\left(f'(\alpha)\right)^2 \tag{10}$$

The potential energy density is given by

$$\frac{dU}{dx} = \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 = \frac{T}{2}\left(\frac{\partial f(x-vt)}{\partial x}\right)^2 = \frac{T}{2}\left(f'(\alpha)\right)^2 \tag{11}$$

The total energy is:

$$E_{\text{tot}} = \left(\frac{T}{2} + \frac{T}{2}\right)\int dx\left(f'(\alpha)\right)^2 = T\int d\alpha\left(f'(\alpha)\right)^2 \tag{12}$$

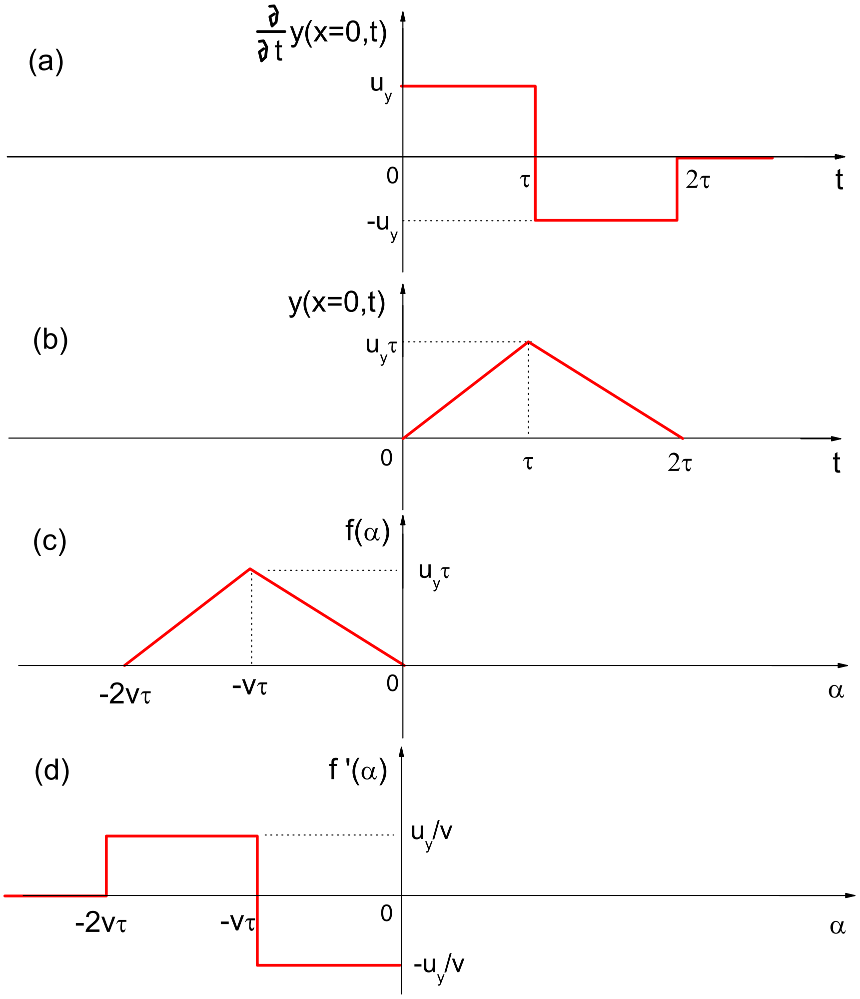

Figure 1: French 7-23

Looking at Fig. 1(d) we may evaluate the integral as:

$$E_{\text{tot}} = T \int_{-2v\tau}^{0} d\alpha \left(\frac{u_y}{v}\right)^2 = 2Tv\tau\frac{u_y^2}{v^2} = 2\mu v\tau u_y^2 \tag{13}$$

Now let us calculate the work done by the transverse force:

$$W = \int \vec{F} \cdot \vec{u_y} dt = -\frac{1}{v} \int \vec{F} \cdot \vec{u_y} d\alpha \tag{14}$$

Here $\vec{F}$ is expressed as

$$F = T\sin\theta \approx T\tan\theta = T\frac{\partial y}{\partial x} = Tf'(\alpha) \tag{15}$$

Although $f'(\alpha)$ changes sign at $\alpha = -v\tau$, $\vec{u_y}$ also changes sign at the same time. This leads to

$$\begin{aligned} W =& \frac{1}{v}\int_{-2v\tau}^{0} T|f'(\alpha)| \cdot u_y d\alpha \\ =& \frac{1}{v} \cdot 2v\tau T\frac{u_y^2}{v} \\ =& 2\mu v\tau u_y^2 \end{aligned} \tag{16}$$

in agreement with Eq. (13)

3 French 7-25

From (7-43) in the text book: we have the general wave equation as

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(r,\theta,\phi) \tag{17}$$

In spherical coordinates the Laplacian can be written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{18}$$

We are dealing with spherically symmetric case, thus the latter two term (which includes θ and ϕ) can be neglected. We have:

$$\nabla^2\psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r)}{\partial r}\right) = \frac{\partial^2}{\partial r^2}\psi(r) + \frac{2}{r}\frac{\partial}{\partial r}\psi(r) \tag{19}$$

Thus the wave equation reads

$$\frac{\partial^2}{\partial r^2}\psi + \frac{2}{r}\frac{\partial}{\partial r}\psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \tag{20}$$

which is presicely (7-44) in the textbook. A simple harmonic wave whose amplitudes fall off inversely with r can be written, in general, as

$$\psi = C\frac{\sin(kr - \omega t + \delta)}{r} \tag{21}$$

We want to verify that this solves Eq. (20) or (19).

$$\frac{\partial}{\partial r}\psi = \frac{C}{r^2}[kr\cos(kr - \omega t + \delta) - \sin(kr - \omega t + \delta)] \tag{22}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) =& C\frac{\partial}{\partial r}[kr\cos(kr - \omega t + \delta) - \sin(kr - \omega t + \delta)] \\ =& C[k\cos(kr - \omega t + \delta) - k^2 r\sin(kr - \omega t + \delta) - k\cos(kr - \omega t + \delta)] \\ =& - Ck^2 r\sin(kr - \omega t + \delta)\end{aligned} \tag{23}$$

Therefore the left hand side of the wave equarion is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) = -Ck^2\frac{\sin(kr - \omega t + \delta)}{r} \tag{24}$$

The right hand side of the wave equation is

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi = -C\frac{\omega^2}{v^2}\frac{\sin(kr - \omega t + \delta)}{r} \tag{25}$$

These two shall be equal given $v = \omega/k$. The wave equation is satisfied.

4 Two wires

(a)

$$v_1 = \sqrt{\frac{\tau}{\mu}}, \quad v_2 = \sqrt{\frac{\tau}{2\mu}} = \frac{v_1}{\sqrt{2}} \tag{26}$$

(b)

Waves in wire1 and wire2 can be described, respectively, as

$$\begin{aligned} y_I &= A_{in}e^{ikx-i\omega t} + A_r e^{-ikx-i\omega t} && (27) \\ y_{II} &= A_t e^{ik'x-i\omega t} && (28) \end{aligned}$$

Note that $v = \omega/k$, therefore $k'/k = v_1/v_2 = \sqrt{2}$

The boundary condition at x=0 (the junction point) is

$$\begin{aligned} y_I(x=0,t) &= y_{II}(x=0,t) && (29) \\ \frac{\partial}{\partial x}y_I(x=0,t) &= \frac{\partial}{\partial x}y_{II}(x=0,t) && (30) \end{aligned}$$

Then we have

$$\begin{aligned} A_{in} + A_r &= A_t && (31) \\ kA_{in} - kA_r &= k'A_t && (32) \end{aligned}$$

Eq. (32) can be rewritten as

$$A_{in} - A_r = \sqrt{2}A_t \tag{33}$$

Simple algebra gives

$$\frac{A_t}{A_{in}} = \frac{2}{1+\sqrt{2}} = 2(\sqrt{2}-1) \tag{34}$$

and

$$\frac{A_r}{A_{in}} = \frac{1-\sqrt{2}}{1+\sqrt{2}} = 2\sqrt{2}-3 \tag{35}$$

(c)

The power can be computed as

$$P = \frac{\mu}{2}(\omega A)^2 v \tag{36}$$

Therefore

$$P_{in} = \frac{\mu}{2}(\omega A_{in})^2 v_1 = \frac{\sqrt{\mu\tau}}{2}(\omega A_{in})^2 \tag{37}$$

$$P_r = \frac{\mu}{2}(\omega A_r)^2 v_1 = \frac{\sqrt{\mu\tau}}{2}(\omega A_{in})^2(2\sqrt{2}-3)^2 \tag{38}$$

$$P_t = \frac{2\mu}{2}(\omega A_t)^2 v_2 = \frac{\sqrt{\mu\tau}}{2}(\omega A_{in})^2\sqrt{2}(2\sqrt{2}-2)^2 \tag{39}$$

It is easily verified that $(2\sqrt{2}-3)^2 + \sqrt{2}(2\sqrt{2}-2)^2 = 1$. Therefore

$$P_t + P_r = P_{in} \tag{40}$$

5 French 8-8

(a)

We first convert 60mi/hr to m/s:

$$60mile/hour = 60 \times 1.609344 \times 10^3 m/(3600s) = 26.8224m/s \tag{41}$$

and we know that the speed of sound at sea level is 340.29m/s.

We use the formula of classical Doppler shift:

$$f' = \frac{c \pm u_r}{c \pm u_s} f \tag{42}$$

Here $u_r = 0$. The police car first approaches the bystander and then goes away. The overall change of frequency is:

$$\begin{aligned}\Delta f =& \frac{c}{c - u_s} f - \frac{c}{c + u_s} f \\ =& \left(\frac{340.29}{340.29 - 26.822} - \frac{340.29}{340.29 + 26.822} \right) \times 2000Hz \\ =& 317.26Hz \approx 320Hz\end{aligned} \tag{43}$$

(b)

After the police car passed, the bystander hear a frequency which becomes lower and lower. However, reflection from a wall in the far end is just like an image of another car approaching the wall on the otherside, symmetric to the actually police car. This source is approaching the bystander thus the corresponding frequency in higher. The bystander should hear a lower and a higher frequencies. If the sound are coherent, there would be "beats" modulated by difference of two frequencies, but in practice this is almost never the case. Therefore the answer is that the bystander can identify two frequencies with one lower and the other higher.

6 Relativistic and classical Doppler shift

(a)

First note that in the formula of relativistic Dopler shift, u is the *relative* speed between the source and the receiver, thus equivalent to $u_r - u_s$ in the classical formula.

In the relativistic case:

$$\begin{aligned}\frac{\Delta f}{f} =&[1 \pm \frac{u}{c}]^{\frac{1}{2}}[1 \mp \frac{u}{c}]^{-\frac{1}{2}} - 1\\ =&[1 \pm \frac{u}{2c} + O\left[\left(\frac{u}{c}\right)^2\right]] \cdot [1 \pm \frac{u}{2c} + O\left[\left(\frac{u}{c}\right)^2\right]] - 1\\ =&\pm \frac{u}{c} + O\left[\left(\frac{u}{c}\right)^2\right]\end{aligned} \tag{44}$$

In the classical case

$$\begin{aligned}\frac{\Delta f}{f} =&[1 \pm \frac{u_r}{c}][1 \pm \frac{u_s}{c}]^{-1} - 1\\ =&[1 \pm \frac{u_r}{c}] \cdot [1 \mp \frac{u_s}{c} + O\left[\left(\frac{u_s}{c}\right)^2\right]] - 1\\ =&\pm \frac{u_r - u_s}{c} + O\left[\left(\frac{u_s}{c}\right)^2, \left(\frac{u_r u_s}{c^2}\right)\right]]\\ =&\pm \frac{u}{c} + O\left[\left(\frac{u}{c}\right)^2\right]\end{aligned} \tag{45}$$

In the last step we make use of $u = u_r - u_s$

(b)

The massive, dark object can be regarded as static. The light star goes around the massive star with a period of 18 hours. What we need to know is the speed u of the light star. When the speed u is pointing towards us, we have a positive frequency shift; and when it is pointing away from us, we have a negative frequency shift. Therefore

$$\frac{\Delta f}{f} = \frac{563 - 539}{0.5 \times (563 + 539)} = 0.043557 = \frac{2u}{c} \tag{46}$$

$u = 0.5 \times 0.043557 \times 3 \times 10^8 m/s = 6.5336 \times 10^6 m/s$

$$T = \frac{2\pi}{\omega} = \frac{2\pi R}{u} \tag{47}$$

$$R = 6.5336\times 10^6 m/s \times 18hr/(2\pi) = \frac{6.5336 \times 10^6 \times 18 \times 3600m}{2\pi} = 6.74\times 10^{10} m \tag{48}$$