Chapter 13

1.4 Write down the extensive form of the game above. How many subgames are there in this game? How many strategies does Coke have?

Answer:

Extensive form given by Figure 5. Single subgame - the entire game. Coke’s strategies are ET, EA, OT, and OA.

2.3 Finally, identify a part of the tree that is not a subgame because it does not contain all the nodes in some information set although it contains some.

Answer: The part of Figure 4 starting at node α.

4.4 What must be the votes cast in the Senate by i) a Democrat, and ii) a Republican if neither plays a dominated strategy?
Answer: The dominant strategy Nash equilibrium voting in the Senate is truthful voting - both parties vote Version 2 over the status quo but only Republicans vote for Version 1.

4.10 What is the subgame perfect equilibrium of this model? Does firm 2 enter the market?

Answer: After Firm 2 enters the market, there is only one possible Nash equilibrium play - with prices (1, 1) and profits $\frac{5}{2}, \frac{5}{2}$. If the costs to entering the market are greater than $\frac{5}{2}$ then firm 2 will not enter and otherwise it will.

Chapter 15

2.1 Consider the infinitely repeated Prisoners Dilemma. Show that the strategy that plays (c, c) in every subgame - regardless of past behavior - is a subgame perfect equilibrium.

Answer: Suppose that player 1 plays n instead at some stage. He loses 2 as a consequence in that stage. Furthermore, he does not influence player 2 after that stage in any way; player 2 continues to play c thereafter. Hence, there is no benefit in the future to playing n in that stage. In other words, each player might as well treat each stage as if it is a one-shot game. The only Nash equilibrium in the stage therefore is to play (c, c).

3.1 Write down the extensive form of the game. Identify the subgames.

Answer: The tree is an infinite unfolding of the Prisoners Dilemma (see Figure 3, Chapter 17 in the text). There are an infinite number of stages to this repeated game. At each of these stages starts a number of subgames; the number of subgames starting at stage 18 is equal to the number of different ways that the first 17 stages might have got played. Each of these subgames must include all of the tree that follows after the start of this subgame.

3.4 Show that there is also a subgame perfect equilibrium in which the price is always 2 dollars but which is sustained by a forgiving trigger. Be explicit about the nature of the forgiving trigger.

Answer: The forgiving trigger strategy is as follows: start by pricing at (2, 2) and continue doing so if each firm has priced in that fashion in the past. The first time that either firm posts a price other than 2, switch to pricing at 1 for the next T periods. Stay at that price of 1 for the T periods regardless of what your rival does during those periods. After T periods of pricing at 1, switch back to 2, i.e., restart the whole procedure: start by pricing at (2, 2) and continue doing so if each firm has priced in that fashion in the past. The first time that either firm posts a price other than 2, switch to pricing at 1 for the next T periods....
any price other than 1. That will only result in a zero profit in that stage (whereas they make some positive profits at a price of 1) and will have no effect on the length of the trigger.

In a non-trigger phase, the payoffs to undercutting a price of 2 is:

\[
5 + \frac{5}{2} \delta + \frac{5}{2} \delta^2 + \frac{5}{2} \delta^3 + \ldots + \frac{5}{2} \delta^T + \delta^{T+1} \cdot \frac{4}{1-\delta}
\]

\[
= 5 + \frac{5}{2} \frac{\delta}{1-\delta} + \delta^{T+1} \cdot \frac{4}{1-\delta}
\]

where I have used the fact that the monopoly profits at a price of 1 is 5 whereas at a price of 2 is 8 and the fact that \(\delta + \delta^2 + \delta^3 + \ldots + \delta^T = \frac{\delta}{1-\delta} \). On the other hand, staying with a price of 2 yields a payoff of

\[
\frac{4}{1-\delta}
\]

Comparing the two payoffs it is beneficial to stay with the strategy if

\[
\frac{4}{1-\delta} > 5 + \frac{5}{2} \frac{\delta}{1-\delta} + \delta^{T+1} \cdot \frac{4}{1-\delta}
\]

or

\[
\frac{4}{1-\delta} (1 - \delta^{T+1}) > 5 + \frac{5}{2} \frac{\delta}{1-\delta}
\]

The last equation is satisfied when \(\delta \) is close to 1 and \(T\) is large because the left-hand side (by L’Hospital’s rule - see text) goes to \(4(T+1)\) whereas the right-hand side goes to \(5 + \frac{5}{2} T\) (as \(\delta\) goes to 1).

3.7 Provide an argument in support of your stated result (in the previous question).

Answer: Mimic the remarks given at the end of the proof of the Folk Theorem for the Prisoners Dilemma in the text.

Chapter 16

2.2 For the specification of demand and supply given by the previous question, what is the collusive ask price? What about the collusive bid price? Hence, what is the collusive spread? Do collusive market-makers sell as much as they buy? (Continue to assume that the market-clearing price is the value of the share.)

Answer: The market-makers profits are

\[(b - a) \times (140 - 5a)\]

where I have used the fact that the quantity demanded will be equal to quantity supplied. In fact that equality implies that

\[140 - 5a = -60 + 5b\]
i.e., \( b = 40 - a \). Substituting the profits become

\[
(40 - 2a) \times (140 - 5a)
\]

Marginal profits are zero if

\[
20a = 480
\]

or \( a = 24 \). Hence \( b = 16 \) and the spread is 8.

3.1 Show that an ask and a bid of 20 is a stage-game Nash equilibrium.

*Answer:* The market-clearing price of 20 is a Nash equilibrium; no dealer would like to undercut this price since they will either buy above value or sell above value (\( = 20 \)) by doing so. Any prices above 20 for an ask or below 20 for a bid is ineffective since it does not bring in any orders.

3.5 Consider, finally, the forgiving trigger strategy: trade at the collusive quote unless some dealer undercuts. Thereafter trade at the market-clearing price for \( T \) stages. After that revert to the collusive quotes. When is this strategy an equilibrium?

*Answer:* Trading according to this strategy yields \( \frac{40}{N(1-\delta)} \). Deviating yields \( 30 + \frac{40\delta^{T+1}}{N(1-\delta)} \). This strategy is an equilibrium if

\[
\frac{40}{N(1-\delta)}(1 - \delta^{T+1}) \geq 30
\]

4.2 Can you show that the same conclusion is true regardless of what demand and supply function we consider?

*Answer:* Consider any supply and demand functions. Denote the collusive profits as \( \Pi \). Hence each dealer gets \( \frac{\Pi}{N} \). Suppose that deviating against the collusive price yields a dealer a profit equal to \( \frac{\pi}{N} \) (shared because there is perfect order preferencing), where, by definition, \( \pi < \Pi \). Not deviating is clearly better.

**Chapter 17**

2.1 Write down the payoff matrix for *good* demand periods.

*Answer:* For these prices, the payoff matrix is:

<table>
<thead>
<tr>
<th>( S )</th>
<th>( A )</th>
<th>( V )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_H )</td>
<td>130, 91</td>
<td>160, 80</td>
<td></td>
</tr>
<tr>
<td>( q_L )</td>
<td>128, 112</td>
<td>152, 95</td>
<td></td>
</tr>
</tbody>
</table>

2.3 Suppose that *good* demand is expected to persist forever. For what values of discount factors does a low production level become an equilibrium?
Answer: For SA the total discounted profits from producing \((Q_L, q_L)\) are \(\frac{152}{1-\delta}\). On the other hand, the profits from expanding output to \(Q_H\) - and then \((Q_H, q_H)\) thereafter - are:

\[
160 + \frac{130\delta}{1-\delta}
\]

Evidently withholding production is better if

\[
152 + \frac{152\delta}{1-\delta} > 160 + \frac{130\delta}{1-\delta}
\]

i.e., if \(\frac{44\delta}{1-\delta} > 16\).

For the smaller producer VA the total discounted profits from producing \((Q_L, q_L)\) are \(\frac{95}{1-\delta}\). On the other hand, the profits from expanding output to \(q_H\) - and then \((Q_H, q_H)\) thereafter - are:

\[
112 + \frac{91\delta}{1-\delta}
\]

Evidently withholding production is better if

\[
95 + \frac{95\delta}{1-\delta} > 112 + \frac{91\delta}{1-\delta}
\]

i.e., if \(\frac{44\delta}{1-\delta} > 187\). It is immediate that if VA will not cheat then neither will SA.

3.2 How would your answer be any different if OPEC used the forgiving trigger instead - and chose to overproduce for \(T\) periods only? Explain.

Answer: Starting in a good period, for the smaller producer VA the total discounted profits from producing \((Q_L, q_L)\) in good periods and \((Q_H, q_H)\) in bad periods are \(95 + \frac{95p + 55(1-p)\delta}{1-\delta}\). On the other hand, the profits from expanding output in that period to \(q_H\) - and then \((Q_H, q_H)\) for \(T\) periods - are:

\[
112 + \frac{91p + 55(1-p)(1-\delta^T)\delta + 95p + 55(1-p)\delta^{T+1}}{1-\delta}
\]

Evidently withholding production is better if

\[
95 + \frac{95\delta p}{1-\delta}(1-\delta^T) > 112 + \frac{91\delta p}{1-\delta}(1-\delta^T)
\]

Check that if VA will not cheat then neither will SA in a good demand period. (And that in bad demand periods neither player has an incentive to cheat.)