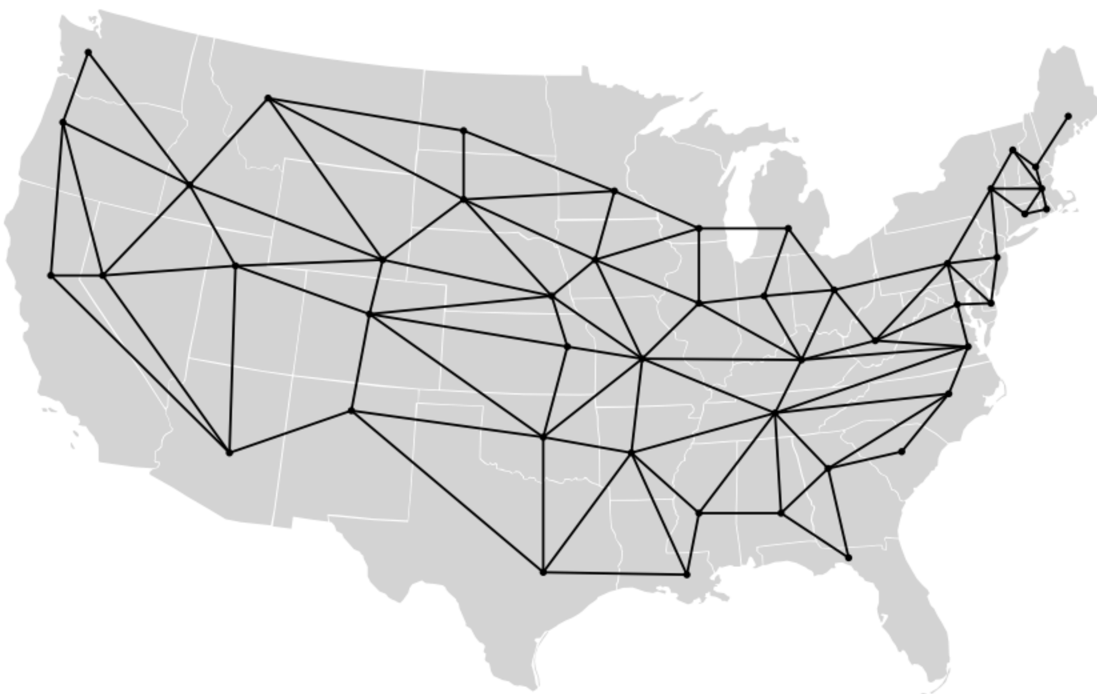
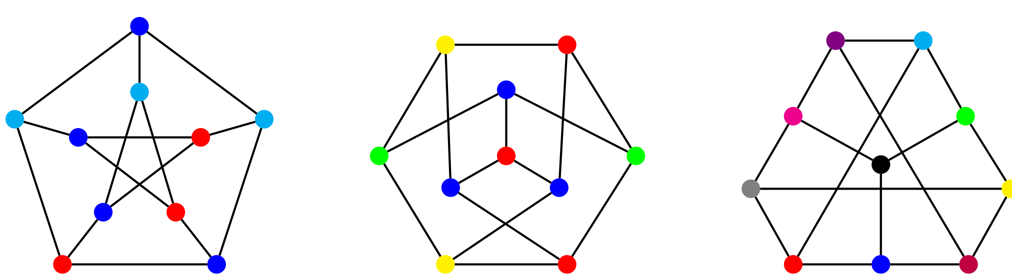
Class Six: Coloring Planar Graphs



A **coloring** of a graph is obtained by assigning every vertex a color such that if two vertices are adjacent, then they receive different colors. Drawn below are three different colorings of three isomorphic drawings of the Petersen graph. An easy way to color a graph is to just assign each vertex a unique color, see for example the far right coloring below. However, this approach is kinda cheating. The interesting and challenging part of this problem is to try and find a coloring of a graph which uses as few colors as possible. We define the **chromatic number** of a graph to be the smallest number of colors necessary to produce a coloring of the graph. We denote the chromatic number of a graph G by $\chi(G)$. The three colorings below show that the chromatic number of the Petersen graph is at most three, in fact this is the best one can do.



Coloring has many applications outside of graph theory. Expand...

Example one: Fireworks/Conflict models.

Example two: scheduling theory.

Draw some small graphs and think about the following questions:

How small can the chromatic number of a graph be? Describe all such graphs.

How large can the chromatic number of a graph be? Describe all such graphs.

What can force the chromatic number of a graph to be large?

How can adding an edge to a graph effect its chromatic number?

How does deleting an edge from a complete graph effect its chromatic number?

Can you attach a new vertex to a graph to increase its chromatic number?

If two graphs are isomorphic, need they have the same chromatic number?

What is the chromatic number of a star?

What is the chromatic number of a path?

What is the chromatic number of a tree?

What is the chromatic number of a cycle?

What is the chromatic number of a wheel?

If a graph G contains a triangle, what can you say about $\chi(G)$?
What if G has four vertices all pairwise adjacent?
What if G has five vertices all pairwise adjacent?

$\boxed{\text{Describe all graphs with chromatic number one.}}$

$\boxed{\text{Give a general description of all graphs with chromatic number two.}}$

$$\boxed{\begin{array}{c}\text{If a graph } G \text{ contains a non-edge, what can you say about } \chi(G)?\\ \text{What if } G \text{ has three vertices all pairwise non-adjacent?}\\ \text{What if } G \text{ has four vertices all pairwise non-adjacent?}\end{array}}$$

$\boxed{\text{Describe a family of graphs with chromatic number three and no triangle.}}$

$\boxed{\text{Is there a graph with chromatic number four and no triangle?}}$

A graph G is called **critical** if $\chi(G - v) < \chi(G)$ for every vertex v of G, that is, if deleting any vertex results in a graph with a smaller chromatic number.

$\boxed{\text{Find all critical graphs with } \chi(G) = 1, 2, 3.}$

$\boxed{\text{Find a critical graphs with } \chi(G) = 4.\ \text{(Hard!)}}$

For a graph G, define $\omega(G)$ to be the size of the largest clique in G and define $\alpha(G)$ to be the size of the largest stable set in G.

Lemma. *For any graph G we have that*

$$\omega(G) \leq \chi(G) \leq n.$$

Lemma. *For any graph G we have that*

$$\alpha(G)\chi(G) \geq n.$$

Lemma. *A graph G has chromatic number two if and only if G is bipartite, which is the case, if and only if G does not contain an odd cycles.*

Theorem (Erdős)**.** *For any positive integer χ, there exists a graph with chromatic number at least χ and no triangles.*

In general, for any arbitrary graph there is no fast way to compute $\chi(G)$. This problem, like the problem of deciding if two graphs are isomorphic, sits at the center of **P** vs **NP** debate, which is basically saying that finding a fast way to optimally color a graph is very, very important, but that most evidence suggests that this is likely impossible.

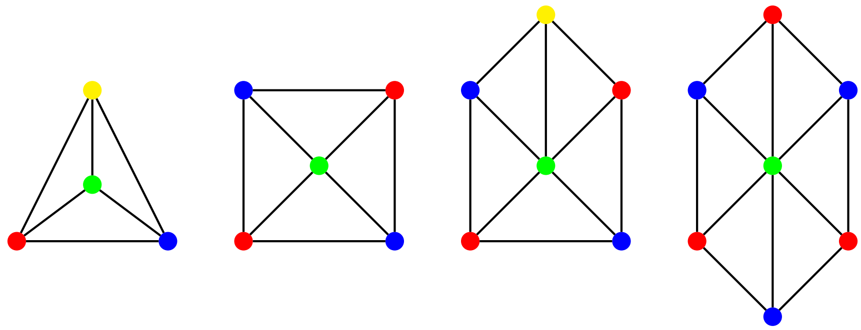


Figure 0.1: Small wheels

Determine the chromatic number of the graph of United States drawn above.

Do there exist nonplanar graphs with chromatic number one? two? three? four?

Lemma. *Every planar graph contains a vertex of degree at most five.*

By successively deleting vertices of degree at most five, the previous result immediately tells us that there exists a coloring which uses at most six colors for any planar graph. With not much more work this can be improved to show that $\chi(G) \leq 5$ for any planar graph G. With a LOT more work one can prove the following famous result, which is known as the Four Color Theorem. So called because given any map we can always color every country's territory one of four colors in such a way so that no two bordering countries receive the same color.

Theorem (Appel, Haken 1976)(Robertson, Sanders, Seymour,Thomas 1996)**.** *If G is a planar graph, then $\chi(G) \leq 4$.*

Since last class we showed that K_4 is planar, while K_5 is not, it follows that the above is the best we could hope for. Additionaly, we saw that K_7 can be drawn on the surface of the torus/donut, while K_8 cannot. A drastically simpler proof, based on the fact that $v-e+f=0$ for a graph embedded on the surface of the torus/donut, yields the following...The Seven Color Theorem if you will.

Lemma. *If G can be embedded on the torus, then $\chi(G) \leq 7$.*

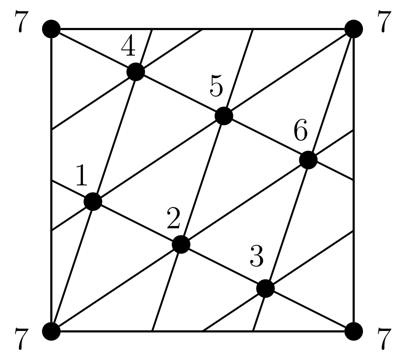


Figure 0.2: K_7 drawn on the torus/donut. Note: $v = 7$, $e = 21$ and $f = 14$.

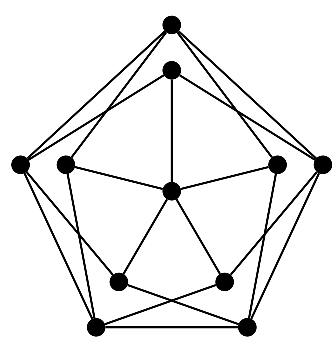


Figure 0.3: Example of a graph with chromatic number four and no triangle.