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Book Reviews

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Featured Review: Nonlocal Modeling, Analysis, and Computation. *By Qiang Du.* SIAM, Philadelphia, 2019. $59.99. xiv+166 pp., softcover. ISBN 978-1-611975-61-1. https://doi.org/10.1137/1.9781611975628.

The monograph *Nonlocal Modeling, Analysis, and Computation* could definitely be a set text for anybody who wants to study nonlocal models. It is the perfect starting point to getting a taste of their intrinsic features and thereby to expanding one's spectrum of modeling tools in any particular application field. I wish I'd had this book three years ago when I started my Ph.D. project on nonlocal models.

Who Is the Author? Qiang Du is the Fu Foundation Professor of Applied Mathematics at Columbia University and an elected fellow of SIAM, AAAS, and AMS. He obtained his Ph.D. under the supervision of Max Gunzburger, who together with Richard Lehoucq first introduced the framework of a nonlocal vector calculus [7]. He is also connected with the Sandia National Laboratories (see, e.g., [6, 3, 8, 4]), where computational expertise in nonlocal models is concentrated and where Steward Silling developed the peridynamics model [9], a nonlocal continuum model for solid mechanics, which initiated an increased focus on nonlocality in the early 2000s. Also, Qiang Du supervised Xiaochuan Tian, who wrote an awarded dissertation on nonlocal models [10]. Thus, this book comes straight from the heart of nonlocal research.

Who Is Addressed? The author aims to address not only advanced researchers in applied mathematics, but also younger researchers from various application fields. With his very nice and infectious introduction to the topic he truly spans a great range of readers. Beyond this introduction, the content grows in its mathematical complexity so that more profound mathematical knowledge is needed. However, mathematical details are omitted whenever they would distract from the main ideas, though mathematical rigor and proofs are available whenever needed to explain the math.

What Is in There? When considering nonlocality, one of its most celebrated properties is the ability to accurately capture singular and anomalous behavior without exogenously imposing further conditions on the model. The author certainly addresses this feature. However, to the best of my knowledge, this is also the first mathematical book to have a focus on nonlocal models with a *finite range of nonlocal interactions*. By considering this particular feature as an additional model parameter, the book stands out from other notable books and surveys such as, e.g., [1]. As the reader learns, treating the interaction horizon as a model parameter allows for the transition from differential equations (infinitely small horizon), through nonlocal continuum mechanics and discrete models (finite horizon), to fractional equations (infinitely large horizon). Despite the deep mathematical results presented in this book,

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its content is largely driven by applications with a strong emphasis on the peridynamics model. The latter leads to a second distinct feature of this book: the rigorous study of vector-valued nonlocal equations.

What Precisely Is in There? Let me now comment in more detail on the content and thereby try to establish a correspondence between the title (*what the reader expects*) and its seven chapters (*what the reader gets*). The nonlocal operators considered in this book are mainly of the form

$$-\mathcal{L}\mathbf{u}(\mathbf{x}) = \int \big(\mathbf{u}(\mathbf{x})\gamma_\delta(\mathbf{x},\mathbf{y}) - \mathbf{u}(\mathbf{y})\gamma_\delta(\mathbf{y},\mathbf{x})\big)d\mathbf{y},$$

where the quantity of interest $\mathbf{u}\colon \mathbb{R}^d \to \mathbb{R}^k$ may be vector-valued and $\gamma_\delta\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a nonlocal interaction kernel having support within a neighborhood depending on the so-called interaction horizon $\delta > 0$, thus accounting for a *finite range of interactions.*

Modeling: Chapters 1 and 2. From the early pages on, the author impresses with a very clear classification of the material. In these two chapters he motivates the notion of nonlocality and introduces a large list of applications and instances in which we are confronted with nonlocality, such as model order reduction and coarse graining, stochastic jump processes and nonlocal diffusion, nonlocal balance laws and nonlocal action-reaction principles, continuum limits of discrete operators (such as Laplace–Beltrami and graph Laplacian), and smoothed-particle hydrodynamics (SPH). A highlight of Chapter 2 is surely the introduction to the peridynamics model. Among others, the reader is familiarized with the linear Navier peridynamic model and the bond-based peridynamic model. For me, having no background in mechanics, Qiang Du has provided a very understandable introduction to this topic.

After these two chapters the author leaves no doubt about the need for further investigations that strengthen our understanding of nonlocality. This leads the reader to the next part of the book.

Analysis: Chapter 3. This part contains a lot of math and proofs, so that a background in functional analysis is of great help. However, with peridynamics in mind, the author often illuminates analytical results with a mechanical interpretation. This chapter aims at providing a systematic framework which facilitates the mathematical analysis of nonlocal models. Thus, it contains an introduction to the nonlocal vector calculus. A nonlocal divergence and a nonlocal gradient operator are defined, and related integral identities (nonlocal Gauss theorem and Green's theorems) are presented. A highlight is certainly the rigorous investigation of nonlocal spaces of vector fields and a connection is drawn to works by Bourgain, Brezis, and Mironescu [2] and Ponce [5]. Further, variational problems on bounded domains are considered, and a nonlocal Poincaré–Korn type inequality serves as the key to well-posedness results.

Some necessary analytical results having been established, it is now time to compute numerical solutions.

Computation: Chapter 4. Let me first point out that the reader will not find implementation details here. In contrast, Du purposely chooses to focus on the robustness study of numerical discretization methods. This is due to the alarming fact that under certain choices of interaction horizon and discretization parameter, different discretization approaches can lead to different solutions. He presents an abstract framework for parametrized variational equations and introduces the notion of asymptotically compatible schemes. By casting nonlocal models (parametrized with the interaction horizon) into this abstract framework, the author derives results about the local limit ($\delta \to 0$) and the fractional limit ($\delta \to \infty$). All in all, this chapter guides the user in the development of robust numerical discretization schemes.

The reader being convinced that nonlocal models are computationally challenging, the author now turns back to the modeling aspect: Why not couple nonlocal models with local models in order to make use of their computational ease?

Back to Modeling: Chapters 5 and 6. Several nonlocal-to-local coupling strategies have already been discussed in the literature. Chapter 5 outlines a coupling approach based on heterogeneous localization, i.e., vanishing nonlocality toward the boundary. However, this necessitates a new type of trace theorem, which is a highlight of this book.

Chapter 6 then presents results on time-dependent diffusion equations which are modeled nonlocally in time. Here, a stochastic perspective comes into play. With the help of the interaction horizon as a modeling parameter, the reader learns how to establish a crossover between anomalous and normal diffusion.

"Think Nonlocally, Act Locally": Chapter 7. The book ends with a list of open problems, thereby inviting the reader to try their hand at nonlocal modeling. I like this part because it again serves as a very good overview and also clarifies the current state of the art.

Bibliography. The remarkably long list of 280 references is truly impressive. In combination with the content, which embeds these references into the whole picture, this list is more than valuable.

My Conclusion. The present monograph should not be regarded as a detailed textbook teaching all aspects of the subject, since it is intended to be a "concise introduction." Beyond Chapter 2 it is research oriented with a bias to the (many) aspects that the author has researched. However, most missing details or aspects are covered by the immense list of references. Each chapter is nearly self-contained, which makes this book very accessible. All in all, I highly recommend it.

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