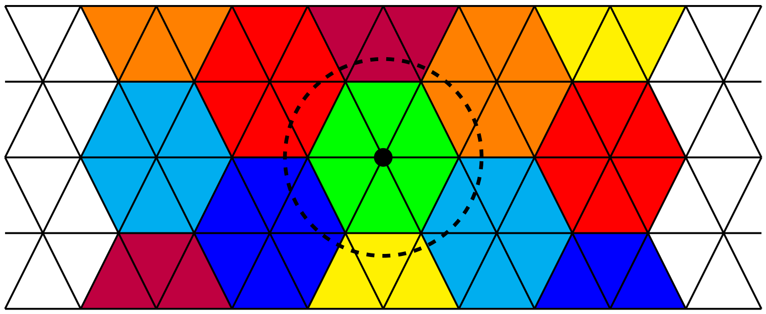
Class Eight: Infinite Graphs



Nothing in the definition of a graph prevented us from considering graphs with an infinite number of vertices. Most of the concepts and ideas we examined with respect to finite graphs have an interesting infinite analogue. We are most interested in graph with vertex set $\mathbb{N} = \{1, 2, 3, ...\}$ and refer to such graphs as **countable**. As before we define the degree of a vertex to be the of vertices adjacent to it, but now we allow for the possibility that there are an infinite such neighbors. If every vertex in a graph on $\mathbb{N}$ has finite degree, then we call the graph **locally finite**. So that we stay in the world of graph theory we mostly examine countable graphs which are locally finite. Lastly, before we begin we recall two fundamental ideas from set theory.

- Removing finitely many points from an infinite set yields an infinite set. This is sometimes know as the infinite pigeonhole principle. Think why...
- A finite or countable union of countable sets is countable. This implies that a countable graph has a countable number of edges.

Draw some "small" graphs and think about the following questions:

How many edges can a countable graphs have?

Describe all countable graphs such that every vertex has degree one.

Describe all countable graphs such that every vertex has degree two.

Describe a countable graph such that every vertex has either degree one or three.

Give a countable graph with a vertex of every degree.
Give a countable graph with a vertex of every degree and no cycles.
If possible describe a countable graph with a unique vertex of every degree.

Give a countable graph such that every vertex has degree three.
If possible describe one which is acyclic and one which is not.

Give a countable graph such that every vertex has degree four.

Give a countable graph such that every vertex has degree six.

What should it mean for a countable graph to be connected?
What should it mean for a locally finite countable graph to be connected?
How connected can a countable graph be?

Give a connected countable graph which contains paths of all length.
Give a connected locally finite countable graph which contains paths of all length.

What does taking the complement of a locally finite countable graph yield?

Given a countable graph, how would you build a self-complementary countable graph?

What should it mean for a infinite graph to be a tree?
Consider a locally finite tree, what can you say about leaves and rays?
What about a countable tree which is not locally finite?

Give an example of a countable graph such that the chromatic number of any subgraph is at most two. Can we say anything about the chromatic number of the entire graph?
What more general result does this suggest?

Lemma (König)**.** *Every locally finite tree contains a ray.*

Lemma (de Bruijn, Erdős)**.** *If every finite subgraph of a countable graph G has chromatic number at most k, then so does G.*

Lemma (Rado)**.** *With high probability, other than the infinite clique and its complement, up to isomorphism there is an unique countable graph.*

We say a graph is a unit distance graph if it can be drawn in the plane so that every edge is a straight line with length one.

Determine which graphs in the following classes are unit distance graphs:
Paths, Stars, Trees, Cycles, Bipartite, Cliques, Wheels, Cube, Grid, Petersen graph.

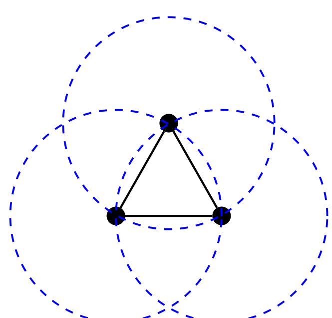
Lemma (Erdős,Chvátal)**.** *Any graph containing K_4 or $K_{2,3}$ is not an unit distance graphs.*

Consider the following uncountable graph. Let the vertices be all points in the plane $\mathbb{R}^2$ and make two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ adjacent if and only if the distance between $\mathbf{x}$ and $\mathbf{y}$ is equal to one. The following is known as the Unit Distance graph problem.

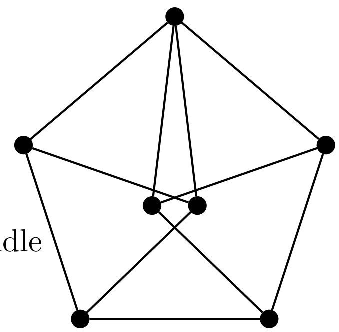
What is the chromatic number of this graph?

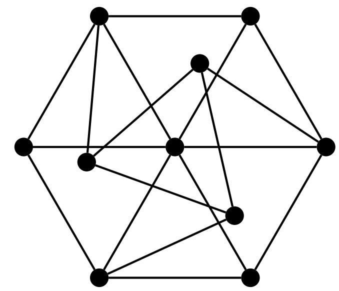
Using finite graphs we see that the chromatic number is at least four. Tiling the plane with sufficiently small hexagons, like at the start of this notes, it follows that the chromatic number is at most seven. Beyond this little progress has be made resolving this question.

Lemma. *The subgraph of $\mathbb{R}^2$ gotten by restricting to rational vertices $\mathbb{Q}^2$ is 2-colorable.*



Moser spindle

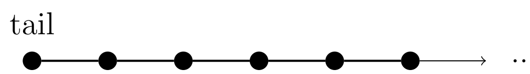




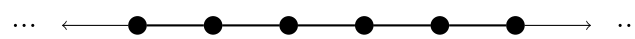
Golomb Graph

Finite graph examples as to why $\chi(\mathbb{R}^2) \geq 4$.

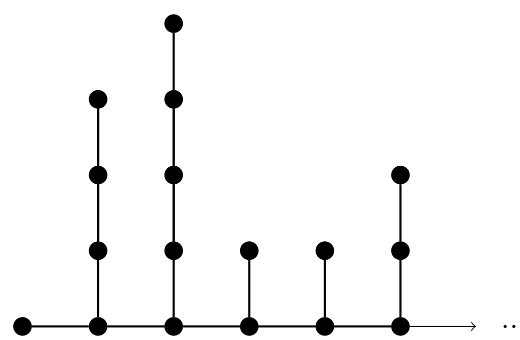
ray



double-ray



comb



ladder