We can think of building a labelled random graph as follows: For each potential edge we flip a coin...If it’s HEADS we include the edge in our random graph and if it’s TAILS we do not. This is known as the Erdős-Rényi model.

Draw some small graphs and think about the following questions:

- In building a random graph on $n$ vertices how many coin flips must we make?
- How many potential random graphs are there on $n$ vertices?
- What are the chances of obtaining a specific graph by our random procedure using a fair coin? What about when using a bias coin?
- On average how many edges will a random graph on $n$ vertices have?
- Draw all random graphs on 3 vertices with their respective probabilities. What is the probability a random graph on 3 vertices is Connected? Bipartite? Connected and bipartite? A path? A tree? A clique?
What is the probability that two random graph on 3 vertices are isomorphic? Both forests? Both trees? Have the same chromatic number?

\[
p^3 \quad p^2(1-p) \quad p^2(1-p) \quad p^2(1-p)
\]

\[
p(1-p)^2 \quad p(1-p)^2 \quad p(1-p)^2 \quad (1-p)^3
\]

What are the chances that a random graph on \( n \) vertices has \( m \) edges?

What are the chances that a vertex in our random graph has a specific degree?
On average what is the degree of a vertex in our random graph?
On average how many isolated vertices are there?

As we vary our coin “fairness” how does our random graph change?

How does the complement of a random graph behave?

On average how many triangles would a random graph have? Complete graphs?

On average how many cycles of length \( k \) would a random graph have?

What are the chances that a random graph on \( n \) vertices has a clique of size \( k \)?
Lemma. Consider a random graph on \( n \) vertices obtained by flipping a biased coin with probability of heads equal to \( p \).

- The average degree is \( (n - 1)p \).
- The average number of edges \( \binom{n}{2}p = \frac{n(n-1)p}{2} \).
- The average number of triangles \( \binom{n}{3}p^3 \).
- The average number of isolated vertices is \( n(1 - p)^{n-1} \).
- The average number of cycles of length \( k \) is \( \binom{n}{k}^2 p^k \), where \( \binom{n}{k} = n(n - 1) \cdots (n - (k - 1)) \) is the falling factorial.

Lemma. Consider a random graph \( G \) on \( n \) vertices obtained by flipping a biased coin with probability of heads equal to \( p \). Then

\[
P[\alpha(G) \geq k] \leq \binom{n}{k} (1 - p)^{\binom{k}{2}} \quad \text{and} \quad P[\omega(G) \geq k] \leq \binom{n}{k} p^{\binom{k}{2}}
\]

Corollary. The Ramsey number \( R(k, k) > 2^{k/2} \).

Theorem (Erdős). For any positive integer \( \chi \), there exists a graph with chromatic number at least \( \chi \) and no triangles.

Evolution of a random graph on \( n \) vertices as the probability \( p \) of an edge existing grows from 0 to 1

- An edge exists \( p \sim \frac{1}{n^2} \)
- An subtree with 3 vertices exists \( p \sim \frac{1}{n^{3/2}} \)
- An subtree with \( k \) vertices exists \( p \sim \frac{1}{n^{k/(k-1)}} \)
- A cycle exists \( p \sim \frac{1}{n} \)
- No isolated vertices/Connected \( p \sim \frac{\ln n}{n} \)

To take a random walk in a graph \( G \) we start at a vertex \( v \) and move to one of its neighbors with probability \( \frac{1}{\deg(v)} \). Repeating this process yields a our notion of a random walk.

What is the long run proportion of time spent in a specific vertex during a random walk in a complete graph? What about a general graph?
In a regular graph, if a random walk just passed through a specific vertex how long till it returns?

What is the expected number of moves it takes a knight to return to its initial position if it starts in a corner of the chessboard? Try other starting positions. Where on the board does it return “fastest” from?

Lemma. The long run proportion of time spent in \( v \) during a random walk in a non-bipartite graph \( G \) is

\[
\pi(v) = \frac{\deg(v)}{\sum_{u \in V(G)} \deg(u)} = \frac{\deg(v)}{2|E(G)|}.
\]

Corollary. The expected number of steps until a random walk returns to its starting point is \( \frac{1}{\pi(v)} \).