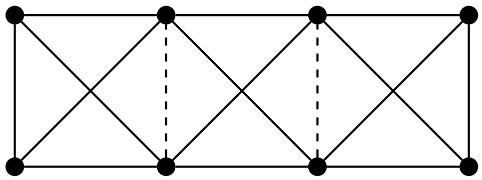
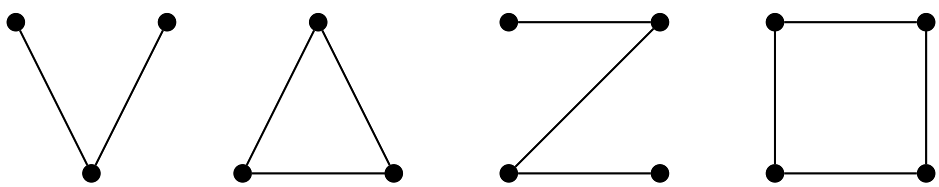
Class Two: Self-Complementarity



When studying any type of mathematical object it is often useful to have some notion for when two objects of this type are equal. For example, when dealing with fractions we want to be able to say things like $\frac{6}{3} = 2 \neq \frac{7}{5}$. What allows us this freedom was the following condition:

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc. \tag{1}$$

Since there is no shortage of different ways to draw a particular graph, we'd like to come up with a similar type of condition which would allow us to say if two graphs are the "same" or "different." On an intuitive level it may seem clear that the four graphs drawn below are "different," but it is not clear how do we go about turning this intuition into mathematics with the hope of obtaining sometype of a checkable procedure akin to (1).



A graph is completely determined by its vertices and edges. And so in drawing a graph we are free to move the vertices around all we want so long as we don't alter the edges. Roughly speaking, we say that two graphs are **isomorphic** if we can deform one into the other, that is, if we can redraw one them so that it appears exactly as the other. Otherwise we say the graphs are **non-isomorphic**. Formally, given two graphs G and H if there exists a bijection $\psi : V(G) \to V(H)$ such that two vertices $u, v \in V(G)$ are adjacent if and only $\psi(u), \psi(v) \in V(H)$ are adjacent. Since there are $n!$ bijections, one way to check if two graphs are isomorphic is to check every single mapping. However, for $n \geq 4$ we have that $n! > 2^n$ and so this is a very inefficient approach to solving the problem. In general, for any arbitrary pair of graphs there is no fast way to decide if the two are isomorphic. The problem sits at the center of **P** vs **NP** debate, which is basically saying that finding a fast way to decide if two graphs are isomorphic is very, very important, but that most evidence suggests that this is an impossible dream. Keep in mind the following question while considering the below examples.

Given two graphs G and H, what are some conditions they should both satisfy in order for G and H to be isomorphic?

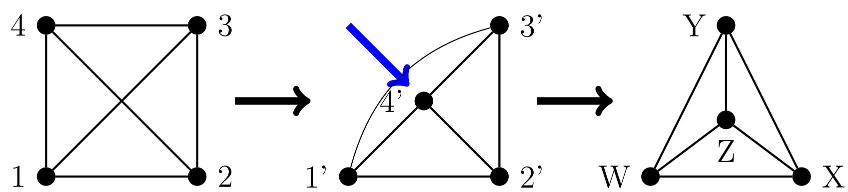


Figure 0.1: The "same" graph and how to deform one drawing into the other.

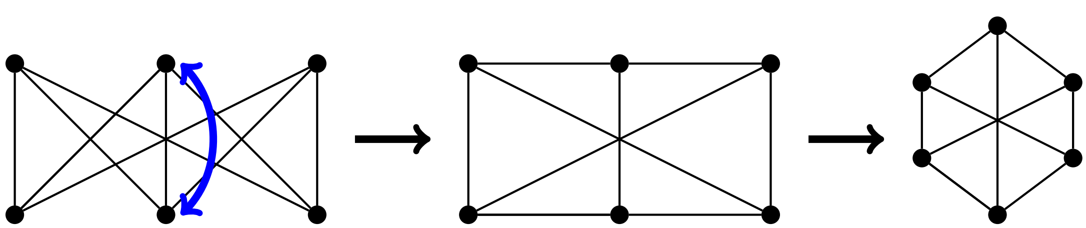


Figure 0.2: The "same" graph and how to deform one drawing into the other.

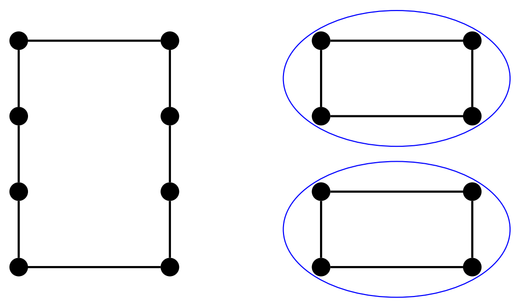


Figure 0.3: One graph is connected, while the other isn't. Hence, same degree sequence and connected, but "different" graphs.

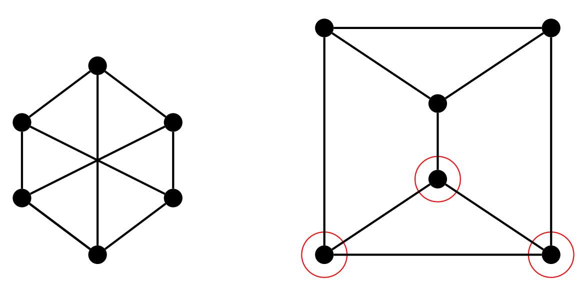


Figure 0.4: One graph contains a triangle, that is, a chordless cycle of length three while the other doesn't. Hence, same degree sequence and connected, but "different" graphs.

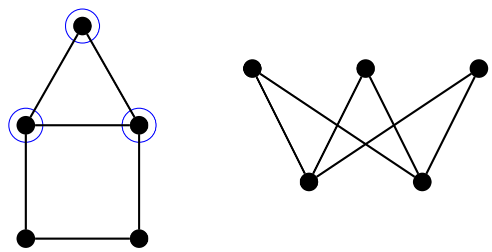


Figure 0.5: One graph contains a triangle, that is, a chordless cycle of length three while the other doesn't. Hence, same degree sequence and connected, but "different" graphs.

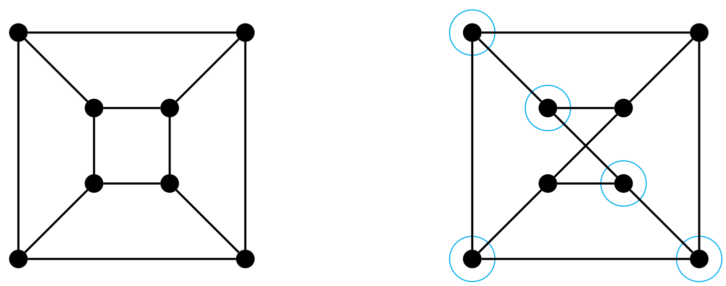


Figure 0.6: One graph contains a chordless cycle of length five while the other doesn't. Hence, same degree sequence but "different" graphs.

Lemma. *Consider two graphs G and H. If G and H are isomorphic, then G and H have to satisfy the follow:*

1. *G and H have the same number of vertices.*
2. *G and H have the same number of edges.*
3. *G and H have the same degree sequence.*
4. *G and H have the same number of connected components.*
5. *G and H have the same number of chordless cycles of a fixed length.*

We call any of the above properties of a graph an **invariant**. Every two isomorphic graphs must both posses every property on the above list. Hence, if two graphs are such that one posses the property and the other doesn't then they cannot be isomorphic. It's important to observe that the above list

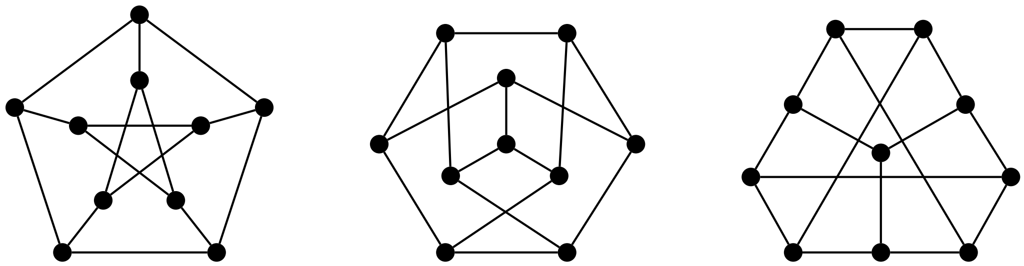


Figure 0.7: Three isomorphic drawings of the infamous Petersen graph!

of invariants is not complete, meaning that there do exist "different" graphs which satisfy all the above conditions.

Draw some small graphs and think about the following questions:

How many non-isomorphic graphs are there with 2 vertices?

How many non-isomorphic graphs are there with 3 vertices?

How many non-isomorphic graphs are there with 4 vertices?

How many non-isomorphic graphs are there with 5 vertices?(Hard! There are 34)

As we let the number of vertices grow things get crazy very quickly! This really is indicative of how much symmetry and finite geometry graphs encode. The sequence of number of non-isomorphic graphs on n vertices for $n = 1, 2, 3, ...$ is as follows: $1, 2, 4, 11, 34, 156, 1044, 12346, 274668, ...$

The **complement** of a graph G is the graph having the same vertex set as G such that two vertices are adjacent if and only the same two vertices are non-adjacent in G. We denote the complement of a graph G by G^c. Note, since the complete graph on n vertices has $\binom{n}{2}$ edges, it follows that if G is a graph on n vertices with m edges, then G^c is also a graph on n vertices but with $\binom{n}{2} - m$ edges. We say that a graph G is **self-complementary** if G is isomorphic to G^c. Observe that the trivial graph on 1 vertex and no edges is clearly self-complementary.

Draw some small graphs and think about the following questions:

Does there exist a self-complementary graph on 4 vertices?

Does there exist a self-complementary graph on 5 vertices?

How many edges can a self-complementary graph on n vertices have?
Could there exist a self-complementary graph on 6 or 7 vertices?

Try and draw all self-complementary graphs on 8 vertices.

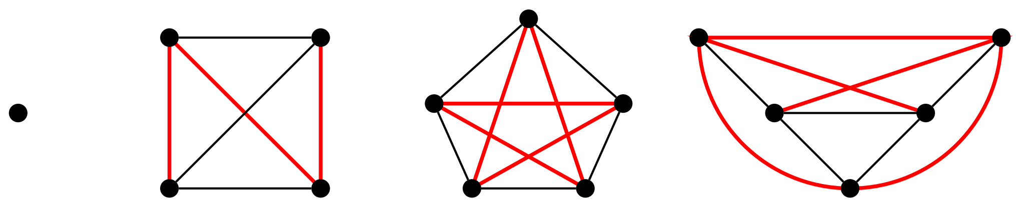
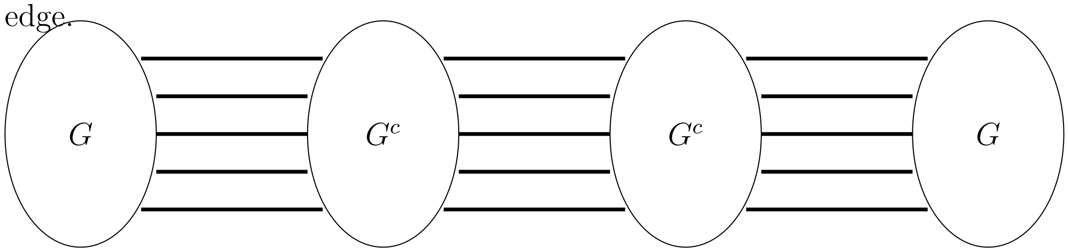


Figure 0.8: Every self-complementary graph with at most seven vertices. ANZ.

Lemma. *A self-complementary graph on n vertices must have $\frac{\binom{n}{2}}{2}$ edges.*

Corollary. *There are no self-complementary on $4k+2$ or $4k+3$ vertices for all $k = 0, 1, 2, \ldots$.*

The sequence of number of non-isomorphic graphs on n vertices for $n = 1, 4, 5, 8, 9, 12, 13, 16...$ is as follows: $1, 1, 2, 10, 36, 720, 5600, 703760, \ldots$ For any graph G on n vertices the below construction produces a self-complementary graph on $4n$ vertices! For an example, look at the graph at the top of the first page. This was obtained by taking G to be the graph on two vertices with one edge.



Modify the above construction to give a self-complementary graph on $4n+1$ vertices.

Figure 3: The 10 sc-graphs on 8 vertices