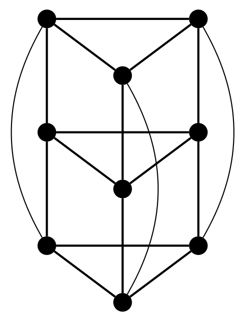
Class Eleven: Excluding Induced Subgraphs



An **induced subgraph** of a graph is any graph which can be obtained by deleting some vertices. Note this is a more restrictive notion of containment than when we consider subgraphs, where we were allowed to delete both vertices and edges. If a graph G has no induced subgraph isomorphic to a graph H, then we say that G is H-**free**. If G is not H-free, we say that G **contains** H. For example, the above graph contains many triangles, i.e. C_3, but is K_4-free. A **split graph** is any graph which can be completely divided into two parts, one of which is a clique and the other stable.

Draw some small graphs and think about the following questions:

What are the induced subgraphs of K_4?

What are the induced subgraphs of a complete graph?
How do things change if we instead consider subgraphs of a complete graph?

When is K_m an induced subgraphs of K_n?

When is $K_{m,n}$ an induced subgraphs of $K_{s,t}$?

When is C_k an induced subgraph of C_ℓ?

Which cycles are induced subgraphs of the Petersen graph?

Which stars are induced subgraphs of wheels?

Describe all P_2-free graphs. What about all P_2^c-free graphs?

Describe all P_3-free graphs. What about all P_3^c-free graphs?

Can you characterize forests in terms of excluding a family of induced subgraphs?

Can you characterize bipartite graphs in terms of excluding a family of induced subgraphs?

Describe all $\{C_3, paw\}$-free graphs.

Describe all $\{C_3, P_4\}$-free graphs. Hint: Are they bipartite?

Describe all P_4-free graphs. Need they always be connected or anticonnected?

Is the complement of a bipartite graph a bipartite graph?
Is the complement of a split graph a split graph?

Try and find three small graphs which are not split.

Lemma. *A graph is a forest if and only if it is $\{C_3, C_4, C_5, ...\}$-free, i.e. contains no cycle.*

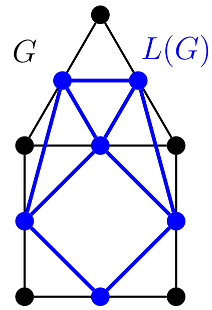
Theorem. *A graph is a bipartite if and only if it is $\{C_3, C_5, C_7, ...\}$-free, i.e. contains no odd cycle.*

Theorem. *A graph is a split graph if and only if it is $\{C_4, C_4^c, C_5\}$-free.*

Theorem. *A graph is a threshold graph if and only if it is $\{C_4, C_4^c, P_4\}$-free.*

The **line graph** of an undirected graph G is another graph $L(G)$ which encodes the adjacencies between edges of G. More specifically, given a graph G, its line graph $L(G)$ is a graph such that

- each vertex of $L(G)$ represents an edge of G, and
- two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G.



Draw some small graphs and think about the following questions:

What is the line graph of the paw? Diamond? K_4?

What is the line graph of a cycle?

What is the line graph of a star?

Can different graphs have the same line graph?

Is there relation between vertex-coloring $L(G)$ and edge-coloring G?

Can a line graph contain a claw?

What is the line graph of $K_{3,3}$? Take the complement. Notice anything?

What is the line graph of K_5? Take the complement. Notice anything?

Lemma. *A graph is isomorphic to its line graph if and only if it is a cycle.*

Theorem (Whitney)**.** *Two connected graphs are isomorphic if and only if their line graphs are isomorphic. (Only Exceptions are the triangle and claw.)*

Theorem. *A graph G is a line graph of a triangle-free graph if and only if G is both claw and diamond-free.*

Theorem. *A graph is a line graph if and only if it does not contain one of the following nine graphs as an induced subgraph.*