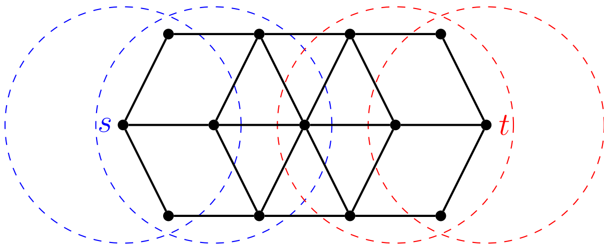
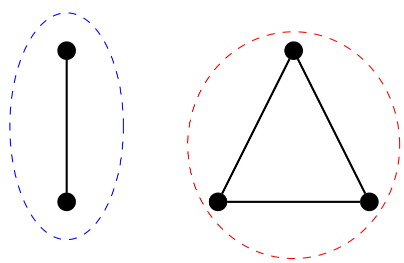
Class Three: Connectivity



We say that a graph is **connected** if it cannot be divide into two parts such that there are no edges between the parts. If a graph is not connected, we say it is **disconnected**. A **component** of a graph is a connected subgraph which is as large as possible. For a graph G, we denote by $c(G)$ the number of components of G. Note, if G is connected, then $c(G) = 1$. For example, in the below graph on five vertices there are no edges between blue vertices and the red vertices, and so the graph is disconnected. Further, the blue and red subgraphs are the two components that make up the graph.



Draw some small graphs and think about the following questions:

How many components can a graph on n vertices have?

Give an example of a disconnected graph with many non-isomorphic components.

If two graphs have the same degree sequence, must they both be connected?

If G and H are isomorphic and G is connected, must H be connected?

How many connected non-isomorphic graphs are there with at most 4 vertices?

Can a connected graph have more vertices than edges?
How many edges must a connected graph have?

How small can the average degree be in a disconnected graph?

By requiring all vertices in a graph to have high degree,
can we force the graph to be connected?
How small can we make this lower bound?

Lemma. *For every graph G on n vertices, we always have that*

$$1 \leq c(G) \leq n.$$

Lemma. *For every connected graph on n vertices with m edges, we always have that*

$$m \geq n - 1.$$

Even we considering only connected graphs, as we let the number of vertices grow things get crazy very quickly! This really is indicative of how much symmetry and finite geometry graphs encode. The sequence of number of connected non-isomorphic graphs on n vertices for $n = 1, 2, 3, ...$ is as follows: $1, 1, 2, 6, 21, 112, 853, 11117, 261080, 11716571, ...$

Lemma. *Any graph on $n \geq 2$ vertices such that every vertex has degree at least $\frac{n-1}{2}$ is connected.(Observe $\frac{n-2}{2}$ doesn't work! Try n=4 or 6)*

A **walk** in a graph is a sequence of vertices $v_1 - v_2 - ... - v_\ell$ such that each vertex is adjacent to both the vertex which precedes and the vertex which follows it in the sequence. We refer to ℓ as the **length** of the walk. A special kind of walk is one where all the vertices are different, we call such a walk a **path**.

Draw some small graphs and think about the following questions:

Given a walk between two vertices in a graph, how do we obtain a path between them?

Is there always a path between any two vertices in a graph?

Is there always a path between any two vertices in a connected graph?

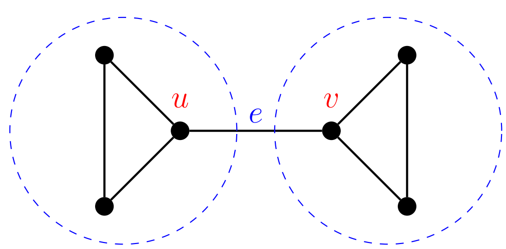
Must a graph be connected, if there is always a path between any two of its vertices?

How would you find a path between two vertices in a graph,
or determine that none exists?

Lemma. *A graph is connected if and only if there is always a path between any two vertices in the graph.*

Lemma. *Breath-first search is an efficient procedures for finding a path between two vertices in a given graph, or determining that none exists.*

Consider the problem of trying to disconnect a connected graph. Here are two potential approaches: "breaking" the graph by deleting vertices or doing so by deleting edges. Let v be a vertex in a graph G on n vertices. We denote by $G - v$ the graph on $n - 1$ vertices obtained by deleting v. We say v is a **cut vertex** if $c(G - v) > c(G)$, that is, if deleting v "breaks" the graphs into more parts. We also define a similar notion looking instead at deleting edges. Let e be an edge in a graph G on n vertices with m edges. We denote by $G - e$ the graph on n vertices with $m - 1$ edges obtained by deleting e. We say v is a **bridge** if $c(G - e) > c(G)$, that is, if deleting e "breaks" the graphs into more parts. For a connected graph, a cut-vertex or bridge is a vertex or edge whose removal disconnects the graph. For example, in the below graph u and v are the only cut-vertices and e is the only bridge.



Draw some small graphs and think about the following questions:

What can happen to the number of components, when deleting a vertex from a graph?

Give an example of a connected graph on n vertices where deleting some vertex results in $n-1$ components.

What can happen to the number of components, when deleting an edge from a graph?

Give an example of a graph with no cut vertices and no bridges.

Give an example of a graph with one cut vertices such that every edge is a bridge.

Give an example of a graph with lots of cut vertices and lots of bridges.

If a graph has a bridge, need there be a cut vertex?

If a graph has a cut vertex, need there be a bridge also?

If every vertex in a connected graph has even degree, can there be a cut-vertex?

If every vertex in a connected graph has even degree, can there be a bridge?

When is a vertex a cut vertex?

When is an edge a bridge?

Lemma. *In a connected graph an edge is a bridge if and only if it does not lie on any cycle of the graph.*