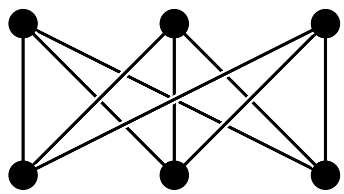
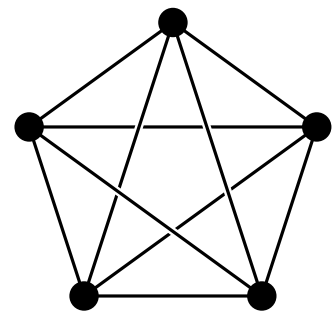
Class Five: Embeddings

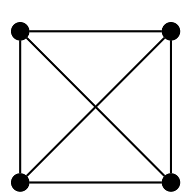


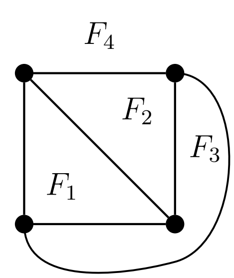
$K_{3,3}$



K_5

It many applications of graph theory it is important to determine how one can draw a particular graph with as few edges overlapping as possible. For example, consider the problem of designing a microchip or building a subway system, overlapping edges here either lead to disfunction or are very costly. A **planar graph** is a graph which can be drawn in the plane without any edges crossing. We refer to a specific drawing of a graph as an **embedding**. A drawing of a graph such that no two edges cross is called a **planar embedding**. The **crossing number** of a graph is the smallest possible number of edge crossings when considering all possible drawings of the graph. Hence, a graph is planar if and only if it has crossing number zero. Below are two different embeddings of the complete graph on four vertices, one with crossing number one and one with crossing number zero. Note, this shows that complete graph on four vertices is indeed a planar graph. When a graph is drawn with no crossing edges, it divides the plane into a set of regions, which we call **faces** and denote by f. This idea follows from something called the Jordan Curve theorem. For example, the planar embedding below results in four regions/faces, that is, $f = 4$.





Draw some small graphs and think about the following questions:

Draw regular planar graphs of degree $d = 0, 1, 2, 3, 4, 5$.

What happens when you delete a vertex from a planar graph?

What happens when you delete an edge from a planar graph?

Are all graphs planar? In particular, is the complete graph on 5 vertices planar?

What is the crossing number of the complete graph on five vertices?
What about the complete graph on five vertices minus an edge?

What is the crossing number of the complete graph on six vertices?
How about seven vertices?

Consider a cycle with some additional chordal edges.
When is such a graph not planar? Try and make as small an example as possible.

A planar graph is called **maximal planar** if the addition of one more edge yields a non-planar graph.

What does a face of a maximal planar graph look like?

How many edges and faces, does a connected maximal planar graph have?

If a graph is planar, is there a planar embedding in which every edge is a straight line segment?

How many faces does a cycle have?

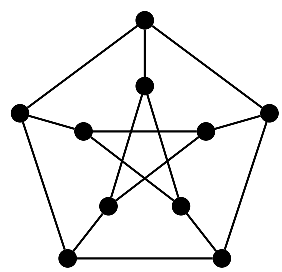
How many faces does a path have?

How many faces does a tree have?

If two planar graphs have the same number of vertices, edges and faces,
need they be isomorphic?
What about two planar graphs with the same degree sequence and number of faces?

For several planar graphs determine the number of faces.
Can different embeddings yield a different number of faces?
Do you notice a pattern?

Is the Petersen graph drawn below planar?



Lemma. *The complete graph on 5 vertices is non-planar, yet deleting any edge yields a planar graph.*

Theorem (Guy's Conjecture)**.**

$$cr(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor.$$

Theorem (Fáry, Wagner)**.** *Every planar graph has a planar embedding in which every edge is a straight line segment.*

Theorem. *Every planar graph is a tangency graph of circles in the plane.*

Theorem (Jordan Curve Theorem)**.** *Any continuous simple closed curve in the plane, separates the plane into two disjoint regions, the inside and the outside.*

Theorem (Euler)**.** *If G is a connected planar graph on v vertices with e edges and f faces, then*

$$v - e + f = 2.$$

Lemma. *The complete bipartite graph $K_{3,3}$ is non-planar, yet deleting any edge yields a planar graph.*

Theorem (Kuratowski)**.** *A graph is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5.*

Using Euler's formula give an upper bound for the number of edges in a connected planar graph on at least three vertices.

Using Euler's formula give an upper bound for the number of edges in a connected planar graph on at least three vertices that contains no triangles.

The following two results show that both K_5 and $K_{3,3}$ are non-planar.

Lemma. *If G is a connected planar graph on at least three vertices, then*

$$|E(G)| \leq 3|V(G)| - 6.$$

Lemma. *If G is a connected planar graph on at least three vertices that contains no triangles, then*

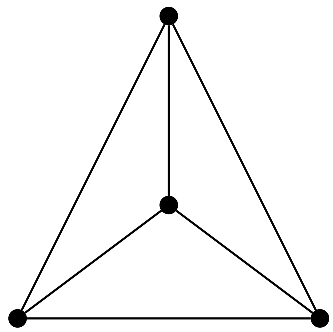
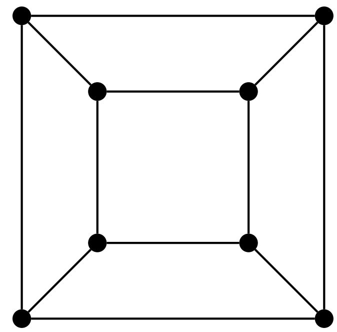
$$|E(G)| \leq 2|V(G)| - 4.$$

The question of whether one is able to draw a graph without crossing edges can be asked with respect to other surfaces...

Draw the following graphs on the surface of a donut (torus) without crossing edges.
The complete bipartite graph $K_{3,3}$.
The complete graph K_5.
The complete graph K_6
The complete graph K_7.

When dealing with the embedding question on the torus, there is indeed a result analogous to Kuratowski for the plane, however instead of two minimal examples there are thousands. The complete graph on 8 vertices K_8 is one of them. Also on the torus $v - e + f = 0$.

A **Platonic graph** is a planar graph such that all vertices have degree d and all faces have the same number of bounding edges b where both $d, b \geq 3$. Examples include the tetrahedron and cube graphs drawn below. This mirrors Euclidean geometry. It was know to the Greeks that there were exactly five platonic solids, that is, convex polyhedron with congruent faces of regular polygons such that the same number of faces meeting at each vertex.

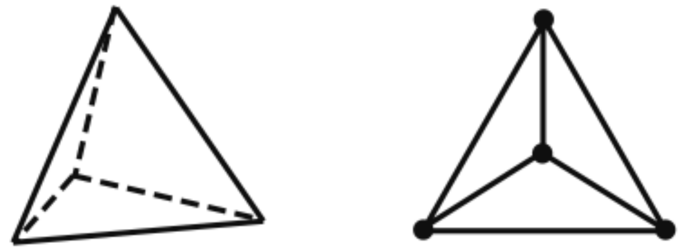
Since each vertex has degree d, the handshaking lemma implies that $dv = 2e$. Since each face has degree b, the handshaking lemma implies that $bf = 2e$. Thus, $v = \frac{2e}{d}$ and $f = \frac{2e}{b}$. And so plugging into $v - e + f = 2$ yields that $\frac{2e}{d} + \frac{2e}{b} = 2 + e > e$. Simplifying yields

$$\frac{1}{d} + \frac{1}{b} > \frac{1}{2}.$$

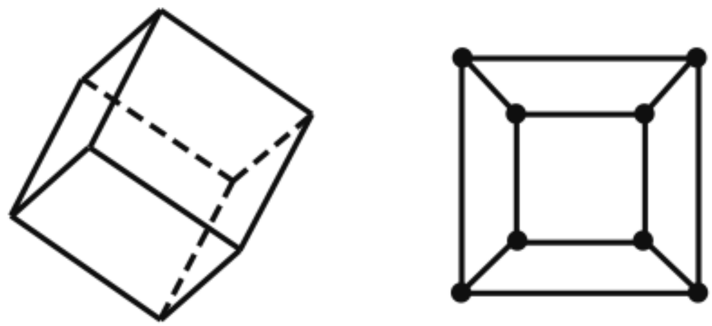
There are only five possible solutions each of which corresponds uniquely to a platonic solid:

- $d = b = 3 \longleftrightarrow$ Tetrahedron.
- $d = 3$ and $b = 4 \longleftrightarrow$ Cube.
- $d = 4$ and $b = 3 \longleftrightarrow$ Octahedron.
- $d = 3$ and $b = 5 \longleftrightarrow$ Dodecahedron.
- $d = 5$ and $b = 3 \longleftrightarrow$ Icosahedron.

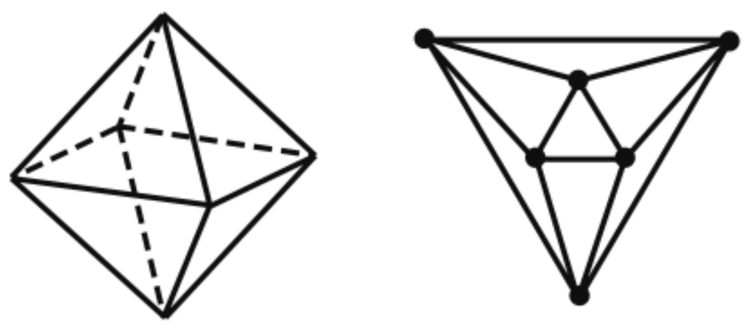
Tetrahedron



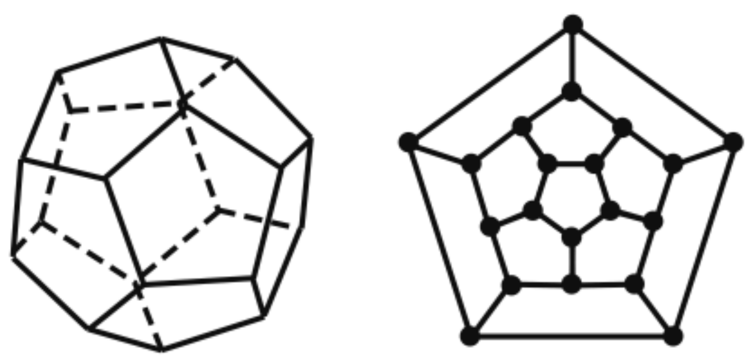
Cube



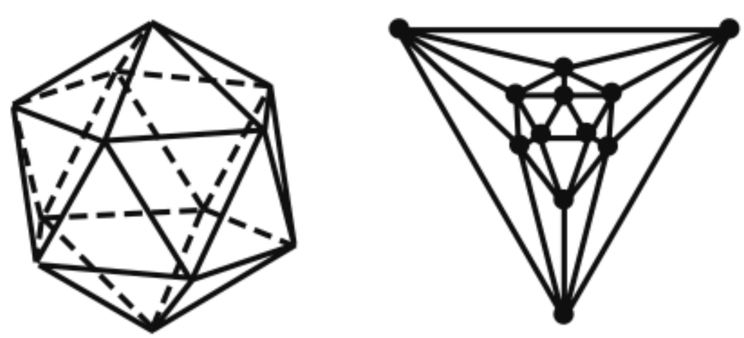
Octahedron

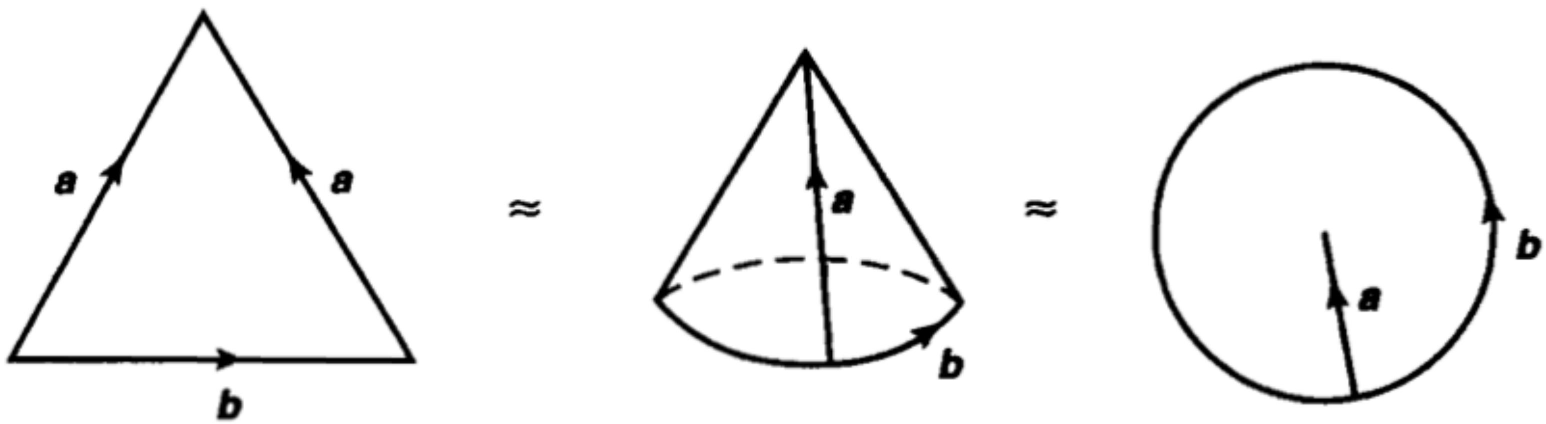


Dodecahedron



Icosahedron




a
a
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a
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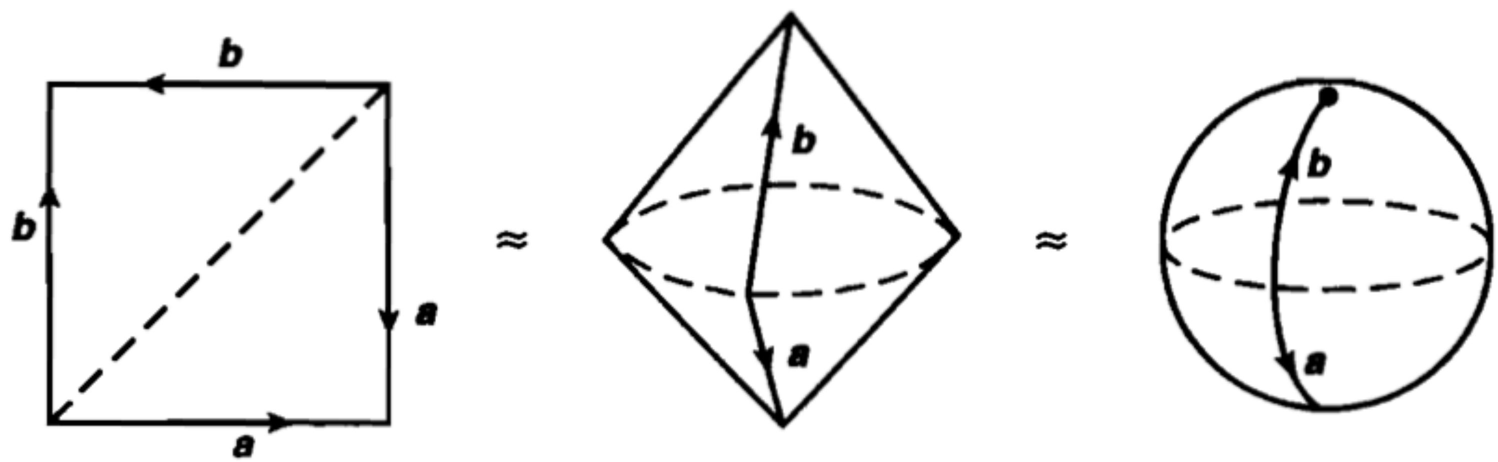
b


b
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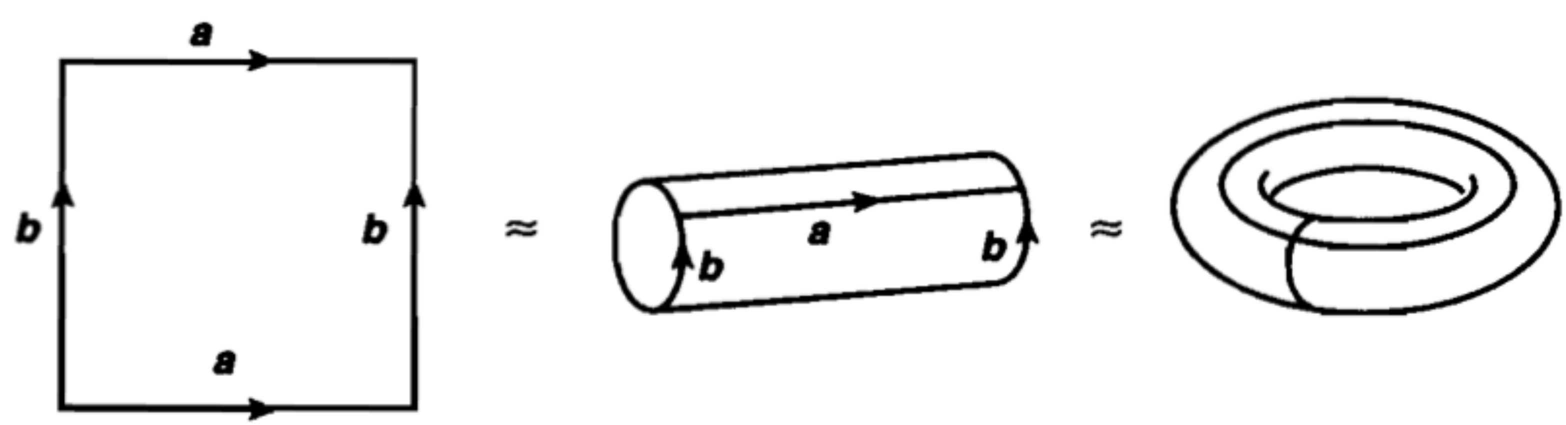
b
a



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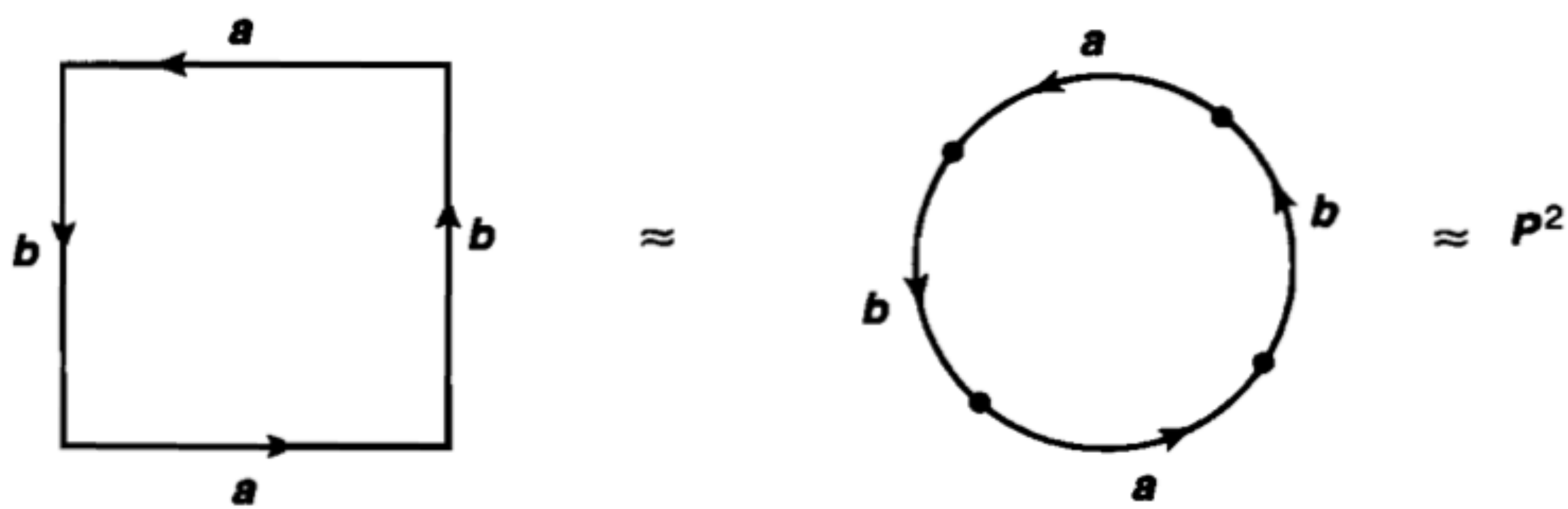
a
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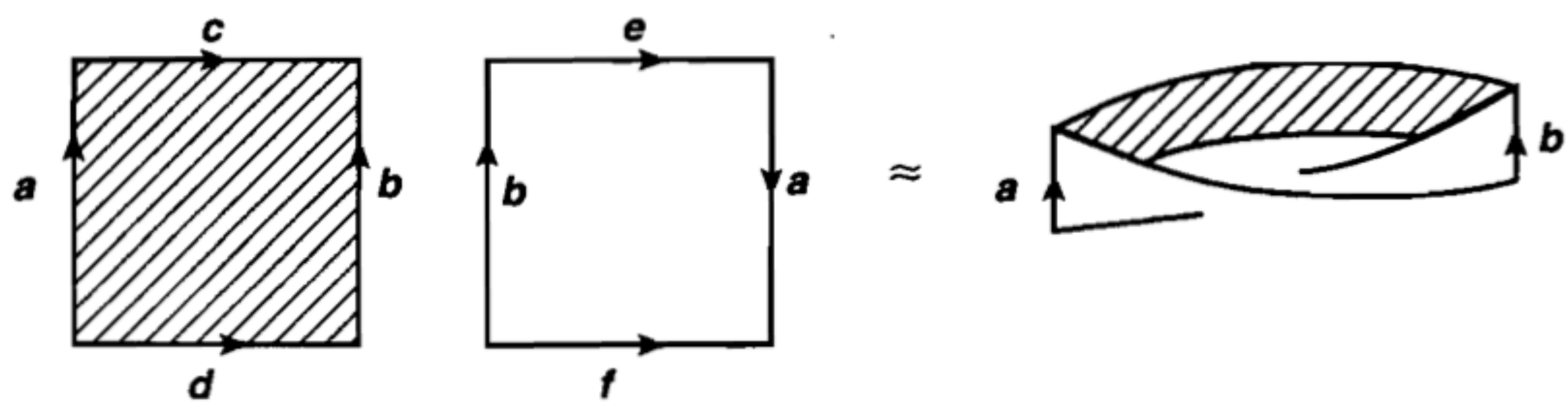
a
b
b
a
≈ P2



c
a
b
d
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b
a
f
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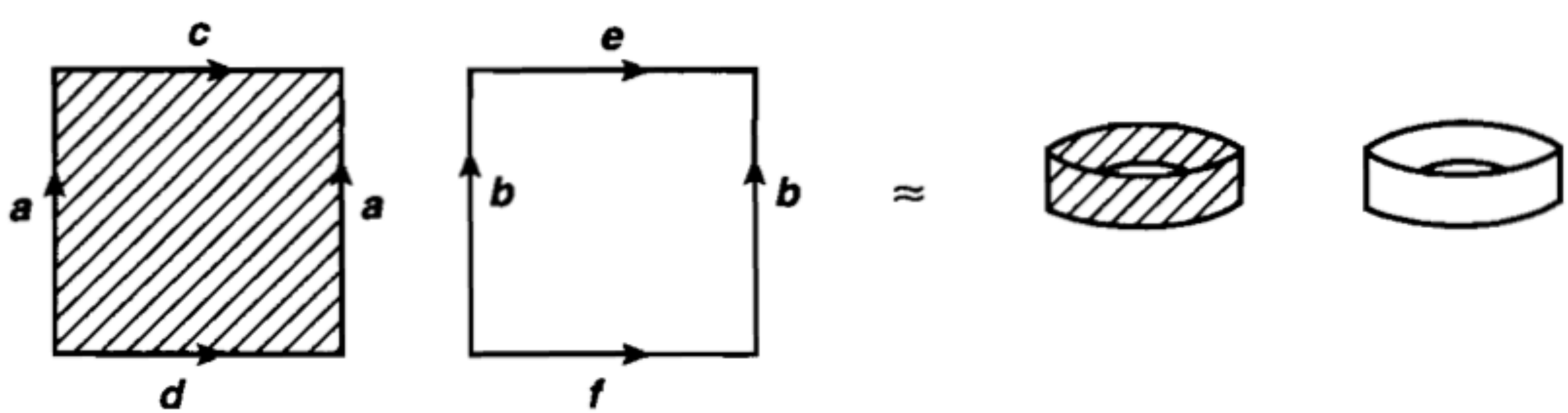
a
b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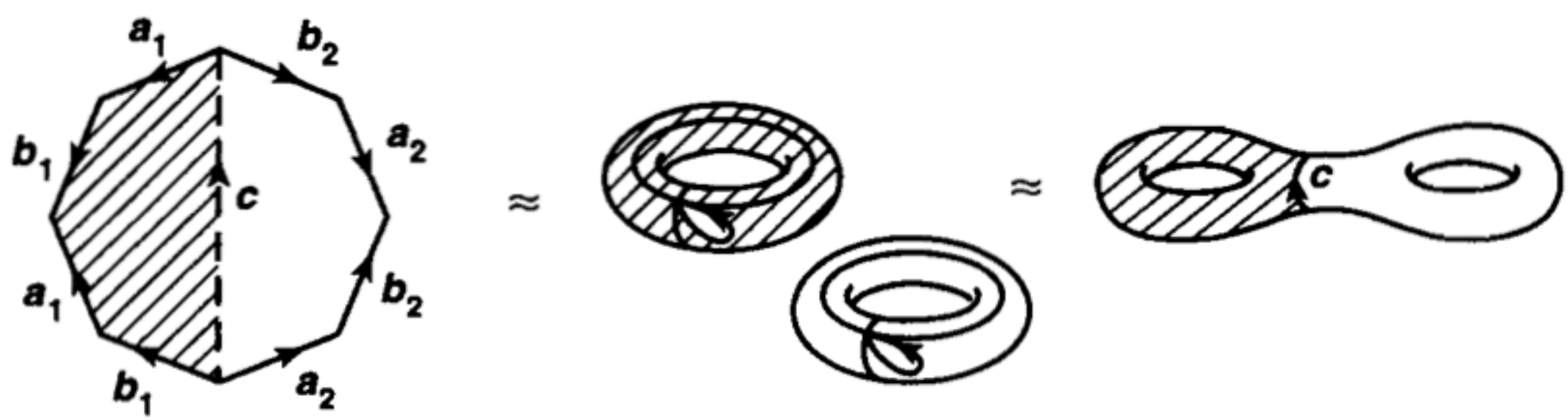
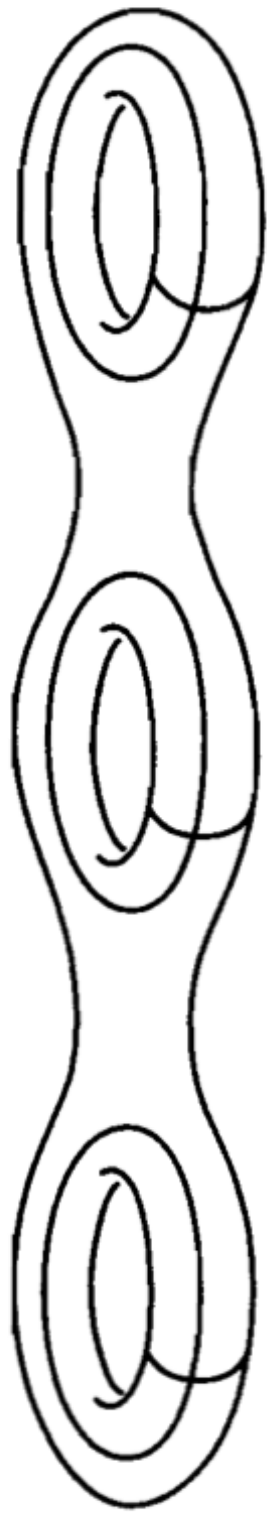
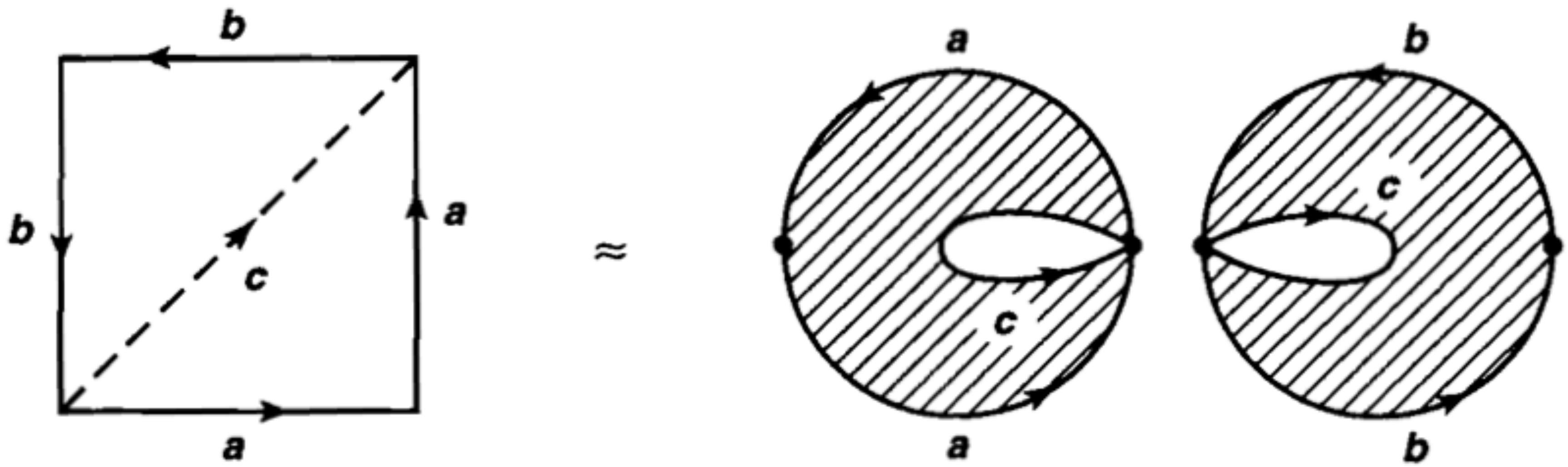


a_1
b_2
b_1
a_2
c
a_1
b_2
b_1
a_2
≈
≈
c





b
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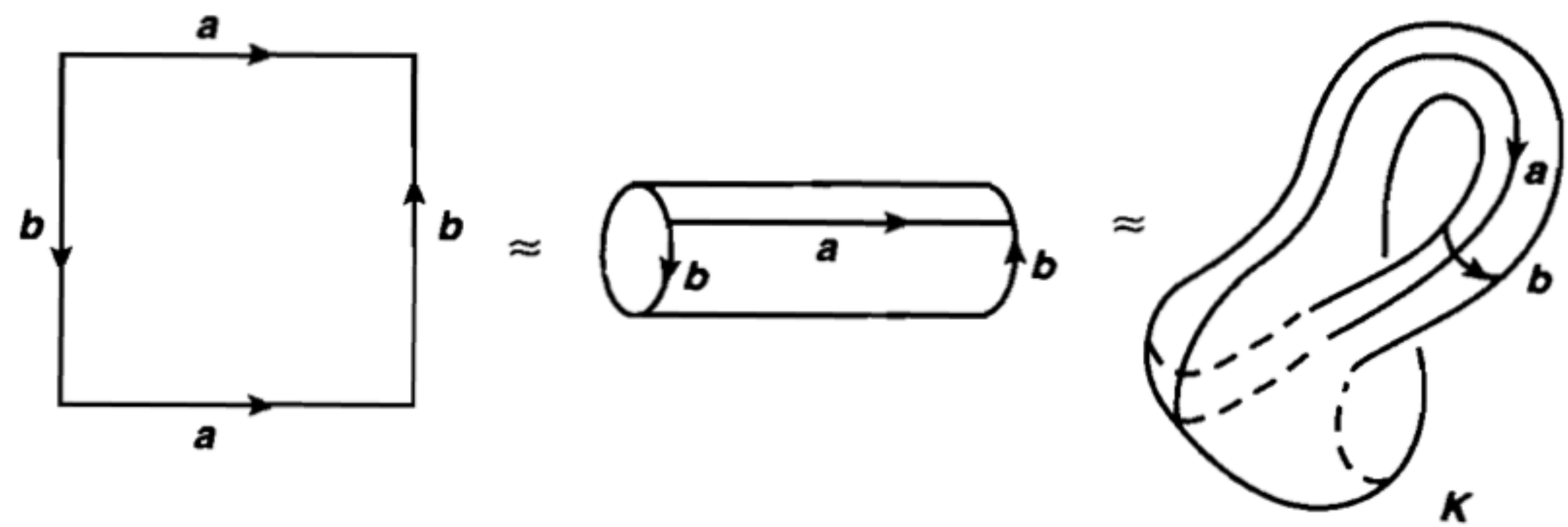
a
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a
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a
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K