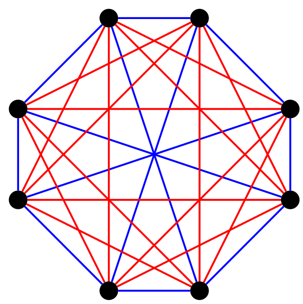
Class Seven: Ramsey Theory



The heart of Ramsey theory lies in the fact that once a mathematical object gets "big" enough it must contain certain special "small" structures. That is, complete disorder is impossible for a large object, as in fact disorder is, in some sense, a pattern in itself.

If $k+1$ objects are placed in k boxes, what can we can we conclude?

Given a set of numbers, is there always one at least the average?
What about at most the average?

More generally, if p objects are placed in h boxes, what can we can we conclude?

Theorem (Pigeonhole Principle)**.** *If $k+1$ pigeons are placed in k holes, then one hole contains at least two pigeons. More generally, if p pigeons are placed in h holes, then one hole contains at least $\lceil \frac{p}{h} \rceil$ pigeons.*

Use the pigeonhole principle to show that $\chi(G) \geq \omega(G)$ for any graph G.
Give examples of graphs where equality does and does not hold.

Use the pigeonhole principle to show that $\alpha(G)\chi(G) \geq |V(G)|$ for any graph G.
Give examples of graphs where equality does and does not hold.

Given a set of nine distinct numbers in order, is there always an increasing or decreasing subsequence of three numbers?
What if you're given a set of ten distinct numbers?

For example, given $10, 3, 2, 1, 6, 5, 4, 9, 8, 7$ in order consider $10 < 9 < 8 < 7$.

Theorem. *Given $n^2 + 1$ distinct numbers in order, there always exists a increasing or decreasing subsidence of length $n + 1$.*

How big must a graph be so it either contains a triangle or a non-edge?
A K_4 or a non-edge?
A K_5 or a non-edge?
A K_s or a non-edge?

We define the Ramsey number $R(s,t)$ to be the minimum size a graph G needs to be so that we are guaranteed that either $\omega(G) \geq s$ or $\alpha(G) \geq t$. It should not be clear why such a number $R(s,t)$ exists, but it can seen to by inductively/iteratively applying the pigeonhole principle. In this new notation, our above observation that every graph is either complete or contains a non-edge translates as $R(s,2) = s$ for all $s \geq 1$. Since $\omega(G) = \alpha(G^c)$ and $\alpha(G) = \omega(G^c)$, it follows that $R(2,t) = t$ for all $t \geq 1$. In general, computing Ramsey numbers is very difficult and few are known.

Famous Party Problem: How many people must be at a party so that either there are three people any two of whom are friends, or there are three people such that no two of them are friends?

Can you determine $R(3,4) = R(4,3)$?

Erdős famously said that if aliens landed and said they'd blow up the earth if humans couldn't determine $R(5,5)$, then would should use all our minds and computers to compute its value. However, if instead they asked for $R(6,6)$, we should figure out how to destroy the aliens. An alternative way to think about the Ramsey number $R(s,t)$ is the following: Color the edges of the complete graph either red or blue, and determine the minimum size a graph G needs to be so that we are guaranteed that either a complete subgraph made up of red edges of size s or a complete subgraph made up of blue edges of size t.

Thinking of coloring edges can you assign a meaning to $R(s,t,u)$
Compute $R(2,2,2)$, $R(2,2,3)$ and $R(2,3,3)$.
Can you give an upper bound on $R(3,3,3)$?

We can also generalize this edge coloring idea to finding any desired monochromatic subgraphs. And so can think of $R(s,t) = R(K_s, K_t)$.

Let P_2 be the two-edge path. Determine $R(P_2, P_2)$ and $R(P_2, K_3)$.

Color the integers $\{1, 2, 3, 4\}$ red and blue. Need there be three integers x, y, z of the same color such that $x + y = z$?
What if we two color $\{1, 2, 3, 4, 5, 6\}$? What if we also require that $x \neq y$?

Theorem (Schur's Theorem)**.** *For any $k \geq 2$ and $n \geq R_k(3, ..., 3)$, it follows that for any k-coloring of $\{1, 2, ..., n\}$ there are three integers x, y, z of the same color such that $x + y = z$.*

Theorem (Fermat's Last Theorem)**.** *The equations $x^n + y^n = z^n$ has no integer solutions for $n > 2$ and $x, y, z \neq 0$.*

Schur tried to use this result to attach Fermat's Last Theorem. However, again using Ramsey theory he was able to prove the following:

Theorem. *For every $m \geq 1$, the equation $x^m + y^m = z^m \pmod{p}$ for all p sufficiently large.*