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Weak Solutions for the Cahn–Hilliard Equation with Degenerate Mobility

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Abstract

In this paper, we study the well-posedness of Cahn–Hilliard equations with degenerate phase-dependent diffusion mobility. We consider a popular form of the equations which has been used in phase field simulations of phase separation and microstructure evolution in binary systems. We define a notion of weak solutions for the nonlinear equation. The existence of such solutions is obtained by considering the limits of Cahn–Hilliard equations with non-degenerate mobilities.

1. Introduction

The Cahn–Hilliard equation

$$\partial_t u = \nabla \cdot (M(u)\nabla\mu) \quad \text{for } x \in \Omega \subset \mathbb{R}^n,\ t \in [0,\infty) \tag{1.1}$$

$$\mu := -\kappa\Delta u + W'(u) \tag{1.2}$$

is a widely used phenomenological diffuse-interface model for phase separation in binary systems [5,6]. Here u is the relative concentration of the two phases, $W(u)$ is a double-well potential with two equal minima at $u^- < u^+$ corresponding to the two pure phases, and $\kappa > 0$ is a parameter whose square root, $\sqrt{\kappa}$, is proportional to the thickness of the transition region between the two phases. The diffusion mobility $M(u)$ is nonnegative and generally depends on u. When the system (1.1)–(1.2) is coupled with either the Neumann type boundary condition $\partial_{\mathbf{n}} u = \partial_{\mathbf{n}}\mu = 0$

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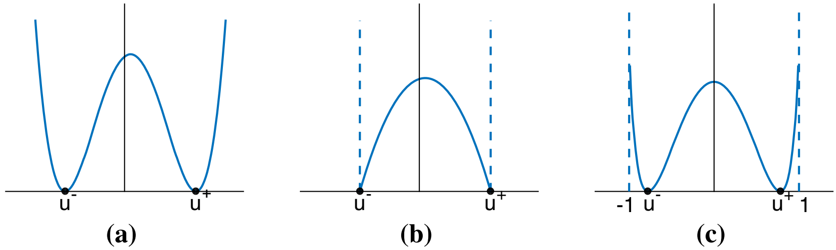

Fig. 1. a A smooth double well potential; **b** a double-barrier potential; **c** a logarithmic potential

or periodic boundary conditions with Ω being the periodic cell, formally the model dissipates the free energy defined by

$$E(u) = \int_{\Omega} \left\{ \frac{\kappa}{2} |\nabla u|^2 + W(u) \right\} \mathrm{d}x. \tag{1.3}$$

In this paper we will concentrate on periodic boundary conditions.

To model the two-phase system, different forms have been used for the double-well potential $W(u)$. The potential may be smooth, such as

$$W(u) = (u - u^+)^2 (u - u^-)^2, \tag{1.4}$$

as depicted in Fig. 1a. This form is simple and convenient for numerical simulations and theoretical analysis, and hence widely used for phase field modeling. Another choice is to put infinite barriers outside of $[u^-, u^+]$ [3], such as

$$W_{\mathrm{db}}(u) = \begin{cases} \frac{1}{2}(u^+ - u)(u - u^-) & \text{if } u^- \leqq u \leqq u^+, \\ \infty & \text{otherwise}, \end{cases} \tag{1.5}$$

as depicted in Fig. 1b. W_{db} does have two equal minima at $u^\pm$ but it has singularities at $u^\pm$, which generate additional difficulties for numerical and theoretical analysis.

The third choice of the potential $W(u)$ has a background from statistical mechanics and involves the logarithms of the relative concentrations due to thermodynamic entropy considerations [5]. It is written as

$$W_{\log}(u) = \frac{\theta}{2}((1+u)\ln(1+u) + (1-u)\ln(1-u)) + \frac{1}{2}(1-u^2), \tag{1.6}$$

and is depicted in Fig. 1c. Here $\theta > 0$ is a parameter representing the temperature of the system. When θ is sufficiently small, $W_{\log}$ has two equal minima at $u^\pm = \pm(1 - \mathcal{R}(\theta))$, where $\mathcal{R}(\theta) > 0$ and approaches zero as $\theta \to 0$. Similar to smooth potentials, $W_{\log}$ is smooth at its minimizers $u^\pm$ but $W_{\log}(u)$ is only defined for $u \in [-1, 1]$, and $\lim_{u \to 1^-} W'_{\log}(u) = +\infty$, and $\lim_{u \to (-1)^+} W'_{\log}(u) = -\infty$, which effectively form infinite barriers at ± 1.

The modeling of the dependence of the diffusion mobility $M(u)$ on the phase concentration u presents additional challenges, especially when $M(u)$ is degenerate in the two pure phases $u^\pm$. For instance, $M(u)$ may take the form [7,11,14]

$$M(u) = |(u - u^+)(u - u^-)|^m \quad \text{for all } u \in \mathbb{R} \tag{1.7}$$

for some $m > 0$. Due to the degeneracy, it has been conjectured that there is no diffusion in the two phases and that the dynamics are governed by diffusion in the transition region [7,14]. In addition, if the initial value of u lies inside $[u^-, u^+]$, then the solution u is believed to remain in $[u^-, u^+]$ for all time.

The non-existence of diffusion in the bulk phases was confirmed by formal asymptotic analysis in [4] in the case when M is of the form (1.7) and W is a double-barrier potential (1.5). As for the boundedness of the solution, it was shown in [10] (see [15] for the 1D case) that, adopting the notation used here, for $M(u)$ given by (1.7) with $m \geqq 1$ and with $\|W\|_{C^2[u^-,u^+]}$ being finite, there exists a weak solution for (1.1)–(1.2) that is bounded in $[u^-, u^+]$ for all time, provided the initial value is in $[u^-, u^+]$ and satisfies some energy conditions. Rather than directly working on $M(u) = |(u - u^+)(u - u^-)|^m$ for all $u \in \mathbb{R}$, the above result of [10] was obtained by considering a cut off in the mobility

$$M_c(u) = \begin{cases} (u^+ - u)^m (u - u^-)^m & \text{if } u \in [u^-, u^+], \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

The cutoff in the mobility pairs perfectly with a double-barrier potential such as (1.5). However, for a potential that is smooth at $u^\pm$, there is a lack of physical justification to impose the cut-off and the weak solution derived in [10] may not be compatible with the Gibbs–Thomson effect. Indeed, the Gibbs–Thomson effect says the concentration of “pure” phases can only be achieved when the interface has zero mean curvature. The concentration of a phase inside small particles of high mean curvature is higher than that of the corresponding pure phase, due to excessive surface tension. This means, mathematically, that if a model accommodates the Gibbs–Thomson effect, then the relative concentration u may not remain inside $[u^-, u^+]$, as long as the interface separating the two phases has nonzero mean curvature. An argument can be made from an energetic point of view that, since there is no barrier at $u^\pm$, one may not be able to exclude perturbations that cause u to go outside of $[u^-, u^+]$.

Remark 1. Special attention is needed for the logarithmic potential (1.6). Due to its singularities at ± 1, it is paired with a mobility

$$M_{\log}(u) = \begin{cases} 1 - u^2 & \text{if } u \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

$M_{\log}$ is indeed degenerate at ± 1, but not at $u^\pm = \pm(1 - \mathcal{R}(\theta))$. It was shown by formal asymptotic analysis in [4] that in the limit when $\theta \to 0$, the motion of the interface is determined by surface diffusion. This is consistent with the fact that $u^\pm \to \pm 1$ as $\theta \to 0$ and $W_{\log}$ approaches the double barrier potential

$$W_{\mathrm{db}}(u) = \begin{cases} \frac{1}{2}(1 - u^2) & \text{if } u \in [-1, 1], \\ +\infty & \text{otherwise.} \end{cases}$$

It was also proved in [10] that there is a weak solution for (1.1)–(1.2) with logarithmic potential $W_{\log}$ and mobility $M_{\log}$, and the weak solution is bounded in $[-1, 1]$, consistent with the barriers at ± 1, but u is not necessarily confined in $[u^-, u^+]$, hence the model accommodates the Gibbs–Thomson effect.

In the case that the double-well potential W is smooth at $u^{\pm}$, it was claimed in [7] that if $M(u)$ is degenerate at $u^{\pm}$, then u remains bounded in $[u^-, u^+]$ but the system does not allow pure phases to exist and hence there is indeed diffusion in the two bulk phases. Our formal results in [9] (see also [8] for the case when the diffusion mobility is degenerate in one phase) show that, if the double-well potential W is smooth at $u^{\pm}$, when the diffusion mobility is of the form (1.7), for a sufficiently smooth solution u, even if its initial value is in $[u^-, u^+]$, later on u will not remain inside $[u^-, u^+]$, as long as the interface has nonzero mean curvature. In addition, there exists a nontrivial porous medium diffusion process in the two phases, which is a consequence of the curvature effect.

Hence, we expect the Cahn–Hilliard equation (1.1)–(1.2) to accommodate the physical Gibbs–Thomson effect, when the double well potential $W(u)$ is smooth at $u^{\pm}$ and the mobility of the form (1.7). However, a related mathematical question to be answered is whether the Cahn–Hilliard equation is well-posed with such choices of the potential and mobility.

1.1. Main Result

The purpose of this paper is to show the existence of a weak solution for (1.1)–(1.2), with a smooth double well potential such as (1.4), and a degenerate mobility (1.7). More precisely, we want to prove the existence of a weak solution that potentially allows the Gibbs–Thomson effect, and hence is different from the one derived in [10]. Due to the degeneracy in $M(u)$, there may exist more than one weak solution.

For the existence result in this paper, it is not the minimization of the potential W at $u^{\pm}$, but the degeneracy of mobility at $u^{\pm}$, that presents the technical difficulties. In the rest of the paper, to simplify notations we will assume

$$u^{\pm} = \pm 1 \quad \text{and} \quad W(u) \text{ is smooth at } \pm 1. \tag{1.10}$$

Consequently we will assume

$$M(u) = |1 - u^2|^m \quad \text{for all } u \in \mathbb{R}. \tag{1.11}$$

The exact form of the potential $W(u)$ is not essential since all we need is its smoothness and some growth conditions as $|u| \to \infty$. In fact, the exact form of $M(u)$ away from the degenerate points ± 1 is not essential either, except for growth conditions as $|u| \to \infty$. However, in applications, $M(u)$ is likely to be of a simple form as (1.11), for simplicity, we assume $M(u) \sim C(|u|^{2m} + 1)$ as $|u| \to \infty$. We will carry out the analysis for $M(u)$ of the form (1.11) but in fact we only need the following conditions on $M(u)$ and $W(u)$:

(i) $M(u) \in C(\mathbb{R}; [0, \infty))$ and there exist $\delta > 0$ and $c_0 > 0$ such that $M(u) = |1-u^2|^m$ for $u \in B_\delta(\pm 1) := (-1-\delta, -1+\delta) \cup (1-\delta, 1+\delta)$ and $M(u) \geqq c_0 > 0$ for $u \in \mathbb{R} \setminus B_\delta(\pm 1)$. In addition, there exist $M_1, M_2 > 0$ such that

$$0 \leqq M(u) \leqq M_1 |u|^{2m} + M_2 \quad \text{for all } u \in \mathbb{R}. \tag{1.12}$$

m can be any positive number $0 < m < \infty$ if $n = 1, 2$ and we require $0 < m < \frac{2}{n-2}$ if $n \geqq 3$.

(ii) $W(u) \in C^2(\mathbb{R};\mathbb{R})$ and there exist $C_i > 0, i = 1, \ldots, 10$ such that for all $u \in \mathbb{R}$

$$C_1|u|^{r+1} - C_2 \leqq W(u) \leqq C_3|u|^{r+1} + C_4, \tag{1.13}$$
$$|W'(u)| \leqq C_5|u|^r + C_6, \tag{1.14}$$
$$C_7|u|^{r-1} - C_8 \leqq W''(u)| \leqq C_9|u|^{r-1} + C_{10} \tag{1.15}$$

for some $1 \leqq r < \infty$ if $n = 1, 2$ and $1 \leqq r \leqq \frac{n}{n-2}$ if $n \geqq 3$.

Remark 2. The conventional double well potential $W(u) = \frac{1}{4}(1-u^2)^2$ gives $r = 3$. This satisfies the requirements (1.13)–(1.15) for $n = 1, 2$ and 3.

Our analysis involves two steps. The first step is to approximate the degenerate mobility $M(u) = |1-u^2|^m$ by a non-degenerate one $M_\theta(u)$ defined for a $\theta > 0$ by

$$M_\theta(u) := \begin{cases} |1-u^2|^m & \text{if } |1-u^2| > \theta, \\ \theta^m & \text{if } |1-u^2| \leqq \theta. \end{cases} \tag{1.16}$$

The uniform lower bound of $M_\theta(u)$ enables us to find a sufficiently regular weak solution for (1.1)–(1.2) with a mobility $M_\theta(u)$ and a smooth potential W.

Theorem 1. *Under assumptions (1.12)–(1.15), for any $u_0 \in H^1(\Omega)$ and $T > 0$, there exists a function u_θ such that*

(1) $u_\theta \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^p(\Omega)) \cap L^2(0,T;H^3(\Omega))$, *where* $1 \leqq p < \infty$ *if* $n = 1, 2$ *and* $1 \leqq p < 2n/(n-2)$ *if* $n \geqq 3$,
(2) $\partial_t u_\theta \in L^2(0,T;(H^2(\Omega))')$,
(3) $u_\theta(x,0) = u_0(x)$ *for all* $x \in \Omega$

which satisfies the Cahn–Hilliard equation in the following weak sense

$$\begin{aligned}&\int_0^T \langle \partial_t u_\theta,\ \phi\rangle_{(H^2(\Omega))',H^2(\Omega)}\,\mathrm{d}t \\ &\quad = -\int_0^T\int_\Omega M_\theta(u_\theta)\Big(-\kappa\nabla\Delta u_\theta + W''(u_\theta)\nabla u_\theta\Big)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{1.17}$$

for all $\phi \in L^2(0,T;H^2(\Omega))$. In addition, the following energy inequality holds for any $t > 0$.

$$\begin{aligned}&\int_\Omega \Big(\frac{\kappa}{2}|\nabla u_\theta(x,t)|^2 + W(u_\theta(x,t))\Big)\,\mathrm{d}x \\ &\quad + \int_0^t\int_\Omega M_\theta(u_\theta(x,\tau))| - \kappa\nabla\Delta u_\theta(x,\tau) + W''(u_\theta(x,\tau))\nabla u_\theta(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqq \int_\Omega \Big(\frac{\kappa}{2}|\nabla u_0|^2 + W(u_0)\Big)\,\mathrm{d}x.\end{aligned} \tag{1.18}$$

The second step is to consider the limit of u_θ as $\theta \to 0$. The limiting value u, of the functions u_θ, does exist and, in a weak sense, solves the Cahn–Hilliard equation (1.1)–(1.2) coupled with the mobility $M(u)$ and smooth potential $W(u)$. It can be interpreted as that u solves the Cahn–Hilliard equation in any open set $U \subset \Omega_T := \Omega \times (0,T)$ where u has enough regularity, namely where $\nabla\Delta u \in L^q(U)$ for some $q > 1$. As for the singular set where $\nabla\Delta u$ fails to satisfy such a regularity condition, the singular set is contained in the set where $M(u)$ is degenerate, plus another set of Lebesgue measure zero.

Theorem 2. *Under assumptions (1.12)–(1.15), for any $u_0 \in H^1(\Omega)$ and $T > 0$, there exists a function $u : \Omega_T \to \mathbb{R}$ satisfying*

(1) $u \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^p(\Omega))$, *where* $1 \leqq p < \infty$ *if* $n = 1, 2$ *and* $1 \leqq p < 2n/(n-2)$ *if* $n \geqq 3$,
(2) *if* $0 < m < 1$, *we have additional regularity* $u \in L^2(0,T;H^2(\Omega))$,
(3) $\partial_t u \in L^2(0,T;(H^2(\Omega))')$,
(4) $u(x,0) = u_0(x)$ *for all* $x \in \Omega$,

which can be considered as a weak solution for the Cahn–Hilliard equation in the following sense.

(i) *Define P to be the set where $M(u)$ is not degenerate, that is*:

$$P := \{(x,t) \in \Omega_T : |1 - u^2| \neq 0\}. \tag{1.19}$$

There exist a set $B \subset \Omega_T$ with $|\Omega_T \backslash B| = 0$ and a function $\zeta : \Omega_T \to \mathbb{R}^n$ satisying $\chi_{B\cap P} M(u)\zeta \in L^2(0,T;L^{2n/(n+2)}(\Omega;\mathbb{R}^n))$, here $\chi_{B\cap P}$ is the characteristic function of $B \cap P$, such that

$$\int_0^T \langle \partial_t u, \phi \rangle_{(H^2(\Omega))', H^2(\Omega)} \,\mathrm{d}t = -\int_{B\cap P} M(u)\zeta \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \tag{1.20}$$

for all $\phi \in L^2(0,T;H^2(\Omega))$.

(ii) *Let $\nabla\Delta u$ be the generalized derivative of u in the sense of distributions. If $\nabla\Delta u \in L^q(U)$ for some open subset $U \subset \Omega_T$ and some $q > 1$, then we have*

$$\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u \quad \text{in } U. \tag{1.21}$$

In addition, the following energy inequality is satisfied for all $t > 0$.

$$\begin{aligned}&\int_\Omega \Big(\frac{\kappa}{2}|\nabla u(x,t)|^2 + W(u(x,t))\Big)\,\mathrm{d}x + \int_{\Omega_t\cap B\cap P} M(u(x,\tau))|\zeta(x,\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad \leqq \int_\Omega \Big(\frac{\kappa}{2}|\nabla u_0|^2 + W(u_0)\Big)\,\mathrm{d}x.\end{aligned} \tag{1.22}$$

Remark 3. Our definition of weak solution was motivated in part by the study of one-dimensional degenerate parabolic equations [1,2]. In our case, due to the lack of regularity in higher space dimensions, we have to resort to different techniques and modify the formulation accordingly. Further study is needed to explore the regularity of our weak solution.

Remark 4. In the convergence results for the regularized solution u_θ and the corresponding chemical potential $\mu_\theta := -\kappa\Delta u_\theta + W'(u_\theta)$, a key challenge is the convergence of $\nabla\mu_\theta$. We can show that, after extracting subsequences,

$$M_\theta(u_\theta)\nabla\mu_\theta \rightharpoonup \sqrt{M(u)}\xi \quad \text{weakly in } L^2(0,T;L^{2n/(n+2)}(\Omega))$$

for some $\xi \in L^2(\Omega_T)$. Consequently we have, for all $\phi \in L^2(0,T;H^2(\Omega))$,

$$\int_0^T \langle \partial_t u, \phi\rangle_{(H^2(\Omega))',H^2(\Omega)}\,\mathrm{d}t = -\int_0^T\int_\Omega \sqrt{M(u)}\xi\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{1.23}$$

Ideally we want

$$\sqrt{M(u)}\xi = M(u)(-\kappa\nabla\Delta u + W''(u)\nabla u),$$

under which (1.23) becomes a weak form of the Cahn–Hilliard equation. Generally this is too much to ask for due to the degeneracy in the set where $u = \pm 1$. We can show that this is almost true in the set where $u \neq \pm 1$. More precisely, let P be the set defined by (1.19); then there exists a set B with $|\Omega_T\backslash B| = 0$, a sequence of increasing sets $\{D_j\}_{j=1}^\infty$ whose limit is $B\cap P$, and a function $\zeta\colon \Omega_T \to \mathbb{R}^n$ satisfying $\chi_{B\cap P}M(u)\zeta \in L^2(0,T;L^{2n/(n+2)}(\Omega;\mathbb{R}^n))$; and

$$\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u$$

in the interior of every D_j and every open set $U \subset \Omega_T$ in which $\nabla\Delta u \in L^q(U)$ for some $q > 1$. In addition, for any $\Psi \in L^2(0,T;L^{2n/(n-2)}(\Omega;\mathbb{R}^n))$,

$$\int_0^T\int_\Omega \sqrt{M(u)}\xi\cdot\Psi\,\mathrm{d}x\,\mathrm{d}t = \int_{B\cap P} M(u)\zeta\cdot\Psi\,\mathrm{d}x\,\mathrm{d}t.$$

Consequently for any $\phi \in L^2(0,T;H^2(\Omega))$,

$$\int_0^T \langle \partial_t u, \phi\rangle_{(H^2(\Omega))',H^2(\Omega)}\,\mathrm{d}t = -\int_{B\cap P} M(u)\zeta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t.$$

The formulation of a weak solution in Theorem 2 is different from that in [10]. Our formulation works for any degenerate mobility $M(u)$ and smooth double–well potential W satisfying the growth conditions (1.12)–(1.15). In contrast, the weak formulation in [10] involves the derivative of $M(u)$ and requires $m \geqq 1$ and W is required to grow at most quadratically as $|u| \to \infty$. With the cut-off of mobility as represented by (1.8) or (1.9), Elliott and Garcke [10] is able to take advantage of the boundedness of the mobility.

In addition to the different formulations, there are other significant differences between our result in Theorem 2 and that in [10]. We choose the domain Ω to be a periodic box, which simplifies the technical presentations with the use of a Fourier series approximation. This approach is similar to the Galerkin approximation used in [10], in which the basis functions are the eigenfunctions of the Laplace operator with Neumann boundary conditions. For our case, we take advantage of the fact that the Fourier series of $\phi \in H^s(\Omega)$ converges to ϕ in H^s norms for $s \geqq 0$ and in particular $s = 2$. Moreover and more importantly, we do not require the initial value u_0 to be in $[u^-, u^+]$ and consequently do not guarantee the solution u to be in $[u^-, u^+]$. This provides the opportunity for the weak solution to satisfy the Gibbs–Thomson effect in space dimension $n \geqq 2$. For example, any sufficiently smooth critical point of the Cahn–Hilliard energy (1.3) belongs to the set of steady solutions, in the sense of Theorem 2, for the degenerate Cahn–Hilliard equation. Specifically, this set includes all steady solutions for the Cahn–Hilliard equation with a constant mobility, which do not necessarily lie in $[u^-, u^+]$ for $n \geqq 2$.

The structure of this paper is as follows. Sections 2 and 3 are devoted, respectively, to the proofs of Theorems 1 and 2. In Section 4 we have a discussion about the results and point out some open problems for further study.

2. Weak Solution for the C–H Equation with Positive Mobilities

In this section we prove Theorem 1. For simplicity we take $\Omega = [0, 2\pi]^n$. Write $\mathbb{Z}_+$ as the set of nonnegative integers. Then

$$\{(2\pi)^{-n/2},\ \mathrm{Re}(\pi^{-n/2}e^{i\xi\cdot x}),\ \mathrm{Im}(\pi^{-n/2}e^{i\xi\cdot x})\colon \xi \in \mathbb{Z}_+^n \backslash \{(0,\dots,0)\}\}$$

form a complete orthonormal basis for $L^2(\Omega)$ that are also orthogonal in $H^k(\Omega)$ for any $k \geqq 1$. Let us label the basis as $\{\phi_j\colon\ j = 1, 2, \dots\}$, with $\phi_1 = (2\pi)^{-n/2}$.

2.1. Galerkin Approximation

The Sobolev embedding theorem says that for $n = 2$, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $1 \leqq p < \infty$ and the embedding is compact; while for $n \geqq 3$, $H^1(\Omega) \hookrightarrow L^p(\Omega)$for $1 \leqq p \leqq 2^* := \frac{2n}{n-2}$ and the embedding is compact if $1 \leqq p < 2^*$. Due to this difference, the case $n \geqq 3$ is more complicated since we have to keep track of the critical exponent 2^*. For this reason we will only write down the proof for $n \geqq 3$, and the proof for $n = 2$ is similar and indeed simpler.

Define

$$u^N(x,t) = \sum_{j=1}^{N} c_j^N(t)\phi_j(x), \quad \mu^N(x,t) = \sum_{j=1}^{N} d_j^N(t)\phi_j(x).$$

We want them to solve the following system of equations for $j = 1, \dots, N$:

$$\int_\Omega \partial_t u^N \phi_j \,\mathrm{d}x = -\int_\Omega M_\theta(u^N)\nabla\mu^N \cdot \nabla\phi_j \,\mathrm{d}x \tag{2.24}$$

$$\int_\Omega \mu^N \phi_j \,\mathrm{d}x = \int_\Omega (\kappa\nabla u^N \cdot \nabla\phi_j + W'(u^N)\phi_j)\mathrm{d}x, \tag{2.25}$$

$$u^N(x,0) = \sum_{j=1}^{N} \left(\int_\Omega u_0\phi_j \,\mathrm{d}x\right)\phi_j(x). \tag{2.26}$$

This is a system of ordinary differential equations for $\{c_j^N(t)\}_{j=1}^N$. Since the right hand side of (2.24) is continuous in c_j, the system has a local solution.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u^N(x,t)) = -\int_\Omega M_\theta(u^N)|\nabla\mu^N|^2\,\mathrm{d}x,$$

integration in time gives the following energy identity:

$$\begin{aligned}
&\int_\Omega \left(\frac{\kappa}{2}|\nabla u^N(x,t)|^2 + W(u^N(x,t))\right)\mathrm{d}x\\
&\quad + \int_0^t\int_\Omega M_\theta(u^N(x,\tau))|\nabla\mu^N(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&= \int_\Omega \left(\frac{\kappa}{2}|\nabla u^N(x,0)|^2 + W(u^N(x,0))\right)\mathrm{d}x\\
&\leqq \int_\Omega \left(\frac{\kappa}{2}|\nabla u_0|^2 + C(|u^N(x,0)|^{r+1}+1)\right)\mathrm{d}x \quad \text{for } 1\leqq r\leqq \frac{n}{n-2} \text{ by (1.13)}\\
&\leqq \frac{\kappa}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + C(\|u^N(\cdot,0)\|^{r+1}_{H^1(\Omega)} + |\Omega|) \quad \text{by } H^1(\Omega)\hookrightarrow L^{r+1}(\Omega)\\
&\leqq \frac{\kappa}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + C(\|u_0\|^{r+1}_{H^1(\Omega)} + |\Omega|) \leqq C < \infty.
\end{aligned}\tag{2.27}$$

Here and below, we use $C > 0$ to denote a generic constant that may depend on $n, T, \Omega, \kappa, u_0$ and the growth conditions of W, M but nothing else, in particular not on the lower bound θ^m of the mobility $M_\theta(u)$. We want to obtain uniform boundedness of u^N from (2.27).

Taking $j = 1$ in (2.24) gives $\int_\Omega \partial_t u^N\,\mathrm{d}x = 0$. Thus $\int_\Omega u^N(x,t)\,\mathrm{d}x = \int_\Omega u^N(x,0)\mathrm{d}x$. Define Π_N as the L^2 projection operator from $L^2(\Omega)$ into $\mathrm{span}\{\phi_j\}_{j=1}^N$, that is, $\Pi_N\phi := \sum_{j=1}^N \left(\int_\Omega \phi\phi_j\,\mathrm{d}x\right)\phi_j$. Then, for any $0 < t < T$,

$$\begin{aligned}
\left|\int_\Omega u^N(x,t)\mathrm{d}x\right| = \left|\int_\Omega u^N(x,0)\,\mathrm{d}x\right| = \left|\int_\Omega \Pi_N u_0\,\mathrm{d}x\right|\\
\leqq \|\Pi_N u_0\|_{L^2(\Omega)}|\Omega|^{1/2} \leqq \|u_0\|_{L^2(\Omega)}|\Omega|^{1/2}.
\end{aligned}\tag{2.28}$$

Combined with (2.27) and (1.13), Poincaré's inequality implies

$$u^N \in L^\infty(0,T;H^1(\Omega))$$

and

$$\|u^N\|_{L^\infty(0,T;H^1(\Omega))} \leqq C \quad \text{for all } N. \tag{2.29}$$

Also, (2.27) implies

$$\|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(\Omega_T)} < C \quad \text{for all } N. \tag{2.30}$$

By the Sobolev embedding theorem and the growth conditions (1.12), (1.14), we have

$$W'(u^N) \in L^\infty(0,T;L^2(\Omega)), \quad M_\theta(u^N) \in L^\infty(0,T;L^{n/2}(\Omega))$$

and for all N,

$$\|W'(u^N)\|_{L^\infty(0,T;L^2(\Omega))} \leqq C, \tag{2.31}$$

$$\|M_\theta(u^N)\|_{L^\infty(0,T;L^{n/2}(\Omega))} \leqq C. \tag{2.32}$$

By (2.29), the coefficients $\{c_j^N : j = 1, 2, \dots, N\}$ are bounded in time and there is a global solution for the system (2.24)–(2.26).

2.2. *Convergence of* $\{u^N\}$

For any $\phi \in L^2(0, T; H^2(\Omega))$, write $\Pi_N \phi(x, t) = \sum_{j=1}^N a_j(t)\phi_j(x)$. Then,

$$\begin{aligned}
\left|\int_\Omega \partial_t u^N \phi \,\mathrm{d}x\right| &= \left|\int_\Omega \partial_t u^N \Pi_N \phi \,\mathrm{d}x\right| = \left|\int_\Omega M_\theta(u^N)\nabla\mu^N \cdot \nabla\Pi_N\phi \,\mathrm{d}x\right| \\
&\leqq \|\sqrt{M_\theta(u^N)}\|_{L^n(\Omega)} \, \|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(\Omega)} \, \|\nabla\Pi_N\phi\|_{L^{2n/(n-2)}} \\
&\leqq C\|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(\Omega)} \, \|\phi\|_{H^2(\Omega)} \quad \text{by (2.32)}.
\end{aligned}$$

Then, by (2.30),

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \partial_t u^N \phi \,\mathrm{d}x \,\mathrm{d}t\right| \\
&\leqq C\left(\int_0^T \int_\Omega M_\theta(u^N)|\nabla\mu^N|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_0^T \|\phi\|_{H^2(\Omega)}^2 \,\mathrm{d}t\right)^{1/2} \\
&\leqq C\|\phi\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned} \tag{2.33}$$

Hence,

$$\|\partial_t u^N\|_{L^2(0,T;(H^2(\Omega))')} \leqq C \quad \text{for all } N. \tag{2.34}$$

Since the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $1 \leqq p < \infty$ if $n = 1, 2$ and $1 \leqq p < 2n/(n-2)$ if $n \geqq 3$, and $L^p(\Omega) \hookrightarrow (H^2(\Omega))'$ is continuous (for $p \geqq 1$ if $n \leqq 3$, $p > 1$ for $n = 4$, and $p \geqq 2n/(n+4)$ if $n \geqq 5$), by the Aubin-Lions Lemma (see, for example [13]), the embeddings

$$\{f \in L^2(0, T; H^1(\Omega)) : \partial_t f \in L^2(0, T; (H^2(\Omega))')\} \hookrightarrow L^2(0, T; L^p(\Omega)),$$

and

$$\{f \in L^\infty(0, T; H^1(\Omega)) : \partial_t f \in L^2(0, T; (H^2(\Omega))')\} \hookrightarrow C([0, T]; L^p(\Omega))$$

are compact for the values of p indicated.

The above boundedness of $\{u^N\}$ and $\{\partial_t u^N\}$ enables us to find a subsequence, not relabeled, and $u_\theta \in L^\infty(0, T; H^1(\Omega))$ such that as $N \to \infty$

$$u^N \rightharpoonup u_\theta \quad \text{weakly-* in } L^\infty(0, T; H^1(\Omega)), \tag{2.35}$$

$$u^N \to u_\theta \quad \text{strongly in } C([0, T]; L^p(\Omega)), \tag{2.36}$$

$$u^N \to u_\theta \quad \text{strongly in } L^2(0, T; L^p(\Omega)) \text{ and almost everywhere in } \Omega \times (0, T), \tag{2.37}$$

$$\partial_t u^N \rightharpoonup \partial_t u_\theta \quad \text{weakly in } L^2(0, T; (H^2(\Omega))'), \tag{2.38}$$

where $1 \leqq p < 2n/(n-2)$ if $n \geqq 3$ and $1 \leqq p < \infty$ if $n = 1, 2$. In addition, we have the following bounds for u_θ:

$$\|u_\theta\|_{L^\infty(0,T;H^1(\Omega))} \leqq C, \tag{2.39}$$

$$\|\partial_t u_\theta\|_{L^2(0,T;(H^2(\Omega))')} \leqq C. \tag{2.40}$$

Since M_θ is continuous and $M_\theta(u^N) \leqq C(1 + |u^N|^{2m})$ and $\sqrt{M_\theta(u^N)} \leqq C(1 + |u^N|^m)$ for $0 < m < 2/(n-2)$, by (2.36) and the general dominated convergence theorem (see, for example, Theorem 17 of Section 4.4 on p. 92 of [12]),

$$M_\theta(u^N) \to M_\theta(u_\theta) \quad \text{strongly in } C([0,T]; L^{n/2}(\Omega)), \tag{2.41}$$

$$\sqrt{M_\theta(u^N)} \to \sqrt{M_\theta(u_\theta)} \quad \text{strongly in } C([0,T]; L^n(\Omega)). \tag{2.42}$$

By (2.37), since $|W'(u^N)| \leqq C(1 + |u^N|^r)$ for $r \leqq n/(n-2)$ when $n \geqq 3$, we have

$$W'(u^N) \to W'(u_\theta) \quad \text{strongly in } C([0,T]; L^q(\Omega)) \tag{2.43}$$

for $1 \leqq q < \infty$ if $n = 1, 2$ and $1 \leqq q < \frac{2n}{r(n-2)}$ if $n \geqq 3$. By (2.31), there exists a $w \in L^\infty(0,T; L^2(\Omega))$ such that $W'(u^N) \rightharpoonup w$ weakly-* in $L^\infty(0,T; L^2(\Omega))$. Combined with (2.43), it turns out $w = W'(u)$, hence

$$W'(u^N) \rightharpoonup W'(u_\theta) \quad \text{weakly-* in } L^\infty(0,T; L^2(\Omega)). \tag{2.44}$$

2.3. Weak Solution

Now we will use the condition that $M_\theta(u^N) \geqq \theta^m$. By (2.30),

$$\|\nabla \mu^N\|_{L^2(\Omega_T)} \leqq C\theta^{-m/2}.$$

Taking $j = 1$ in (2.25), by (2.31) we have

$$\left| \int_\Omega \mu^N \, dx \right| = \left| \int_\Omega W'(u^N) \, dx \right| \leqq C < \infty.$$

So, by Poincaré's inequality,

$$\|\mu^N\|_{L^2(0,T;H^1(\Omega))} \leqq C(\theta^{-m/2} + 1). \tag{2.45}$$

Hence we can find a subsequence of μ^N, not relabeled, and

$$\mu_\theta \in L^2(0,T; H^1(\Omega)),$$

such that

$$\mu^N \rightharpoonup \mu_\theta \quad \text{weakly in } L^2(0,T; H^1(\Omega)). \tag{2.46}$$

Combining (2.46) with (2.42), we have

$$\sqrt{M_\theta(u^N)}\nabla\mu^N \rightharpoonup \sqrt{M_\theta(u_\theta)}\nabla\mu_\theta \quad \text{weakly in } L^2(0,T; L^{2n/(n+2)}(\Omega)).$$

Noticing that $\sqrt{M_\theta(u^N)}\nabla\mu^N$ is bounded in $L^2(\Omega_T)$ by (2.30), we can extract a further sequence, not relabeled, such that the above weak convergence can be improved

$$\sqrt{M_\theta(u^N)}\nabla\mu^N \rightharpoonup \sqrt{M_\theta(u_\theta)}\nabla\mu_\theta \quad \text{weakly in } L^2(\Omega_T). \tag{2.47}$$

Consequently we have the following bound

$$\int_{\Omega_T} M_\theta(u_\theta)|\nabla\mu_\theta|^2 \,\mathrm{d}x\,\mathrm{d}t \leqq C < \infty. \tag{2.48}$$

Since $M_\theta(u^N)\nabla\mu^N = \sqrt{M_\theta(u^N)}\sqrt{M_\theta(u^N)}\nabla\mu^N$, for any $\Phi \in L^2(0,T;L^{2^*}(\Omega))$ where $2^* = 2n/(n-2)$,

$$\begin{aligned}
&\left|\int_0^T \int_\Omega (M_\theta(u^N)\nabla\mu^N - M_\theta(u_\theta)\nabla\mu_\theta)\Phi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&= \left|\int_0^T \int_\Omega (\sqrt{M_\theta(u^N)} - \sqrt{M_\theta(u_\theta)})\sqrt{M_\theta(u^N)}\nabla\mu^N\Phi \right.\\
&\qquad \left. +\sqrt{M_\theta(u_\theta)}(\sqrt{M_\theta(u^N)}\nabla\mu^N - \sqrt{M_\theta(u_\theta)}\nabla\mu_\theta)\Phi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leqq \int_0^T \|\sqrt{M_\theta(u^N)} - \sqrt{M_\theta(u_\theta)}\|_{L^n(\Omega)}\|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(\Omega)} \\
&\qquad \cdot\|\Phi\|_{L^{2^*}(\Omega)}\,\mathrm{d}t \\
&\qquad +\left|\int_{\Omega_T} (\sqrt{M_\theta(u^N)}\nabla\mu^N - \sqrt{M_\theta(u_\theta)}\nabla\mu_\theta)\sqrt{M_\theta(u_\theta)}\Phi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leqq \sup_{t\in[0,T]} \|\sqrt{M_\theta(u^N)} - \sqrt{M_\theta(u_\theta)}\|_{L^n(\Omega)}\|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(\Omega_T)} \\
&\qquad \cdot\|\Phi\|_{L^2(0,T;L^{2^*}(\Omega))} \\
&\qquad +\left|\int_{\Omega_T} (\sqrt{M_\theta(u^N)}\nabla\mu^N - \sqrt{M_\theta(u_\theta)}\nabla\mu_\theta)\sqrt{M_\theta(u_\theta)}\Phi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&=: I + II.
\end{aligned} \tag{2.49}$$

By (2.42) and the boundedness of $\sqrt{M_\theta(u^N)}\nabla\mu^N$ in $L^2(\Omega_T)$, $I \to 0$ as $N \to \infty$. To estimate II, since

$$\begin{aligned}
\int_{\Omega_T} |\sqrt{M_\theta(u_\theta)}\Phi|^2 \,\mathrm{d}x\,\mathrm{d}t &\leqq \int_0^T \|M_\theta(u_\theta)\|_{L^{n/2}}\ \|\Phi\|_{L^{2^*}}^2 \,\mathrm{d}t \\
&\leqq \big\|M_\theta(u_\theta)\big\|_{L^\infty(0,T;L^{n/2}(\Omega))}\big\|\Phi\big\|_{L^2(0,T;L^{2^*}(\Omega))}^2,
\end{aligned}$$

we have $\sqrt{M_\theta(u_\theta)}\Phi \in L^2(\Omega_T)$ and consequently $II \to 0$ as $N \to \infty$ by (2.47). Hence

$$M_\theta(u^N)\nabla\mu^N \rightharpoonup M_\theta(u_\theta)\nabla\mu_\theta \quad \text{weakly in } L^2(0,T;L^{2n/(n+2)}(\Omega)).$$

For any $\alpha(t) \in L^2(0,T)$, since $\alpha(t)\nabla\phi_j \in L^2(0,T; L^{2^*}(\Omega_T))$, multiplying (2.24) by $\alpha(t)$, integrating in time over $(0,T)$, and taking limit as $N \to \infty$, we have, for all $j \in \mathbb{N}$,

$$\begin{aligned}\int_0^T \langle \partial_t u_\theta,\ \alpha(t)\phi_j(x)\rangle_{(H^2(\Omega))', H^2(\Omega)}\,\mathrm{d}t \\ = -\int_{\Omega_T} M_\theta(u_\theta)\nabla\mu_\theta \cdot \alpha(t)\nabla\phi_j\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.50}$$

For any $\phi \in L^2(0,T; H^2(\Omega))$, its Fourier series $\sum_{j=1}^{\infty} a_j(t)\phi_j$ converges strongly to ϕ in $L^2(0,T; H^2(\Omega))$. So $\sum_{j=1}^{\infty} a_j(t)\nabla\phi_j$ converges strongly to $\nabla\phi$ in $L^2(0,T; L^{2^*}(\Omega))$. Consequently by (2.50), we have

$$\int_0^T \langle \partial_t u_\theta,\ \phi\rangle_{(H^2(\Omega))', H^2(\Omega)}\,\mathrm{d}t = -\int_{\Omega_T} M_\theta(u_\theta)\nabla\mu_\theta \cdot \nabla\phi\,\mathrm{d}x\,\mathrm{d}t \tag{2.51}$$

for all $\phi \in L^2(0,T; H^2(\Omega))$. As for the initial value, by (2.26),

$$u^N(x,0) \to u_0(x) \quad \text{as } N \to \infty \text{ in } L^2(\Omega).$$

By (2.36), we see that $u_\theta(x,0) = u_0(x)$ in $L^2(\Omega)$.

2.4. Regularity of u_θ

Now we consider the regularity of u_θ. By (2.25), for any $a_j(t) \in L^2(0,T)$, since $a_j(t)\phi_j \in L^2(0,T; C(\bar{\Omega}))$, by (2.35), (2.44), and (2.46), in the limit when $N \to \infty$ we have

$$\int_0^T\int_\Omega \mu_\theta a_j(t)\phi_j\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega (\kappa\nabla u_\theta \cdot a_j(t)\nabla\phi_j + W'(u_\theta)a_j(t)\phi_j)\,\mathrm{d}x\,\mathrm{d}t$$

for all $j \in \mathbb{N}$. Then for any $\phi \in L^2(0,T; H^1(\Omega))$, since its Fourier series strongly converges to ϕ in $L^2(0,T; H^1(\Omega))$, we have

$$\int_0^T\int_\Omega \mu_\theta\phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega (\kappa\nabla u_\theta \cdot \nabla\phi + W'(u_\theta)\phi)\,\mathrm{d}x\,\mathrm{d}t.$$

Since $W'(u_\theta) \in L^\infty(0,T; L^2(\Omega))$, and $\mu_\theta \in L^2(0,T; H^1(\Omega))$, by regularity theory we see that $u_\theta \in L^2(0,T; H^2(\Omega))$. Hence

$$\mu_\theta = -\kappa\Delta u_\theta + W'(u_\theta) \quad \text{almost everywhere in } \Omega_T. \tag{2.52}$$

Since (1.15) implies $|W''(u_\theta)| \leqq C(1 + |u_\theta|^{r-1})$, and

$$u_\theta \in L^\infty(0,T; H^1(\Omega)) \hookrightarrow L^\infty(0,T; L^q(\Omega))$$

for $1 \leqq q < \infty$ if $n = 1, 2$ and $1 \leqq q \leqq 2^* = 2n/(n-2)$ if $n \geqq 3$,

$$\begin{aligned}
\int_\Omega |\nabla W'(u_\theta)|^2 \,\mathrm{d}x &= \int_\Omega |W''(u_\theta)|^2 |\nabla u_\theta|^2 \,\mathrm{d}x \\
&\leqq \left(\int_\Omega |W''(u_\theta)|^n \,\mathrm{d}x\right)^{2/n} \left(\int_\Omega |\nabla u_\theta|^{2n/(n-2)} \,\mathrm{d}x\right)^{(n-2)/n} \\
&\leqq C\left(1 + \int_\Omega |u_\theta|^{n(r-1)} \,\mathrm{d}x\right)^{2/n} \|\nabla u_\theta\|^2_{L^{2n/(n-2)}(\Omega)} \\
&\leqq C\left(1 + \|u_\theta\|^{2(r-1)}_{L^{n(r-1)}(\Omega)}\right) \|\nabla u_\theta\|^2_{L^{2n/(n-2)}(\Omega)} \\
&\leqq C\left(\|u_\theta\|^{2(r-1)}_{L^\infty(0,T;H^1(\Omega))} + 1\right) \|\nabla u_\theta\|^2_{H^1(\Omega)}.
\end{aligned}$$

The last inequality above uses $n(r-1) \leqq 2n/(n-2)$. Hence

$$\begin{aligned}
&\int_0^T \int_\Omega |\nabla W'(u_\theta)|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\leqq C\left(\|u_\theta\|^{2(r-1)}_{L^\infty(0,T;H^1(\Omega))} + 1\right) \int_0^T \|\nabla u_\theta\|^2_{H^1(\Omega)} \,\mathrm{d}t \\
&\leqq C\left(\|u_\theta\|^{2(r-1)}_{L^\infty(0,T;H^1(\Omega))} + 1\right) \|u_\theta\|^2_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

So $\nabla W'(u_\theta) = W''(u_\theta)\nabla u_\theta \in L^2(\Omega_T)$ and $W'(u_\theta) \in L^2(0, T; H^1(\Omega))$. Combined with $\mu_\theta \in L^2(0, T; H^1(\Omega))$, by (2.52) we have $u_\theta \in L^2(0, T; H^3(\Omega))$ and

$$\nabla\mu_\theta = -\kappa\nabla\Delta u_\theta + W''(u_\theta)\nabla u_\theta \quad \text{almost everywhere in } \Omega \times (0, T). \tag{2.53}$$

In summary, combining (2.51) and (2.53) we have

$$\begin{aligned}
&\int_0^T \langle \partial_t u_\theta,\ \phi\rangle_{(H^2(\Omega))', H^2(\Omega)} \,\mathrm{d}t \\
&= -\int_0^T \int_\Omega M_\theta(u_\theta)(-\kappa\nabla\Delta u_\theta + W''(u_\theta)\nabla u_\theta) \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}t
\end{aligned} \tag{2.54}$$

for all $\phi \in L^2(0, T; H^2(\Omega))$.

2.5. *Energy Inequality*

By (2.35)–(2.37) and (2.47), since u^N and μ^N satisfy the energy identity

$$\begin{aligned}
&\int_\Omega \left(\frac{\kappa}{2}|\nabla u^N(x,t)|^2 + W(u^N(x,t))\right) \mathrm{d}x \\
&\quad + \int_0^t \int_\Omega M_\theta(u^N(x,\tau)) |\nabla\mu^N(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau \\
&= \int_\Omega \left(\frac{\kappa}{2}|\nabla u^N(x,0)|^2 + W(u^N(x,0))\right) \mathrm{d}x,
\end{aligned} \tag{2.55}$$

taking limit as $N \to \infty$ gives the energy inequality (1.18).

3. C–H Equation with Degenerate Mobility

In this section we prove the main theorem. The proof consists of three parts. First, we prove the weak convergence of approximate solutions u_{θ_i} defined in Section 2 for a sequence of positive numbers $\theta_i \to 0$. Second, we show that the weak limit u solves the degenerate Cahn–Hilliard equation in the weak sense and satisfies the energy inequality (1.22). Finally we prove additional regularity of u for the weak degenerate case $0 < m < 1$.

3.1. Weak Convergence of Approximate Solutions $\{u_{\theta_i}\}$

Fix $u_0 \in H^1(\Omega)$ and a sequence $\theta_i > 0$ that monotonically decreases to 0 as $i \to \infty$. By Theorem 1, for any $\theta_i > 0$, there exists a

$$u_i \in L^\infty(0, T; H^1(\Omega)) \cap C([0, T]; L^p(\Omega)) \cap L^2((0, T); H^3(\Omega))$$

whose weak derivative is

$$\partial_t u_i \in L^2(0, T; (H^2(\Omega))'),$$

where $1 \leqq p < \infty$ if $n = 1, 2$ and $1 \leqq p < 2n/(n-2)$ if $n \geqq 3$, such that for all $\phi \in L^2(0, T; H^2(\Omega))$

$$\int_0^T \langle \partial_t u_i, \phi \rangle_{(H^2(\Omega))', H^2(\Omega)} \, \mathrm{d}t = -\int_0^T \int_\Omega M_i(u_i) \nabla \mu_i \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t, \quad (3.56)$$

$$\mu_i = -\kappa \Delta u_i + W'(u_i). \quad (3.57)$$

Here, for simplicity, we write $u_i = u_{\theta_i}$ and $M_i(u_i) := M_{\theta_i}(u_i)$. By the arguments in the proof of Theorem 1, the bounds on the right hand side of (2.29), (2.34), (2.48) depend only on the growth conditions of the mobility and potential, so there exists a constant $C > 0$ independent of θ_i such that

$$\|u_i\|_{L^\infty(0,T;H^1(\Omega))} \leqq C, \quad (3.58)$$

$$\|\partial_t u_\theta\|_{L^2(0,T;(H^2(\Omega))')} \leqq C. \quad (3.59)$$

$$\|\sqrt{M_i(u_i)} \nabla \mu_i\|_{L^2(\Omega_T)} \leqq C, \quad (3.60)$$

$$\|\mu_i\|_{L^\infty(0,T;(H^1(\Omega))')} \leqq C. \quad (3.61)$$

By the growth conditions on M_i and W, the Sobolev embedding theorem gives

$$\|W'(u_i)\|_{L^\infty(0,T;L^2(\Omega))} \leqq C, \quad (3.62)$$

$$\|M_i(u_i)\|_{L^\infty(0,T;L^{n/2}(\Omega))} \leqq C. \quad (3.63)$$

Similar to the proof of Theorem 1, the above boundedness of $\{u_i\}$ and $\{\partial_t u_i\}$ enables us to find a subsequence, not relabeled, and $u \in L^\infty(0, T; H^1(\Omega)) \cap$

$C([0,T];L^p(\Omega))$ for any $1 \leqq p < \infty$ if $n = 1, 2$ and $1 \leqq p < 2n/(n-2)$ if $n \geqq 3$, such that, as $i \to \infty$,

$$u_i \rightharpoonup u \quad \text{weakly-* in } L^\infty(0,T;H^1(\Omega)), \tag{3.64}$$

$$u_i \to u \quad \text{strongly in } C([0,T];L^p(\Omega)), \tag{3.65}$$

$$u_i \to u \quad \text{strongly in } L^2(0,T;L^p(\Omega)) \text{ and almost everywhere in } \Omega \times (0,T), \tag{3.66}$$

$$\partial_t u_i \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;(H^2(\Omega))'). \tag{3.67}$$

By (3.65) and (3.66), the general dominated convergence theorem, and the uniform convergence of $M_i \to M$ and $\sqrt{M_i} \to \sqrt{M}$ in $\mathbb{R}$ as $i \to \infty$, we have

$$M_i(u_i) \to M(u) \quad \text{strongly in } C([0,T];L^{n/2}(\Omega)), \tag{3.68}$$

$$\sqrt{M_i(u_i)} \to \sqrt{M(u)} \quad \text{strongly in } C([0,T];L^n(\Omega)). \tag{3.69}$$

By (3.60), there exists a $\xi \in L^2(\Omega_T)$ such that

$$\sqrt{M_i(u_i)}\nabla\mu_i \rightharpoonup \xi \tag{3.70}$$

weakly in $L^2(\Omega_T)$. Combined with (3.69) we have

$$M_i(u_i)\nabla\mu_i \rightharpoonup \sqrt{M(u)}\xi \quad \text{weakly in } L^2(0,T;L^{2n/(n+2)}(\Omega)). \tag{3.71}$$

So taking the limit as $i \to \infty$ in (3.56), we obtain

$$\int_0^T \langle \partial_t u, \phi\rangle_{(H^2(\Omega))',H^2(\Omega)}\,\mathrm{d}t = -\int_0^T\int_\Omega \sqrt{M(u)}\xi\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \tag{3.72}$$

for all $\phi \in L^2(0,T;H^2(\Omega))$.

As for the initial value, since $u_i(x,0) = u_0(x)$, by (3.65), we have $u(x,0) = u_0(x)$.

3.2. Weak Solution for the Degenerate Cahn–Hilliard Equation

Now we consider the relation between $\sqrt{M(u)}\xi$ and u. This boils down to convergence properties of $\nabla\mu_i = -\kappa\nabla\Delta u_i + W''(u_i)\nabla u_i$. We consider the convergence of $W''(u_i)\nabla u_i$ first.

3.2.1. Weak Convergence of $W''(u_i)\nabla u_i$. By (1.15), since $1 \leqq r \leqq n/(n-2)$, $0 \leqq r-1 \leqq 2/(n-2)$,

$$\begin{aligned}\int_\Omega |W''(u_i)|^n\,\mathrm{d}x &\leqq C\int_\Omega (|u_i|^{r-1}+1)^n\,\mathrm{d}x \leqq C\left(\int_\Omega |u_i|^{2n/(n-2)} + |\Omega|\right)\\ &\leqq C(\|u_i\|_{H^1(\Omega)}+1).\end{aligned}$$

That is,

$$\|W''(u_i)\|_{L^\infty(0,T;L^n(\Omega))} \leqq C.$$

Combined with (3.58), we have

$$\|W''(u_i)\nabla u_i\|_{L^\infty(0,T;L^{2n/(n+2)}(\Omega))} \leqq C.$$

Thus, we can find a subsequence, not relabeled, and $\Phi \in L^\infty(0,T;L^{2n/(n+2)}(\Omega;\mathbb{R}^n))$ such that

$$W''(u_i)\nabla u_i \rightharpoonup \Phi \quad \text{weakly-* in } L^\infty(0,T;L^{2n/(n+2)}(\Omega;\mathbb{R}^n)).$$

However, we also know that $W''(u_i) \to W''(u)$ strongly in $C([0,T];L^q(\Omega))$ for any $1 \leqq q < \infty$ if $n = 1, 2$, $1 \leqq q < n$ if $n \geqq 3$; and $\nabla u_i \rightharpoonup \nabla u$ weakly-* in $L^\infty(0,T;L^2(\Omega))$. In any case $W''(u_i)\nabla u_i \rightharpoonup W''(u)\nabla u$ weakly-* in $L^\infty(0,T;L^1(\Omega))$. So in fact $\Phi = W''(u)\nabla u$ and

$$W''(u_i)\nabla u_i \rightharpoonup W''(u)\nabla u \quad \text{weakly-* in } L^\infty(0,T;L^{2n/(n+2)}(\Omega)). \tag{3.73}$$

3.2.2. Weak Convergence of $\nabla\mu_i$ as a Whole. Choose a sequence of positive numbers δ_j that monotonically decreases to 0. By (3.66) and Egorov's theorem, for every $\delta_j > 0$, there exists a subset $B_j \subset \Omega_T$ with $|\Omega_T\backslash B_j| < \delta_j$ such that

$$u_i \to u \quad \text{uniformly in } B_j. \tag{3.74}$$

We may as well take

$$B_1 \subset B_2 \subset \cdots \subset B_j \subset B_{j+1} \subset \cdots \subset \Omega_T. \tag{3.75}$$

Define $B := \cup_{j=1}^\infty B_j$, then $|\Omega_T\backslash B| = 0$. Define

$$P_j := \{(x,t) \in \Omega_T : |1-u^2| > \delta_j\}.$$

Then

$$P_1 \subset P_2 \subset \cdots \subset P_j \subset P_{j+1} \subset \cdots \subset \Omega_T \tag{3.76}$$

and $\cup_{j=1}^\infty P_j = P$. For each j, B_j can be split into two parts:

$$D_j := B_j \cap P_j, \quad \text{where } |1-u^2| > \delta_j \quad \text{and} \quad u_i \to u \text{ uniformly},$$
$$\hat{D}_j := B_j\backslash P_j, \quad \text{where } |1-u^2| \leqq \delta_j \quad \text{and} \quad u_i \to u \text{ uniformly}.$$

By (3.75) and (3.76), we have

$$D_1 \subset D_2 \subset \cdots \subset D_j \subset D_{j+1} \subset \cdots \subset D := B \cap P, \tag{3.77}$$

and indeed $D = \cup_{j=1}^\infty D_j$.

For any $\Psi \in L^2(0,T;L^{2n/(n-2)}(\Omega_T;\mathbb{R}^n))$,

$$\begin{aligned}
&\int_{\Omega_T} M_i(u_i)\nabla\mu_i \cdot \Psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{\Omega_T\backslash B_j} M_i(u_i)\nabla\mu_i \cdot \Psi \,\mathrm{d}x\,\mathrm{d}t + \int_{B_j\backslash P_j} M_i(u_i)\nabla\mu_i \cdot \Psi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{B_j\cap P_j} M_i(u_i)\nabla\mu_i \cdot \Psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.78}$$

As $i \to \infty$, by (3.71) the left hand side of (3.78) has limit

$$\lim_{i\to\infty} \int_{\Omega_T} M_i(u_i)\nabla\mu_i \cdot \Psi \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} \sqrt{M(u)}\xi \cdot \Psi \, \mathrm{d}x \, \mathrm{d}t.$$

The three terms on the right hand side of (3.78) need a delicate analysis:
(a) *First term of* (3.78). For the first term, since $\lim_{j\to\infty} |\Omega_T \backslash B_j| = 0$,

$$\begin{aligned} &\lim_{j\to\infty}\lim_{i\to\infty} \int_{\Omega_T\backslash B_j} M_i(u_i)\nabla\mu_i \cdot \Psi \, \mathrm{d}x \, \mathrm{d}t \\ &= \lim_{j\to\infty} \int_{\Omega_T\backslash B_j} \sqrt{M(u)}\xi \cdot \Psi \, \mathrm{d}x \, \mathrm{d}t \\ &= 0. \end{aligned} \tag{3.79}$$

To analyze the second and third terms, we write $u_{0,k} := u_k$ and $\mu_{0,k} := \mu_k$. Then for $j \geqq 1$, by the uniform convergence of $u_{j-1,k} \to u$ in B_j, there exists an index N_j such that, for all $k \geqq N_j$,

$$|1 - u_{j-1,k}^2| > \frac{\delta_j}{2} \quad \text{in } B_j \cap P_j, \quad |1 - u_{j-1,k}^2| \leqq 2\delta_j \quad \text{in } B_j\backslash P_j.$$

(b) *Second term of* (3.78). By taking u_i and μ_i as the subsequences $u_{j-1,k}$ and $\mu_{j-1,k}$ in (3.78), and considering the limit, we have

$$\begin{aligned} &\lim_{j\to\infty}\lim_{k\to\infty} \left| \int_{B_j\backslash P_j} M_{j-1,k}(u_{j-1,k})\nabla\mu_i \cdot \Psi \, \mathrm{d}x \, \mathrm{d}t \right| \\ &\leqq \lim_{j\to\infty}\lim_{k\to\infty} \left\{ \left(\sup_{B_j\backslash P_j} \sqrt{M_{j-1,k}(u_{j-1,k})} \right) \left(\int_{B_j\backslash P_j} |\Psi|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \right. \\ &\quad \left. \cdot \left(\int_{B_j\backslash P_j} M_{j-1,k}(u_{j-1,k}) |\nabla\mu_{j-1,k}|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \right\} \\ &\leqq \lim_{j\to\infty}\lim_{k\to\infty} \left(\sup_{B_j\backslash P_j} \sqrt{M_{j-1,k}(u_{j-1,k})} \right) \|\sqrt{M_{j-1,k}(u_{j-1,k})}\nabla\mu_{j-1,k}\|_{L^2(\Omega_T)} \\ &\quad \cdot |\Omega|^{1/n} \, \|\Psi\|_{L^2(0,T;L^{2n/(n-2)}(\Omega))} \\ &\leqq C \lim_{j\to\infty}\lim_{k\to\infty} \max\{(2\delta_j)^{m/2}, \quad \theta_{j-1,k}^{m/2}\} \\ &= 0. \end{aligned} \tag{3.80}$$

(c) *Third term of* (3.78). By the boundedness of

$$\sqrt{M_{j-1,k}(u_{j-1,k})}\nabla\mu_{j-1,k} \quad \text{in } L^2(\Omega_T) \quad \text{(that is (3.60))},$$

we see that

$$
\begin{aligned}
&\left(\frac{\delta_j}{2}\right)^m \int_{B_j \cap P_j} |\nabla \mu_{j-1,k}|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leqq \int_{B_j \cap P_j} M_{j-1,k}(u_{j-1,k}) |\nabla \mu_{j-1,k}|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leqq \int_{\Omega_T} M_{j-1,k}(u_{j-1,k}) |\nabla \mu_{j-1,k}|^2 \,\mathrm{d}x\,\mathrm{d}t \leqq C.
\end{aligned}
$$

Hence $\nabla \mu_{j-1,k}$ is bounded in $L^2(B_j \cap P_j)$ and there is a further subsequence, labeled as $\nabla \mu_{j,k}$ with $k = 1, 2, \ldots$, that weakly converges to some function $\zeta_j \in L^2(B_j \cap P_j)$.

Since $\{B_j \cap P_j\}_{j=1}^{\infty}$ is an increasing sequence of sets with a limit $B \cap P$, we have $\zeta_j = \zeta_{j-1}$ almost everywhere in $B_{j-1} \cap P_{j-1}$. In addition, we may extend $\zeta_j \in L^2(B_j \cap P_j)$ into a function $\hat{\zeta}_j \in L^2(B \cap P)$ by

$$
\hat{\zeta}_j := \begin{cases} \zeta_j & \text{if } x \in B_j \cap P_j, \\ 0 & \text{if } x \in (B \cap P) \backslash (B_j \cap P_j). \end{cases}
$$

So for almost every $x \in B \cap P$, there exists a limit of $\hat{\zeta}_j(x)$ as $j \to \infty$. We write

$$
\zeta(x) = \lim_{j \to \infty} \hat{\zeta}_j(x) \quad \text{almost everywhere in } B \cap P.
$$

Clearly $\zeta(x) = \zeta_j(x)$ almost everywhere $x \in B_j \cap P_j$ for all j.

Using a standard diagonal argument, we can extract a subsequence such that

$$
\nabla \mu_{k,N_k} \rightharpoonup \zeta \quad \text{weakly in } L^2(B_j \cap P_j) \quad \text{for all } j. \tag{3.81}
$$

By the strong convergence

$$
\sqrt{M_i(u_i)} \to \sqrt{M(u)} \quad \text{in } L^{\infty}(0, T; L^n(\Omega)) \quad \text{(that is (3.69))}
$$

we obtain

$$
\chi_{B_j \cap P_j} \sqrt{M_{k,N_k}(u_{k,N_k})} \nabla \mu_{k,N_k} \rightharpoonup \chi_{B_j \cap P_j} \sqrt{M(u)} \zeta
$$

weakly in $L^2(0, T; L^{2n/(n+2)}(\Omega))$ for all j. Here $\chi_{B_j \cap P_j}$ is the characteristic function of $B_j \cap P_j \subset \Omega_T$. However, since we know $\sqrt{M_i(u_i)} \nabla \mu_i \rightharpoonup \xi$ weakly in $L^2(\Omega_T)$ (see the discussion below (3.69)), we see in fact that $\xi = \sqrt{M(u)} \zeta$ in every $B_j \cap P_j$, and hence

$$
\xi = \sqrt{M(u)} \zeta \quad \text{in } B \cap P. \tag{3.82}
$$

Consequently, by (3.71),

$$
\chi_{B \cap P} M_{k,N_k}(u_{k,N_k}) \nabla \mu_{k,N_k} \rightharpoonup \chi_{B \cap P} M(u) \zeta
$$

weakly in $L^2(0, T; L^{2n/(n+2)}(\Omega))$.

In (3.78), replacing u_i by the above mentioned subsequence u_{k,N_k}, and taking limits first as $k \to \infty$ and then as $j \to \infty$, by (3.79) and (3.80), we obtain that for every $\Psi \in L^2(0,T; L^{2n/(n-2)}(\Omega; \mathbb{R}^n))$

$$\begin{aligned}\int_{\Omega_T} \sqrt{M(u)}\xi \cdot \Psi \,\mathrm{d}x\,\mathrm{d}t &= \lim_{j\to\infty} \int_{B_j \cap P_j} M(u)\zeta \cdot \Psi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{B\cap P} M(u)\zeta \cdot \Psi \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.83}$$

Compared with (3.72), we find that u and ζ solve the following weak equation

$$\int_0^T \langle \partial_t u, \phi\rangle_{(H^2(\Omega))', H^2(\Omega)} \,\mathrm{d}t = -\int_{B\cap P} M(u)\zeta \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \tag{3.84}$$

for all $\phi \in L^2(0,T; H^2(\Omega))$.

3.2.3. The Relation Between ζ and u. The desired relation between ζ and u is $\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u$. This is a delicate question to be studied here. Indeed, while $W''(u)\nabla u \in L^\infty(0,T; L^{2n/(n+2)}(\Omega))$ by (3.73), the term $\nabla\Delta u$ is only defined in the sense of distributions and may not even be a function, given the known regularity $u \in L^\infty(0,T; H^1(\Omega))$.

Claim *If for some j, the interior of $B_j \cap P_j$, denoted by $(B_j \cap P_j)^\circ$, is not empty, then*

$$\nabla\Delta u \in L^{2n/(n+2)}((B_j \cap P_j)^\circ),$$

and

$$\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u \quad \textit{in } (B_j \cap P_j)^\circ.$$

To prove this, we write in $(B_j \cap P_j)^\circ$

$$-\kappa\nabla\Delta u_{k,N_k} = \nabla\mu_{k,N_k} - W''(u_{k,N_k})\nabla u_{k,N_k}.$$

By (3.73) and (3.81), taking the limit as $k \to \infty$ in the sense of distributions, we obtain

$$-\kappa\nabla\Delta u = \zeta - W''(u)\nabla u \quad \text{in } (B_j \cap P_j)^\circ.$$

Since $\zeta - W''(u)\nabla u$ in $L^{2n/(n+2)}(B_j \cap P_j)$, we have the regularity

$$-\kappa\nabla\Delta u \in L^{2n/(n+2)}((B_j \cap P_j)^\circ),$$

and consequently $\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u$ in $(B_j \cap P_j)^\circ$

Another issue is that ζ is not defined in $\Omega_T \backslash (B \cap P)$. Indeed the value of ζ in $\Omega_T \backslash (B \cap P)$ does not matter as it does not appear in the right hand side of (3.84). This ambiguity can be removed in every open subset of Ω_T in which $\nabla\Delta u$ has enough regularity.

Claim *For any open set $U \subset \Omega_T$ in which $\nabla\Delta u \in L^p(U)$ for some $p > 1$, where p may depend on U, we have*

$$\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u \quad \textit{in } U.$$

To show this, we need to consider the limit of

$$\nabla\mu_{k,N_k} = -\kappa\nabla\Delta u_{k,N_k} + W''(u_{k,N_k})\nabla u_{k,N_k}.$$

The right hand side weakly converges to $-\kappa\nabla\Delta u + W''(u)\nabla u$ in $L^q(U)$ for $q = \min\{p,\ 2n/(n+2)\} > 1$. Hence

$$\nabla\mu_{k,N_k} \rightharpoonup -\kappa\nabla\Delta u + W''(u)\nabla u \quad \text{weakly in } L^q(U).$$

So in $B \cap P \cap U$, we have $\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u$. And we may extend the definition of ζ into $U\backslash(B \cap P)$ by defining it to be $-\kappa\nabla\Delta u + W''(u)\nabla u$.

Define

$$\tilde{\Omega}_T := \cup\{U \subset \Omega_T\colon \nabla\Delta u \in L^p(U) \quad \text{for some } p > 1,\ p \text{ depending on } U\}.$$

Then $\tilde{\Omega}_T$ is open and

$$\zeta = -\kappa\nabla\Delta u + W''(u)\nabla u \quad \text{in } \tilde{\Omega}_T.$$

ζ is now defined in $(B \cap P) \cup \tilde{\Omega}_T$. To extend the definition of ζ to Ω_T, notice that

$$\Omega_T\backslash((B \cap P) \cup \tilde{\Omega}_T) \subset (\Omega_T\backslash P) \cup (\Omega_T\backslash B).$$

Since $|\Omega_T\backslash B| = 0$ and $M(u) = 0$ in $\Omega_T\backslash P$, the value of ζ outside of $(B\cap P)\cup\tilde{\Omega}_T$ does not contribute to the integral on the right hand side of (3.84) so we may just let $\zeta = 0$ outside of $(B \cap P) \cup \tilde{\Omega}_T$.

3.2.4. Energy Inequality. By (1.18) we have

$$\begin{aligned}
&\int_\Omega \left(\frac{\kappa}{2}|\nabla u_{k,N_k}(x,t)|^2 + W(u_{k,N_k}(x,t))\right) \mathrm{d}x \\
&\quad + \int_{\Omega_t\cap B\cap P} M_{k,N_k}(u_{k,N_k}(x,\tau))|\nabla\mu_{k,N_k}(x,\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqq \int_\Omega \left(\frac{\kappa}{2}|\nabla u_0|^2 + W(u_0)\right) \mathrm{d}x. \qquad (3.85)
\end{aligned}$$

Taking the limit as $k \to \infty$ and using (3.64)–(3.66), (3.70) and (3.82), we obtain the energy inequality (1.22).

3.3. *Weak Degeneracy* $(0 < m < 1)$

Now we will consider additional regularities if the degeneracy is weak, in the sense that $0 < m < 1$. Since $M(u) = |1-u^2|^m$ we define a corresponding entropy density as

$$\Phi(u) := \int_0^u \int_0^r \frac{1}{|1-s^2|^m}\,\mathrm{d}s\,\mathrm{d}r. \tag{3.86}$$

Then $\Phi \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{\pm 1\})$ is convex and $\Phi(u) \geqq 0$ for all u. For any $\theta > 0$, define

$$\Phi_\theta(u) = \int_0^u \int_0^r \frac{1}{M_\theta(s)}\,\mathrm{d}s\,\mathrm{d}r.$$

Since $M_\theta(s) \geqq M(s)$ for all $s \in \mathbb{R}$ and $M_\theta(s) \geqq \theta^m > 0$, $\Phi_\theta \in C^2(\mathbb{R})$ is convex and $0 \leqq \Phi_\theta(u) \leqq \Phi(u)$ for all $u \in R$.

Since $u_\theta \in L^2(0,T;H^3(\Omega))$, we have $\Phi_\theta'(u_\theta) \in L^2(0,T;H^2(\Omega))$ and we can use $\Phi_\theta'(u_\theta)$ as a test function in (1.17),

$$\begin{aligned}
&\int_0^T \langle \partial_t u_\theta, \Phi_\theta'(u_\theta)\rangle\,\mathrm{d}t \\
&\quad= -\int_{\Omega_T} M_\theta(u_\theta)(-\kappa\nabla\Delta u_\theta + W''(u_\theta)\nabla u_\theta)\cdot \Phi_\theta''(u_\theta)\nabla u_\theta\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\kappa\int_{\Omega_T} |\Delta u_\theta|^2\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} W''(u_\theta)|\nabla u_\theta|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_\Omega \Phi_\theta(u_\theta(T,x))\,\mathrm{d}x + \int_{\Omega_T} W''(u_\theta)|\nabla u_i|^2\,\mathrm{d}x\,\mathrm{d}t + \kappa\int_{\Omega_T}|\Delta u_\theta|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_\Omega \Phi_\theta(u_0)\,\mathrm{d}x.
\end{aligned}$$

By growth condition (1.15) we have

$$\begin{aligned}
&\int_\Omega \Phi_\theta(u_\theta(T,x))\,\mathrm{d}x + C\int_{\Omega_T} |u_\theta|^{r-1}|\nabla u_\theta|^2\,\mathrm{d}x\,\mathrm{d}t + \kappa\int_{\Omega_T}|\Delta u_\theta|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqq \int_\Omega \Phi_\theta(u_0)\,\mathrm{d}x + C\int_{\Omega_T}|\nabla u_\theta|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqq C\left\{\int_\Omega \Phi(u_0)\,\mathrm{d}x + \frac{2T}{\kappa}\int_\Omega\left(\frac{\kappa}{2}|\nabla u_0|^2 + W(u_0)\right)\mathrm{d}x\right\} \\
&\quad\leqq C(\|u_0\|_{H^1(\Omega)}^{\max\{2,r+1\}} + 1).
\end{aligned}$$

As long as $u_0 \in H^1(\Omega)$, u_θ is bounded in $L^2(0,T;H^2(\Omega))$ and hence the limit u is also in $L^2(0,T;H^2(\Omega))$.

Remark 5. This entropy estimate does not work for $m \geqq 1$, because in the definition (3.86), we have to overcome the singularity at $s = \pm 1$, since we do not guarantee our solution u to be bounded between $[-1, 1]$. This is a genuine difference compared with the result in [10]. The weak solution in [10] only makes sense for $m \geqq 1$, since it involves the derivative of $M(u)$. In contrast, our definition of weak solution does not impose such a restriction.

4. Discussion

The degeneracy of diffusion mobility in the Cahn–Hilliard equation presents many technical challenges and counter-intuitive behavior, and the relation between degenerate diffusion mobility and the double–well potential is a delicate problem as shown by recent studies. Smooth double–well potentials and mobility functions are usually used in phase field modeling of realistic physical, biological and material systems, and it is of much interest to explore the geometric evolutions of the underlying interfaces and to provide rigorous mathematical justification for the well-posedness of the problem. For a degenerate mobility (1.7), even though previous results in the literature concentrate on weak solutions that are bounded between the two pure phases $u^{\pm}$, the bounded solutions are not always compatible with the physical features of the system, such as the Gibbs–Thomson effect. Our formal results in [9] predicted the existence of a solution that is compatible with the Gibbs–Thomson effect and such a solution will not be confined within $[u^-, u^+]$.

In this paper we rigorously proved the existence of a solution in a weak sense, as long as the initial value is in $H^1(\Omega)$. Such a weak solution potentially allows the physical Gibbs–Thomson effect. When the space dimension $n \geqq 2$, we expect our weak solution not to stay within $[u^-, u^+]$, even if the initial value is. Further numerical evidence of such claims will be given in a separate work. Additional theoretical exploration is needed for a rigorous justification. There are some other open problems for further study. One is about better characterization of the regularity of the weak solution and its singular set. Our result shows that the singular set is concentrated on the set where the mobility is degenerate, plus a set of Lebesgue measure zero. It is interesting to study the conditions under which such a singular set disappears, and the weak solution becomes classical. Also, since the degenerate mobility (1.7) formally approaches the constant mobility as $m \to 0^+$, it would be interesting to study the relation between the weak solutions for the degenerate Cahn–Hilliard equation and the solution for the Cahn–Hilliard equation with a constant mobility.

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