For our purposes a graph is a just a bunch of points, called vertices, together with lines or curves, called edges, joining certain pairs of vertices. Three small examples of graphs are drawn in Figure 0.1. It is important to note that there is no geometry in play here. These drawings are just to help us visualize things.

We say two vertices are adjacent if they are joined by an edge, and that two vertices are non-adjacent if they are not joined by an edge. Drawn below on the left is a pair of adjacent vertices, and on the right is a pair of non-adjacent vertices.

The only requirements we make of our graphs are the following (Figure 0.2):

- No loops, that is, no edge can start and end at the same vertex.
- No parallel edges, that is, there can be no more than one edge between every pair of vertices.
Figure 0.2: No loops or parallel edges are allowed in our graphs.

We give things names so as to distinguish and describe them. A graph is completely determined by its vertices and edges, and so we write $G = (V, E)$ to denote the graph $G$ with vertex set $V$ and edge set $E$. When discussing a specific graph we name the vertices, which in turn allows us to describe the edges, since an edge is determined by the two vertices it links. For practice let’s label and explicitly describe the three graphs drawn in Figure 0.1:

$$G_1 = (V_1, E_1)$$

$$G_2 = (V_2, E_2)$$

$$G_3 = (V_3, E_3)$$

<table>
<thead>
<tr>
<th>Vertices $V_1$</th>
<th>A, B, C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edges $E_1$</td>
<td>AB, BC, CD</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Vertices $V_2$</th>
<th>1, 2, 3, 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edges $E_2$</td>
<td>12, 13, 14, 23, 34</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Vertices $V_3$</th>
<th>W, X, Y, Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edges $E_3$</td>
<td>WX, WY, WZ, XY, XZ, YZ</td>
</tr>
</tbody>
</table>

Considering a graph $G = (V, E)$, we will always denote by $n$ the number of vertices in $G$, and denote by $m$ the number of edges in $G$. Observe that $n$ equals the size of $V$ and that $m$ equals the size of $E$. In turn we say that $G$ is a graph on $n$ vertices with $m$ edges. So in the above, we see that $G_1$ is a graph on 3 vertices with 3 edges, $G_2$ is a graph on 4 vertices with 5 edges, and $G_3$ is a graph on 4 vertices with 6 edges.

Draw some small graphs and think about the following questions:

| What is the least number of edges a graph on $n$ vertices can have? |
| What is the most number of edges a graph on $n$ vertices can have? |
| How would you build such a graph? |
Figure 0.3: What vertices of degree 0, 1, 2, 3, 4 look like.

Is there an easy way to distinguish two graphs from one another? How many possible graphs are there on \( n \) vertices?

**Lemma.** For every graph \( G \) on \( n \) vertices with \( m \) edges, we always have that

\[
0 \leq m \leq \binom{n}{2} = \frac{n \times (n-1)}{2}.
\]

**Corollary.** There are at most \( 2^{\binom{n}{2}} \) possible graphs on \( n \) vertices.

The degree of a vertex \( v \) in a graph \( G \) is the number of edges which meet at \( v \). For instance, in \( G_1 \) each vertex has degree 2 and in \( G_3 \) each vertex has degree 3, whereas in \( G_2 \) vertices 1 and 3 have degree 3, while vertices 2 and 4 have degree 2. For a specific vertex \( v \), we denote its degree by \( \text{deg}(v) \).

**Lemma.** For every vertex \( v \) in a graph \( G \) on \( n \) vertices, we always have that

\[
0 \leq \text{deg}(v) \leq n - 1.
\]

We say a vertex is **even** if its degree is an even number and that a vertex is **odd** if its degree is an odd number. And so, we see that every vertex in \( G_1 \) is even, every vertex in \( G_3 \) is odd, and in \( G_2 \) vertices 1 and 3 are odd, while vertices 2 and 4 are even. The **degree sequence** of a graph \( G = (V, E) \) is just a list of the degrees of each vertex in \( V \). For instance, the degree sequence of \( G_1 \) is \((2, 2, 2)\), the degree sequence of \( G_2 \) is \((2, 2, 3, 3)\), and the degree sequence of \( G_3 \) is \((3, 3, 3, 3)\).
Draw some small graphs and think about the following questions:

- How small can the degree of a vertex in a graph be?
- How large can the degree of a vertex in a graph be?
- Is there a family of graphs such that every vertex has degree 0?
- Is there a family of graphs such that every vertex has degree 1?
- Is there a family of graphs such that every vertex has degree 2?
- Is there a family of graphs such that every vertex either has degree 1 or degree 2?
- Is there a graph on \( n \) vertices such that every vertex has degree \( n - 1 \)?
- Given integers \( p \) and \( q \), is there a graph such that every vertex has either degree \( p \) or \( q \)?
- Is there a graph such that all of its vertices have the different degrees?
- Does every graph have an even vertex?
- Does every graph have an odd vertex?
- Is there a graph with exactly two even vertices?
Is there a graph with exactly two odd vertices?

Is there a graph with exactly one even vertex?

Is there a graph with exactly one odd vertex?

If you know the degree sequence of a graph, what can you say about the number of edges?

Is there a graph with degree sequence (0,1)?

Is there a graph with degree sequence (1,1,1)?

Is there a graph with degree sequence (1,1,1,3)?

Is there a graph with degree sequence (1,2,2,3)?

Is there a graph with degree sequence (1,1,2,2,4)?

Is there a graph with degree sequence (2,2,2,3,3)?

Is there a graph with degree sequence (1,3,3,3)?

Is there a graph with degree sequence (1,2,3,4,4)?

Can “different” graphs have the same degree sequence?

**Lemma.** The sum of the degrees of all the vertices in a graph is equal to twice the number of edges.

**Corollary.** At every party the total number of hands shaken is even.

**Corollary.** The number of odd vertices in a graph is always even.
**Corollary.** For every graph $G$ the following hold:

- There is always a vertex of degree at least $\left\lfloor \frac{2m}{n} \right\rfloor$.
- There is always a vertex of degree at most $\left\lceil \frac{2m}{n} \right\rceil$.

There’s always more to say...

**Theorem** (Erdős-Gallai). Let $d_1 \geq d_2 \geq ... \geq d_n$ be a sequence of integers.

1. If we allow loops and parallel edges, then there exists a graph with degree sequence $(d_1, d_2, ..., d_n)$ if and only if $d_1 + d_2 + ... + d_n$ is even.

2. If we allow parallel edges, then there exists a graph with degree sequence $(d_1, ..., d_n)$ if and only if $d_1 + ... + d_n$ is even and $d_1 \leq d_2 + ... + d_n$.

3. There exists a graph with degree sequence $(d_1, ..., d_n)$ if and only if $d_1 + ... + d_n$ is even and for each $1 \leq k \leq n$ we have that $d_1 + ... + d_k \leq k(k - 1) + \min(d_{k+1}, k) + ... + \min(d_n, k)$. 