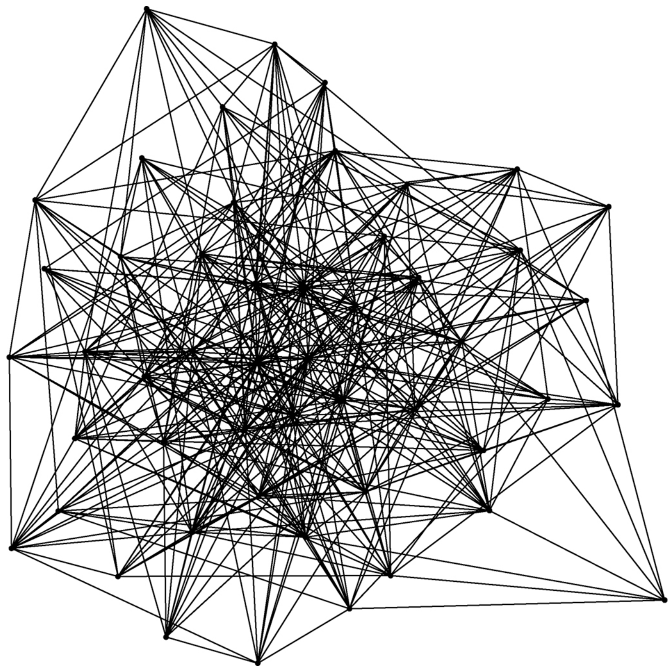
Class Nine: Random Graphs



We can think of building a labelled **random graph** as follows: For each potential edge we flip a coin...If it's HEADS we include the edge in our random graph and if it's TAILS we do not. This is known as the Erdős-Rényi model.

Draw some small graphs and think about the following questions:

In building a random graph on n vertices how many coin flips must we make?

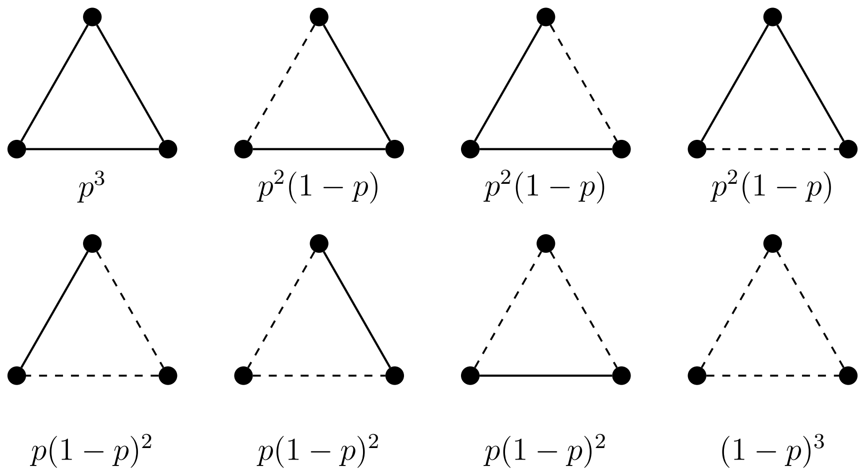
How many potential random graphs are there on n vertices?

What are the chances of obtaining a specific graph by our random procedure using a fair coin? What about when using a bias coin?

On average how many edges will a random graph on n vertices have?

Draw all random graphs on 3 vertices with their respective probabilities. What is the probability a random graph on 3 vertices is Connected? Bipartite? Connected and bipartite? A path? A tree? A clique?

What is the probability that two random graph on 3 vertices are isomorphic?
Both forests? Both trees? Have the same chromatic number?



What are the chances that a random graph on n vertices has m edges?

What are the chances that a vertex in our random graph has a specific degree?
On average what is the degree of a vertex in our random graph?
On average how many isolated vertices are there?

As we vary our coin "fairness" how does our random graph change?

How does the complement of a random graph behave?

On average how many triangles would a random graph have? Complete graphs?

On average how many cycles of length k would a random graph have?

What are the chances that a random graph on n vertices has a clique of size k?

Lemma. *Consider a random graph on n vertices obtained by flipping a biased coin with probability of heads equal to p.*

- *The average degree is $(n-1)p$.*
- *The average number of edges $\binom{n}{2}p = \frac{n(n-1)p}{2}$.*
- *The average number of triangles $\binom{n}{3}p^3$.*
- *The average number of isolated vertices is $n(1-p)^{n-1}$.*
- *The average number of cycles of length k is $\frac{(n)_k}{2k}p^k$, where $(n)_k = n(n-1)...(n-(k-1))$ is the falling factorial.*

Lemma. *Consider a random graph G on n vertices obtained by flipping a biased coin with probability of heads equal to p. Then*

$$P[\alpha(G) \geq k] \leq \binom{n}{k}(1-p)^{\binom{k}{2}} \quad \text{and} \quad P[\omega(G) \geq k] \leq \binom{n}{k}p^{\binom{k}{2}}$$

Corollary. *The Ramsey number $R(k,k) > 2^{k/2}$.*

Theorem (Erdős)**.** *For any positive integer χ, there exists a graph with chromatic number at least χ and no triangles.*

Evolution of a random graph on n vertices as the probability p of an edge existing grows from 0 to 1

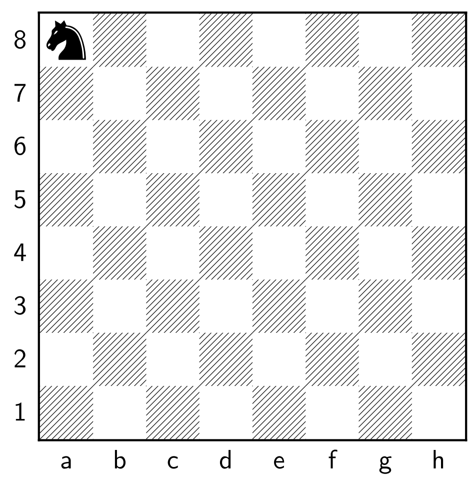
- An edge exists $p \sim \frac{1}{n^2}$
- An subtree with 3 vertices exists $p \sim \frac{1}{n^{3/2}}$
- An subtree with k vertices exists $p \sim \frac{1}{n^{k/(k-1)}}$
- A cycle exists $p \sim \frac{1}{n}$
- No isolated vertices/Connected $p \sim \frac{\ln n}{n}$

To take a **random walk** in a graph G we start at a vertex v and move to one of its neighbors with probability $\frac{1}{deg(v)}$. Repeating this process yields a our notion of a random walk.

What is the long run proportion of time spent in a specific vertex during a random walk in a complete graph? What about a general graph?

In a regular graph, if a random walk just passed through a specific vertex how long till it returns?

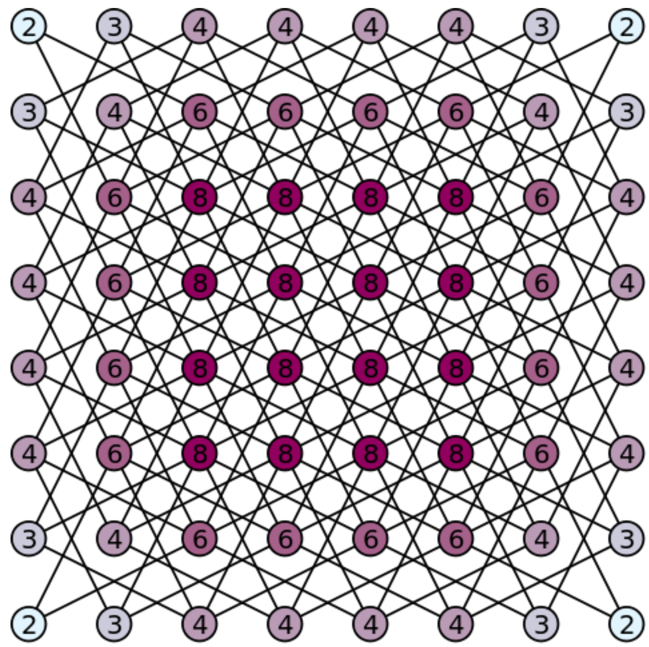
What is the expected number of moves it takes a knight to return to its initial position if it starts in a corner of the chessboard?
Try other starting positions. Where on the board does it return "fastest" from?



Lemma. *The long run proportion of time spent in v during a random walk in a non-bipartite graph G is*

$$\pi(v) = \frac{deg(v)}{\sum_{u \in V(G)} deg(u)} = \frac{deg(v)}{2|E(G)|}.$$

Corollary. *The expected number of steps until a random walk returns to its starting point is* $\frac{1}{\pi(v)}$.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