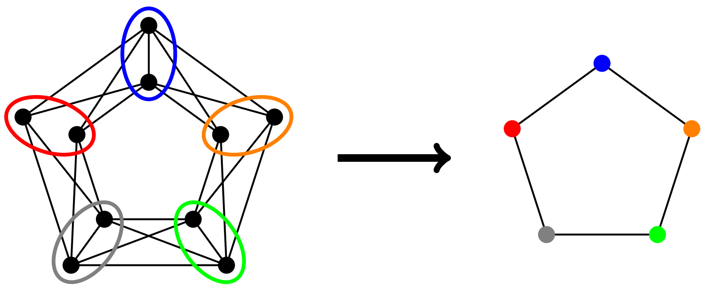
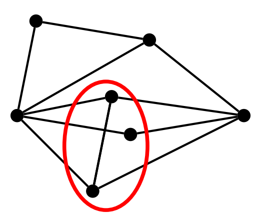
Class Twelve: Prime Graphs and Substitution



Prime numbers are of great mathematical interest because of the fundamental theorem of arithmetic, which shows that every integer can be uniquely factored into primes. That is to say, because prime numbers are the basic building blocks of all other integers. This idea of breaking apart a mathematical object into its most basic and fundamental pieces is everywhere in mathematics. In graph theory, we develop our notion of primality from thinking about how a vertex behaves in a graph. With respect to a specific vertex in a graph the remaining vertices can be divide into to two parts based on whether or not they are adjacent to this specific vertex. More generally, a **factor** in a graph is a chunk of vertices so that the remaining vertices see either all or none of the chunk. For example, in the below graph the three circled vertices are a factor. We say a graph is **prime** if it has no factors with more than one vertex.



Draw some small graphs and think about the following questions:

How would you generalize a vertex? a stable set? a complete graph?

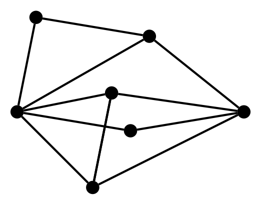
Try and find factors in all graphs on at most three vertices. What about four vertices?

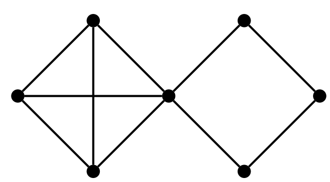
Are cycles prime? paths? stars? complete graphs? trees?

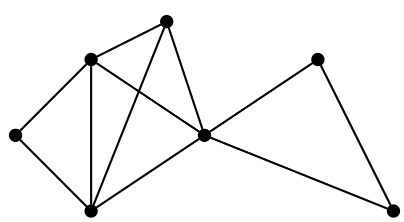
If a graph is prime, need it be connected? anticonnected?

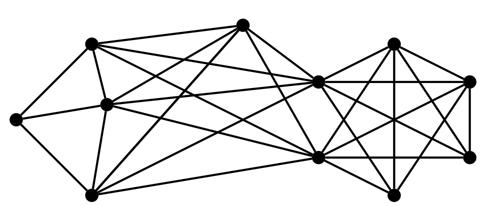
If a graph is prime, what about its complement?

Factor the following graphs into primes



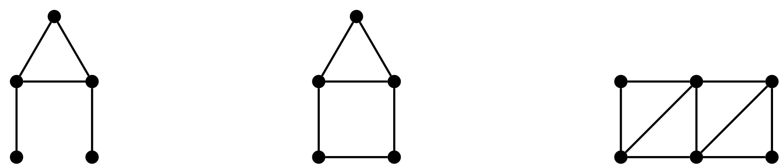






The **prime graph** is the infinite graph on the counting numbers $\mathbb{N}$ such that two vertices $x, y \in \mathbb{N}$ are adjacent if and only if $x + y$ is a prime integer.

Is the prime graph connected?



Triangle prime extensions

Substitution...Prime extensions...modular decomposition aka “factoring”