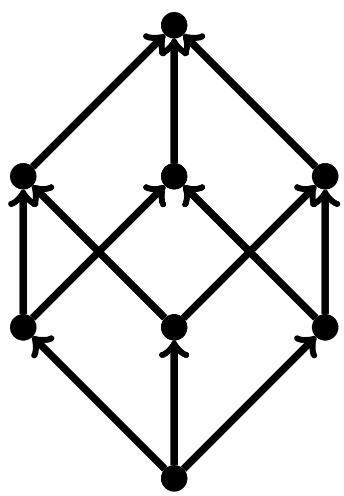
Class Ten: Directed Graphs



When exploring finite and infinite simple graphs we were in a sense exploring all possible symmetric relations between any set of objects. We now consider the situation where this relation is one sided. A **directed graph** is a graph whose edges have been oriented. We call these edges **arcs**, and when referring to the endpoints of an arc, say an arc is directed from **head** to **tail**. An **orientation** of an undirected graph is obtained by assigning a direction to all its edges. We say an undirected graph is orientable if it has an orientation in which there is a **directed path** linking every pair of vertices.
Draw some small graphs and think about the following questions:

How many arcs can a directed graph have?

How many ways can we orient a graph with m edges?

What should it mean for a directed graph to be connected?
Think about paths, partitions, and the underlying undirected graph.

When is an undirected graph orientable?

How would you define the degree of a vertex in a directed graph?
What would the handshaking lemma look like now?

What should it mean for two directed graphs to be isomorphic?

How many non-isomorphic directed graphs are there with 1 vertices?

How many non-isomorphic directed graphs are there with 2 vertices?

How many non-isomorphic graphs are there with 3 vertices?

How many non-isomorphic graphs are there with 4 vertices?(Hard! There are 218)

Two directed graphs are **isomorphic** if their respect underlying undirected graphs are isomorphic and are oriented the same. As we let the number of vertices grow things get crazy very quickly! This really is indicative of how much symmetry and finite geometry graphs encode. The sequence of number of non-isomorphic directed graphs on n vertices for $n = 1, 2, 3, ...$ is as follows: $1, 3, 16, 218, 9608, 1540944, 882033440, 1793359192848...$

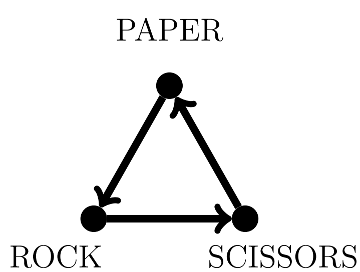
Lemma. *A graph G has $2^{|E(G)|}$ possible orientations.*

Lemma. *A graph is orientable if and only if it contains no bridges.*

Lemma. *The sum of the in-degree of all the vertices in a directed graph is equal to the number of arcs. Similarly, the sum of the out-degree of all the vertices in a graph is equal to the number of arcs.*

Theorem. *A directed graph is strongly connected if and only if there is always an arc leaving an chunk.*

Theorem. *A directed graph G always contains a directed path of length $\chi(G) - 1$.*



DeBruijn Sequences, Towers of Hanoi and walks in hypercubes. (expand)

A **tournament** is an orientation of a complete graph. It is so called as we can imagine this as an encoding of the results of a Round-Robin tournament. That is, where every team plays one another and by convention the winning team in each match is the head of the corresponding arc. The above Rock-Paper-Scissors directed graph is the smallest non-transitive relation/tournament.

Draw all possible tournament outcomes with up to four teams?
Is there always a champion?
Can we always fairly rank the teams?

When does a tournament have a champion?

When does a tournament allow us to fairly rank all the teams?

Theorem. *A tournament has a complete ranking if and only if it is acyclic.*