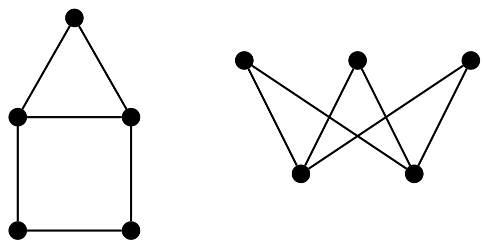
Class One: Degree Sequences



For our purposes a **graph** is a just a bunch of points, called **vertices**, together with lines or curves, called **edges**, joining certain pairs of vertices. Three small examples of graphs are drawn in Figure 0.1. It is important to note that there is no geometry in play here. These drawings are just to help us visualize things.

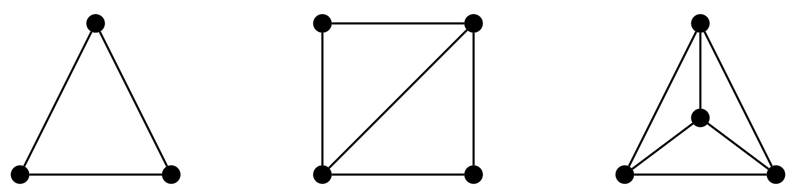


Figure 0.1: Some small examples of graphs.

We say two vertices are **adjacent** if they are joined by an edge, and that two vertices are **non-adjacent** if they are not joined by an edge. Drawn below on the left is a pair of adjacent vertices, and on the right is a pair of non-adjacent vertices.

The only requirements we make of our graphs are the following (Figure 0.2):

- No **loops**, that is, no edge can start and end at the same vertex.
- No **parallel edges**, that is, there can be no more than one edge between every pair of vertices.

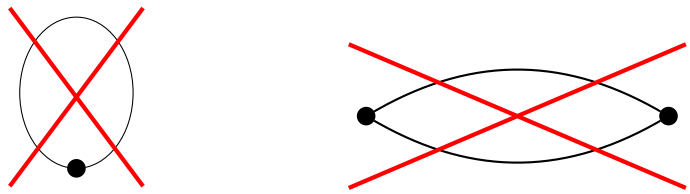
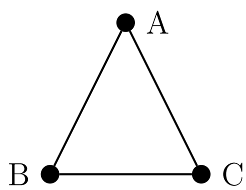
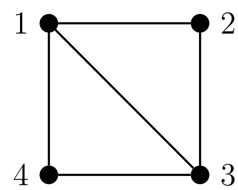


Figure 0.2: No loops or parallel edges are allowed in our graphs.

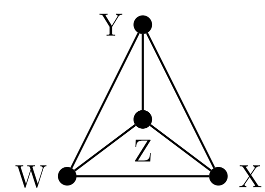
We give things names so as to distinguish and describe them. A graph is completely determined by its vertices and edges, and so we write $G = (V, E)$ to denote the graph G with **vertex set** V and **edge set** E. When discussing a specific graph we name the vertices, which in turn allows us to describe the edges, since an edge is determined by the two vertices it links. For practice lets label and explicitly describe the three graphs drawn in Figure 0.1:



$G_1 = (V_1, E_1)$



$G_2 = (V_2, E_2)$



$G_3 = (V_3, E_3)$

Vertices V_1	A, B, C
Edges E_1	AB BC CD

Vertices V_2	1, 2, 3, 4
Edges E_2	12 13 14 23 34

Vertices V_3	W,X,Y,Z
Edges E_3	WX WY WZ XY XZ YZ

Considering a graph $G = (V, E)$, we will always denote by n the **number of vertices in** G, and denote by m the **number of edges in** G. Observe that n equals the size of V and that m equals the size of E. In turn we say that G is a **graph on** n **vertices with** m **edges**. So in the above, we see that G_1 is a graph on 3 vertices with 3 edges, G_2 is a graph on 4 vertices with 5 edges, and G_3 is a graph on 4 vertices with 6 edges.

Draw some small graphs and think about the following questions:

> What is the least number of edges a graph on n vertices can have?

> What is the most number of edges a graph on n vertices can have?
> How would you build such a graph?

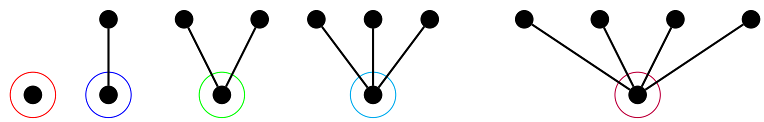


Figure 0.3: What vertices of degree 0,1,2,3,4 look like.

Is there an easy way to distinguish two graphs from one another?
How many possible graphs are there on n vertices?

Lemma. *For every graph G on n vertices with m edges, we always have that*

$$0 \leq m \leq \binom{n}{2} = \frac{n \times (n-1)}{2}.$$

Corollary. *There are at most $2^{\binom{n}{2}}$ possible graphs on n vertices.*

The **degree** of a vertex v in a graph G is the number of edges which meet at v. For instance, in G_1 each vertex has degree 2 and in G_3 each vertex has degree 3, whereas in G_2 vertices 1 and 3 have degree 3, while vertices 2 and 4 have degree 2. For a specific vertex v, we denote its degree by $deg(v)$.

Lemma. *For every vertex v in a graph G on n vertices, we always have that*

$$0 \leq deg(v) \leq n-1.$$

We say a vertex is **even** if its degree is an even number and that a vertex is **odd** if its degree is an odd number. And so, we see that every vertex in G_1 is even, every vertex in G_3 is odd, and in G_2 vertices 1 and 3 are odd, while vertices 2 and 4 are even. The **degree sequence** of a graph $G = (V, E)$ is just a list of the degrees of each vertex in V. For instance, the degree sequence of G_1 is $(2, 2, 2)$, the degree sequence of G_2 is $(2, 2, 3, 3)$, and the degree sequence of G_3 is $(3, 3, 3, 3)$.

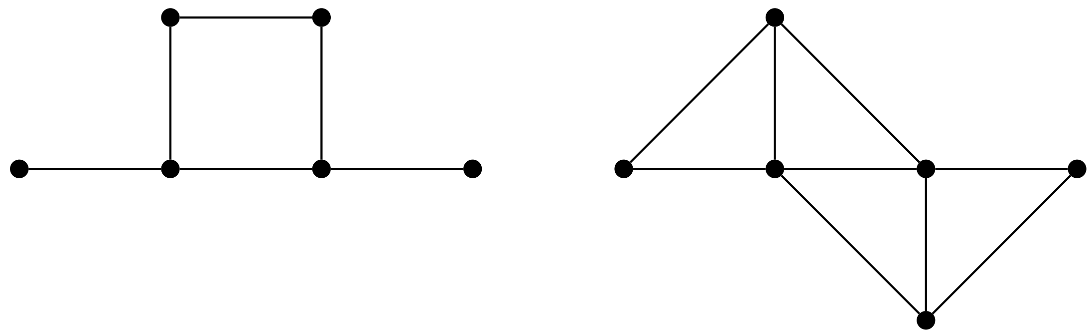


Figure 0.4: Graphs with degree sequences $(1, 1, 2, 2, 3, 3)$ and $(2, 2, 3, 3, 4, 4)$.

Draw some small graphs and think about the following questions:

How small can the degree of a vertex in a graph be?

How large can the degree of a vertex in a graph be?

Is there a family of graphs such that every vertex has degree 0?

Is there a family of graphs such that every vertex has degree 1?

Is there a family of graphs such that every vertex has degree 2?

Is there a family of graphs such that every vertex either has degree 1 or degree 2?

Is there a graph on n vertices such that every vertex has degree $n - 1$?

Given integers p and q, is there a graph such that every vertex has either degree p or q?

Is there a graph such that all of its vertices have the different degrees?

Does every graph have an even vertex?

Does every graph have an odd vertex?

Is there a graph with exactly two even vertices?

Is there a graph with exactly two odd vertices?

Is there a graph with exactly one even vertex?

Is there a graph with exactly one odd vertex?

If you know the degree sequence of a graph, what can you say about the number of edges?

Is there a graph with degree sequence $(0, 1)$?

Is there a graph with degree sequence $(1, 1, 1)$?

Is there a graph with degree sequence $(1, 1, 1, 3)$?

Is there a graph with degree sequence $(1, 2, 2, 3)$?

Is there a graph with degree sequence $(1, 1, 2, 2, 4)$?

Is there a graph with degree sequence $(2, 2, 2, 3, 3)$?

Is there a graph with degree sequence $(1, 3, 3, 3)$?

Is there a graph with degree sequence $(1, 2, 3, 4, 4)$?

Can "different" graphs have the same degree sequence?

Lemma. *The sum of the degrees of all the vertices in a graph is equal to twice the number of edges.*

Corollary. *At every party the total number of hands shaken is even.*

Corollary. *The number of odd vertices in a graph is always even.*

Corollary. *For every graph G the following hold:*

- *There is always a vertex of degree at least* $\lfloor \frac{2m}{n} \rfloor$.
- *There is always a vertex of degree at most* $\lceil \frac{2m}{n} \rceil$.

There's always more to say...

Theorem (Erdős-Gallai)**.** *Let* $d_1 \geq d_2 \geq ... \geq d_n$ *be a sequence of integers.*

1. *If we allow loops and parallel edges, then there exists a graph with degree sequence* $(d_1, d_2, ..., d_n)$ *if and only if* $d_1 + d_2 + ... + d_n$ *is even.*
2. *If we allow parallel edges, then there exists a graph with degree sequence* $(d_1, ..., d_n)$ *if and only if* $d_1 + ... + d_n$ *is even and* $d_1 \leq d_2 + ... + d_n$.
3. *There exists a graph with degree sequence* $(d_1, ..., d_n)$ *if and only if* $d_1 + ... + d_n$ *is even and for each* $1 \leq k \leq n$ *we have that* $d_1 + ... + d_k \leq k(k-1) + min(d_{k+1}, k) + ... + min(d_n, k)$.