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A Class of High Order Nonlocal Operators

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Abstract

We study a class of nonlocal operators that may be seen as high order generalizations of the well known nonlocal diffusion operators. We present properties of the associated nonlocal functionals and nonlocal function spaces including nonlocal versions of Sobolev inequalities such as the nonlocal Poincaré and nonlocal Gagliardo–Nirenberg inequalities. Nonlocal characterizations of high order Sobolev spaces in the spirit of Bourgain–Brezis–Mironescu are provided. Applications of nonlocal calculus of variations to the well-posedness of linear nonlocal models of elastic beams and plates are also considered.

1. Introduction

The focus of this paper is to investigate the following class of nonlocal energy functionals for a scalar function $u : \mathbb{R}^N \to R$:

$$E_n(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \gamma_n(|s|)|D_n^s[u](x)|^2 \mathrm{d}s\mathrm{d}x, \quad n \in \mathbb{N}, \tag{1}$$

the associated variational problems, and the corresponding nonlocal operators. Here D_n^s denotes an nth order difference operator, and $\gamma_n : \mathbb{R}_+ \to \mathbb{R}_+$ is a compactly supported function (kernel) corresponding to the index n. A special instance, namely E_1, given by

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \gamma_1(|s|)|u(x+s) - u(x)|^2 \mathrm{d}s\mathrm{d}x$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \gamma_1(|y-x|)|u(y) - u(x)|^2 \mathrm{d}y\mathrm{d}x,$$

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is a nonlocal functional that, together with its associated nonlocal operators and nonlocal problems defined in nonlocal function spaces, has attracted much attention in recent literature, see for example [1–3,5,8,10,12,16,26]. We refer to [28] for a survey and a more extensive list of references on related mathematical analysis. Important to our study here is the fact that while the functional E_1 defines a natural nonlocal function space V_1, containing $u \in L^2$ with bounded $E_1(u)$ [9], it has also been used for nonlocal characterizations of conventional Sobolev spaces [5,26]. As a matter of fact, for suitably scaled kernel functions having their support shrinking to the origin, the local limit of the respective sequence of energy functionals involving the first order difference operator D_1^s becomes the Dirichelet integral, that is, the square of the standard H^1 semi-norm. The sequence of function spaces may then be seen as a nonlocal characterization of H^1 [5]. The corresponding nonlocal operator becomes an analog of the classical, local, Laplacian. Thus, functionals $\{E_n\}$ corresponding to a higher order difference operator D_n^s may be seen as the nonlocal analog of local functionals involving higher order derivatives.

There have been various other studies of nonlocal operators related to high order derivatives such as high order fractional Laplacian operators discussed in [29,32] that are an extension to the study of [8], and the work in [4] as a higher order extension of [5] on the nonlocal characterization of Sobolev spaces. It is worth noting that our present work is not only driven by the large mathematical interest in studying how properties of E_1 may be extended to the more general class of functionals $\{E_n\}$, but also finds motivation in their applications to the modeling of various physical processes. Indeed, nonlocality is ubiquitous in nature. In recent years, there have been many works that apply nonlocal models to give better descriptions of the physical realities in many areas including materials and biological sciences, mechanics, stochastic processes, data and image analysis [6,7,11,13–15,17,23,30]. Studies of $\{E_n\}$ not only enrich the mathematical theory but also have potentially broader applications in these and other important fields. In practice, the nature of nonlocal interactions depends on the particular physical state and may also be a result of the coarse graining process; this means that one may potentially encounter very different types of nonlocal kernels associated with nonlocal operators of different orders. A particular example is the nonlocal peridynamic plate model considered in Section 6.1, which is derived in [24,25] using a general non-ordinary state-based formulation of peridynamics. The kernel function used for the fourth order nonlocal operator there can be modeled differently, say, from that used for the second order nonlocal operators used in the ordinary bond based peridynamics model of Navier elasticity equations [20,30]. Moreover, to be directly applicable to [24,25], the kernels cannot be taken as those usual ones leading to typical fractional Sobolev spaces. Furthermore, for coupled systems of nonlocal models, the formulation may involve nonlocal interactions that are described by matrix-valued or higher order tensor valued kernels [21]. As a first attempt, we do not consider the tensor kernels here but elect to work with scalar-valued kernels that are more general than those corresponding to fractional differential operators considered by many existing studies. While these generalizations provide a more physically desirable flexibility in treating different nonlocal interactions on different orders and make the analyst readily applicable to recently studied nonlocal mechanical models, it becomes evident that additional technical challenges also arise in mathematical analysis, for example, the Fourier symbols of the associated nonlocal operators no longer enjoy explicit forms so that more refined mathematical estimates on different symbols for operators of different orders have to be worked out.

The main results of this paper contain the characterizations of the nonlocal function spaces associated with the functionals $\{E_n\}$, including the various embedding and compactness properties. Analogus to the classical counterpart, nonlocal versions of Poincaré inequalities are also presented, including a version valid for more general nonlocal kernels and a second one for more specialized kernels that leads to sharper control on the Poincaré constant. In addition, we can naturally consider nonlocal Gagliardo–Nirenberg type inequalities for norms associated with the class of nonlocal spaces. Such results, in their general forms that go beyond the conventional forms related to fractional Sobolev spaces, have not appeared in the literature before to the best of our knowledge. These results offer extensions to some previous works given initially in [5,26] and more recently in [12,18,19,22]. Moreover, localizations of the related nonlocal spaces also offer nonlocal characterizations of high order Sobolev spaces in the spirit of Bourgain–Brezis–Mironescu are provided. Our results are established for functionals defined in the whole space $\mathbb{R}^N$ as well as in an open and bounded domain $\Omega \subset \mathbb{R}^N$. The latter is made possible through the consideration of functions that vanish outside Ω. This study can be viewed as part of the ongoing effort to establish a systematic framework of nonlocal calculus of functions and nonlocal calculation of variations for nonlocal models [10,11,18,19]. As an illustration of its many possible applications, the framework is used here to analyze a recently developed nonlocal bending elasticity model of linear elastic beams [24,25]. The mathematical well-posedness of such a model does not readily follow from known results on nonlocal function spaces already given in the literature but relies on the new compactness and localization properties of the corresponding function space and the nonlocal Poincare inequalities established here with the more general kernels under our considerations.

The paper is organized as follows. In Section 2, the class of nonlocal operators of interested are introduced and some preliminary results are given. In Section 3, we present Nonlocal Sobolev type inequalities which are stated as Theorems 2, 3, and 4 respectively. Theorems 2 and 4 are proved there, but the proof of theorem 3 is postponed to Section 4 where compactness properties are established. In fact, Section 4 contains several compactness results, stated and proved as Theorems 5, 6 and 7, which further allow us to investigate limiting properties of nonlocal variational problems with vanishing nonlocality. Such properties are presented as Theorem 9 in Section 5. Then, in Section 6, we discuss a particular application of the high order nonlocal operator theory to a nonlocal model of a linear elastic beam, establishing the well-posedness and convergence to the local PDE models in theorem 5. Some final remarks are given in Section 7. Detailed proofs of a couple of technical Lemmas 1 and 6 are given in the appendix.

2. Energy spaces and nonlocal operators

We first discuss some notations and terms used in (1). The associated nonlocal function spaces, or energy spaces, as we later refer to, them and operators are then introduced. The energy spaces are shown to be Hilbert spaces with norms defined by the nonlocal functionals.

2.1. Definition and notation

First, with regard to the kernel functions $\{\gamma_n\}$, we assume that for any positive integer $n \in \mathbb{N}$,

$$\begin{cases} \gamma_n \text{ is nonnegative, compactly supported, } |x|^{2n}\gamma_n(|x|) \in L^1_{loc}(\mathbb{R}^N), \\ \text{and there exist a constant } \eta > 0, \text{ such that } B_\eta(0) \subset \mathrm{Supp}\{\gamma_n(|\cdot|)\}. \end{cases} \tag{K}$$

Next, for any $s \in \mathbb{R}$, D^s_n denotes the n-th difference operator acting on any function u: $\mathbb{R}^N \to \mathbb{R}$ given by

$$D^s_n[u](x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} u(x + a^j_n s) \quad \text{where } \binom{n}{j} = \frac{n!}{j!(n-j)!} \tag{2}$$

and

$$\begin{cases} a^j_n = \frac{n+1}{2} - j & \text{if n is odd, or} \\ a^j_n = \frac{n}{2} - j & \text{if n is even.} \end{cases} \tag{3}$$

The difference operators defined above allow us to interpret a higher order difference operator by the composition of lower order difference operators. This simple fact together with two other basic equalities are given in a lemma that will be useful in subsequent calculations. The proof of the lemma is rudimentary and is included in the Appendix for completeness.

Lemma 1. *For $n \geqq 2$, $s \in \mathbb{R}^N$, D^s_n satisfies the following:*

$$D^s_n = D^s_1 \circ D^s_{n-1} \text{ if } n \text{ is odd, or } D^s_n = -D^{-s}_1 \circ D^s_{n-1} \text{ if } n \text{ is even.} \tag{4}$$

$$\left| \sum_{j=0}^{n} (-1)^j \binom{n}{j} e^{i\xi \cdot a^j_n s} \right|^2 = (2 - 2\cos(\xi \cdot s))^n, \tag{5}$$

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (a^j_n)^n = n!\,. \tag{6}$$

With the kernels and difference operators defined, we now introduce the associated function spaces given by

$$\mathcal{S}^{n,\gamma_n} = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \gamma_n(|s|)|D_n^s[u](x)|^2 \mathrm{d}s\mathrm{d}x < \infty \right\}. \tag{7}$$

Obviously $\mathcal{S}^{n,\gamma_n}$ is a subspace of $L^2(\mathbb{R}^N)$. For $0 < \alpha < 1$, if the kernel $\gamma_1(|s|)$ behaves like $|s|^{-N-2\alpha}$ near the origin, the space associated with $\mathcal{S}^{1,\gamma_1}$ corresponds to the usual fractional Sobolev space H^α. We note that the norm equivalence may depend on how $\gamma_1(|s|)$ behaves away from the origin and its compact support, as well as how it is normalized. Moreover, although our study covers such fractional cases, our focus is on kernels that may behave more generally at the origin, including those $\gamma_n(|s|)$ in $L^1_{loc}(\mathbb{R}^N)$ along with their $2n$-moments as specified in assumption (K).

For simplicity, whenever there is no notational confusion, we use $\mathcal{S}^n$ to denote $\mathcal{S}^{n,\gamma_n}$. Let the bilinear form $((\cdot,\cdot))_{\mathcal{S}^n} : \mathcal{S}^n \times \mathcal{S}^n \to \mathbb{R}$ be defined by

$$((u,v))_{\mathcal{S}^n} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \gamma_n(|s|) \left(D_n^s[u](x)\right) \left(D_n^s[v](x)\right) \mathrm{d}s\mathrm{d}x.$$

Then the space $(\mathcal{S}^n, (\cdot,\cdot)_{\mathcal{S}^n})$ is a real inner product space with $(\cdot,\cdot)_{\mathcal{S}^n}$ defined as

$$(u,v)_{\mathcal{S}^n} = (u,v)_{L^2} + ((u,v))_{\mathcal{S}^n}.$$

Now we use $|u|_{\mathcal{S}^n}$ to denote the semi-norm $\sqrt{((u,u))_{\mathcal{S}^n}}$. Then $\mathcal{S}^n$ is equipped with a norm $\|\cdot\|_{\mathcal{S}^n}$ given by

$$\|u\|^2_{\mathcal{S}^n} = \|u\|^2_{L^2} + |u|^2_{\mathcal{S}^n}.$$

Next, we consider a simple property of the difference operators which may be viewed as an analog of the integration by parts formula. Similar formulae have been discussed in many earlier works such as [10,11]. For convenience, we drop the domain of integration in the integral whenever there is no ambiguity, in particular, when it is an integral over the whole space.

Lemma 2. (Integration by parts formula) *The following is true for functions $u, v \in L^2(\mathbb{R}^N)$ and $\rho \in L^1(\mathbb{R}^N)$.*

$$\iint \rho(|s|)u(x)D_1^s[v](x)\mathrm{d}s\mathrm{d}x = \iint \rho(|s|)D_1^{-s}[u](x)v(x)\mathrm{d}s\mathrm{d}x.$$

Proof. It is very easy to check the following equalities.

$$\begin{aligned}\iint \rho(|s|)u(x)D_1^s[v](x)\mathrm{d}s\mathrm{d}x &= \iint \rho(|s|)u(x)(v(x+s)-v(x))\mathrm{d}s\mathrm{d}x \\ &= \iint \rho(|s|)(u(x-s)-u(x))v(x)\mathrm{d}s\mathrm{d}x \\ &= \iint \rho(|s|)D_1^{-s}[u](x)v(x)\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Another lemma is on characterizing the norm $|\cdot|_{\mathcal{S}^n}$ by the Fourier transform.

Lemma 3. *Suppose u is a function defined on $\mathbb{R}^N$, and $\mathscr{F}$ is denoted as the Fourier transform of u, then for any $n \geqq 1$*

$$\iint \gamma_n(|s|)|D_n^s[u](x)|^2 \mathrm{d}s\mathrm{d}x = \iint \gamma_n(|s|)(2-2\cos(\xi\cdot s))^n |\hat{u}(\xi)|^2 \mathrm{d}\xi\mathrm{d}s. \quad (8)$$

Proof. By the Plancherel Formula, we have

$$\begin{aligned}\iint \gamma_n(|s|)|D_n^s[u](x)|^2 \mathrm{d}s\mathrm{d}x &= \iint \gamma_n(|s|)|D_n^s[u](x)|^2 \mathrm{d}x\mathrm{d}s \\ &= \int_{\mathbb{R}^N} \gamma_n(|s|)\|D_n^s[u](\cdot)\|^2_{L^2(R^N)}\mathrm{d}s \\ &= \int_{\mathbb{R}^N} \gamma_n(|s|)\|\mathscr{F}\left(D_n^s[u](\cdot)\right)\|^2_{L^2(R^N)}\mathrm{d}s.\end{aligned}$$

Now by the definition of D_n^s, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \gamma_n(|s|)\|\mathscr{F}\left(D_n^s[u](\cdot)\right)\|^2_{L^2(R^N)}\mathrm{d}s \\ &= \iint \gamma_n(|s|)\left|\sum_{j=0}^n (-1)^j \binom{n}{j} e^{i\xi\cdot a_n^j s}\right|^2 |\hat{u}(\xi)|^2 \mathrm{d}\xi\mathrm{d}s.\end{aligned}$$

Hence, by Lemma 1,

$$\left|\sum_{j=0}^n (-1)^j \binom{n}{j} e^{i\xi\cdot a_n^j s}\right|^2 = (2-2\cos(\xi\cdot s))^n,$$

we thus get the desired result.

Finally, to help establishing desired compactness properties in Section 4, we quote here a result from [5, Lemma 2] as a lemma.

Lemma 4. *Let $g, h : (0,\delta) \to \mathbb{R}_+$. Assume $g(t) \leqq g(t/2), t \in (0,\delta)$, and that $h = h(t)$ is non-increasing. Then, for some $C = C(d) > 0$,*

$$\int_0^\delta t^{d-1} g(t)h(t)\mathrm{d}t \geqq C\delta^{-d} \int_0^\delta t^{d-1} g(t)\mathrm{d}t \int_0^\delta t^{d-1} h(t)\mathrm{d}t.$$

2.2. The energy spaces

We first show that the inner product space $\mathcal{S}^n = \mathcal{S}^{n,\gamma_n}$ is complete, thus a Hilbert space. Analogous results for the special case of $n = 1$, but for more general vector fields, can be found in, for example, [18,19].

Theorem 1. *For $n \in \mathbb{Z}^+$, assume that γ_n satisfies (K). Then $(\mathcal{S}^n, (\cdot,\cdot)_{\mathcal{S}^n})$ is a Hilbert space.*

Proof. It suffices to check that the space is complete under the norm $\|\cdot\|_{\mathcal{S}^n}$. Given a Cauchy sequence $\{u_k\} \in \mathcal{S}^n$, it is also Cauchy in $L^2(\mathbb{R}^N)$. So $\{u_k\}$, up to a subsequence, converges to some $u \in L^2(\mathbb{R}^N)$. We show that $|u_k - u|_{\mathcal{S}^n} \to 0$ as $k \to \infty$.

For any $\varepsilon > 0$, we may choose M large enough such that $|u_k - u_m|^2_{\mathcal{S}^n} \leqq \varepsilon^2$ for any $k, m \geqq M$. We then follow similar techniques as that in [18,19] to define a cut-off of kernel γ_n by $\gamma_n^\tau(r) = \gamma_n(r)\chi_{[\tau,\infty)}(r)$ to make γ_n^τ integrable for any given $\tau > 0$, and

$$\iint \gamma_n^\tau(|s|)(D_n^s[u_k - u_m](x))^2 \mathrm{d}s\mathrm{d}x \leqq |u_k - u_m|^2_{\mathcal{S}^n}.$$

We now first claim that, for any $w, v \in \mathcal{S}^n$,

$$\begin{aligned}&\iint \gamma_n^\tau(|s|) D_n^s[w](x) D_n^s[v](x)\mathrm{d}s\mathrm{d}x \\ &\quad = (-1)^n \iint \gamma_n^\tau(|s|) D_{2n}^s[w](x) v(x)\mathrm{d}s\mathrm{d}x. \qquad (9)\end{aligned}$$

In fact, by Lemma 1, $D_n^s[v]$ can be decomposed into $D_1^s \circ D_{n-1}^s[v]$ or $-D_1^{-s} \circ D_{n-1}^s[v]$. Then using the integration by parts formula in Lemma 2, we may throw the first order difference operator to the term involving $D_n^s[w](x)$ and apply Lemma 1 again to get $-D_{n+1}^s[w](x)$. Repeating this procedure we can finally get (9).

Subsequently, we get

$$\begin{aligned}&\iint \gamma_n^\tau(|s|)(D_n^s[u_k - u_m](x))^2 \mathrm{d}s\mathrm{d}x \\ &\quad = (-1)^n \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u_m](x)\mathrm{d}s \right) (u_k - u_m)(x)\mathrm{d}x.\end{aligned}$$

Since $u_m \to u$ in $L^2(\mathbb{R}^N)$, we have for a fixed k, and all $x \in \mathbb{R}^N$,

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u_m](x)\mathrm{d}s = \int_{\mathbb{R}^N} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u](x)\mathrm{d}s.$$

Therefore by dominated convergence theorem,

$$\begin{aligned}&(-1)^n \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u](x)\mathrm{d}s \right) (u_k - u)(x)\mathrm{d}x \\ &\quad = \lim_{m\to\infty} (-1)^n \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \gamma_n^\tau(|s|) D_{2n}^s[u_k - u_m](x)\mathrm{d}s \right) (u_k - u_m)(x)\mathrm{d}x \leqq \varepsilon^2\end{aligned}$$

for $k \geqq M$. Now we can apply Equation (9) again and obtain

$$\iint \gamma_n^\tau(|s|)(D_n^s[u_k - u](x))^2 \mathrm{d}s\mathrm{d}x \leqq \varepsilon^2.$$

In the end, by letting $\tau \to 0$ and applying Fatou's lemma, we have

$$|u_k - u|_{\mathcal{S}^n} = \iint \gamma_n(|s|)(D_n^s[u_k - u](x))^2 \mathrm{d}s\mathrm{d}x \leqq \varepsilon^2,$$

which completes the proof.

For $\Omega \subset \mathbb{R}^N$, we let $\mathcal{S}^n_\Omega = \mathcal{S}^{n,\gamma_n}_\Omega$ denote the closure of $C^\infty_c(\Omega)$ in $\mathcal{S}^n$, that is,

$$\mathcal{S}^n_\Omega = \{u \in L^2(\mathbb{R}^N) : u_k \to u \text{ in } \mathcal{S}^n \text{ as } k \to \infty \text{ for some } u_k \in C^\infty_c(\Omega)\}.$$

It is important to note that, for the later discussion on problems defined in a bounded domain, if $u \in \mathcal{S}^n_\Omega$ and $supp(\gamma_n) \subset B_\delta(\mathbf{0})$, then

$$\iint \gamma_n(|s|)|D^s_n[u](x)|^2 \mathrm{d}s\mathrm{d}x = \int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})} \gamma_n(|s|)|D^s_n[u](x)|^2 \mathrm{d}s\mathrm{d}x$$

where $\Omega_\delta := \{x \in \mathbb{R}^N \backslash \overline{\Omega} : \mathrm{dist}(x, \partial\Omega) < \delta\}$.

We may see that $|\cdot|_{\mathcal{S}^n}$ is indeed a norm as demonstrated in the following lemma.

Lemma 5. *Suppose $u \in \mathcal{S}^n_\Omega$ with γ_n satisfying (K) and $supp(\gamma_n) \subset B_\delta(\mathbf{0})$, and*

$$\iint \gamma_n(|s|)|D^s_n[u](x)|^2 \mathrm{d}s\mathrm{d}x = 0,$$

then $u = 0$.

Proof. The conditions of the lemma imply that

$$D^s_n[u](x) = 0 \quad \text{for almost everywhere } x \in \Omega \cup \Omega_\delta \quad \text{and} \quad s \in B_\delta(\mathbf{0}).$$

Thus, $u = u(x)$ must be a polynomial of degree $(n-1)$ almost everywhere in $\Omega \cup \Omega_\delta$. Now since $u|_{\Omega_\delta} = 0$ by assumption, we have $u(x) \equiv 0$ for almost everywhere $x \in \Omega \cup \Omega_\delta$.

2.3. The nonlocal operators

It is easy to see that $\mathcal{S}^n_\Omega$ is also a Hilbert space with respect to the same inner product, we can define naturally via the Riesz representation theorem a linear operator $\mathcal{L}_n$ from $\mathcal{S}^n_\Omega$ to its dual $(\mathcal{S}^n_\Omega)^*$ by

$$\langle \mathcal{L}_n u, v\rangle = ((u, v))_{\mathcal{S}^n}, \quad \forall\, u, v \in \mathcal{S}^n_\Omega. \tag{10}$$

First, $\mathcal{L}_n$ is a bounded linear operator on $\mathcal{S}^n_\Omega$. Next, if in addition to (K), we also have $\gamma_n = \gamma_n(|x|) \in L^1_{loc}(\mathbb{R}^N)$, then we have already seen from Equation (9) that

$$\mathcal{L}_n u(x) = (-1)^n \int_{\mathbb{R}^N} \gamma_n(|s|) D^s_{2n}[u](x)\mathrm{d}s, \quad \text{almost everywhere } x \in \Omega.$$

However, $\mathcal{L}_n$ is an unbounded operator if $\gamma_n(|\cdot|) \notin L^1_{loc}(\mathbb{R}^N)$. In this case, we proceed in the way suggested by [19]. By introducing the sequence of operator

$$\mathcal{L}^\tau_n u(x) = (-1)^n \int_{\mathbb{R}^N} \gamma^\tau_n(|s|) D^s_{2n}[u](x)\mathrm{d}s,$$

for $\gamma^\tau_n(|s|)$ defined earlier, we can show that $\mathcal{L}^\tau_n u \to \mathcal{L}_n u$, where $\mathcal{L}_n u$ is defined as

$$\mathcal{L}_n u(x) = (\mathrm{P.V.})\, (-1)^n \int_{\mathbb{R}^N} \gamma_n(|s|) D^s_{2n}[u](x)\mathrm{d}s. \tag{11}$$

3. Nonlocal Sobolev type inequalities

In this section, we show several nonlocal Sobolev type inequalities, including Poincaré type and Gagliardo–Nirenberg type inequalities. First, we prove two kinds of nonlocal Poincaré type inequalities. The first kind says that for every function u of the space $\mathcal{S}^n_\Omega$, the L^2 norm of u can be bounded in terms of $|u|_{\mathcal{S}^n}$.

Theorem 2. (The 1st nonlocal Poincaré inequality) *For $n \in \mathbb{Z}^+$ and kernel γ_n satisfying (K), there exists $C = C(n, \gamma_n, \Omega)$ such that*

$$\|u\|_{L^2} \leqq C|u|_{\mathcal{S}^n} \quad \forall\, u \in \mathcal{S}^n_\Omega.$$

The second kind of nonlocal Poincaré type inequalities extends the results above and shows that a lower order norm (say, the n-th order norm $|u|_{\mathcal{S}^n}$) can be bounded by a higher order norm (say, $|u|_{\mathcal{S}^{n+1}}$) for any function u in the space defined by the latter (that is, $\mathcal{S}^{n+1}_\Omega$). Obviously, this cannot be true for arbitrary kernel functions γ_n and γ_{n+1}. Hence, besides the assumption that $\gamma_1, \gamma_n, \gamma_{n+1}$ are kernels satisfying (K) respectively, we assume further that,

$$\gamma_k \text{ is non-increasing, } \operatorname{supp}\{\gamma_k(|\cdot|)\} \subset B_1(\mathbf{0}) \quad \text{for } \ k = 1, n, \quad \gamma_{n+1} = \gamma_1\,\gamma_n, \tag{12}$$

and there is a constant C such that

$$\mathcal{I}_n(\xi)\,\mathcal{I}_1(\xi) \leqq C\,\mathcal{I}_{n+1}(\xi),\ \forall \xi \in \mathbb{R}^N, \tag{13}$$

where

$$\mathcal{I}_k(\xi) = \int \gamma_k(|s|)(1 - \cos(\xi \cdot s))^k \mathrm{d}s,\ \forall k. \tag{14}$$

We remark that while the requirement (13) might not appear very intuitive at the first sight, it actually can be satisfied by a large class of kernels. For instance, if

$$\gamma_n(|s|) = (\gamma_1)^n(|s|), \quad \text{and} \quad \gamma_{n+1}(|s|) = (\gamma_1)^{n+1}(|s|), \tag{15}$$

then (13) is true. This is a simple consequence of the fact that for $A, B : X \to \mathbb{R}$ and μ a positive measure on X, if

$$(A(x) - A(y))(B(x) - B(y)) \geqq 0, \quad \forall\, x, y \in X$$

then

$$\int_X AB\mathrm{d}\mu \geqq \frac{1}{\mu(X)} \int_X A\mathrm{d}\mu \int_X B\mathrm{d}\mu.$$

We may apply $A(x) = (B(x))^n$ with $B(x) = \gamma_1(|x|)(1 - \cos(\xi \cdot x))$ and μ the Lebesgue measure, then we see that (13) holds with C being the volume of the unit ball in $\mathbb{R}^N$.

To present another class of examples different from the above, we first offer an alternative characterization of (13) in the following lemma, whose proof is left to the appendix.

Lemma 6. (Another characterization of (13)) *Assume that γ_1, γ_n and $\gamma_{n+1} = \gamma_1\gamma_n$ are kernels that satisfy (K) respectively and without loss of generality that $B_\eta(\mathbf{0}) \subset Supp\{\gamma_k(|\cdot|)\} \subset B_1(\mathbf{0})$ for all k. In addition, assume that the following properties are satisfied.*

(i) $$\int_{|s|<\varepsilon} \gamma_n(|s|)|s|^{2n}\mathrm{d}s \int_{|s|<\varepsilon} \gamma_1(|s|)|s|^2\mathrm{d}s \leqq C \int_{|s|<\varepsilon} \gamma_{n+1}(|s|)|s|^{2n+2}\mathrm{d}s$$

for any $\varepsilon \leqq 1$ with C independent of ε.

(ii) $$\left(\int_{|s|<\varepsilon} \gamma_k(|s|)|s|^{2k}\mathrm{d}s\right)\left(\int_{\varepsilon<|s|<1} \gamma_k(|s|)\mathrm{d}s\right)^{-1} \leqq C\varepsilon^{2k}, \quad k = 1, n,$$

for any $\varepsilon \leqq \eta$ with C independent of ε.

Then we have (13) satisfied.

Let us now check that if γ_1 and γ_n satisfy

$$m_1|x|^{-\beta_1} \leqq \gamma_1(|x|) \leqq M_1|x|^{\beta_1} \quad \text{and} \quad m_2|x|^{-\beta_2} \leqq \gamma_n(|x|) \leqq M_2|x|^{\beta_2}$$

for some $\beta_1 \in [0, N+2)$ and $\beta_2 \in [0, N+2n)$, then (13) holds. Indeed, by direct integration, we get the condition (i) of Lemma 6 since

$$\int_{|s|<\varepsilon} \gamma_n(|s|)|s|^{2n}\mathrm{d}s \int_{|s|<\varepsilon} \gamma_1(|s|)|s|^2\mathrm{d}s \leqq CM_1M_2\varepsilon^{2N+2n+2-\beta_2-\beta_1},$$

and

$$\int_{|s|<\varepsilon} \gamma_n(|s|)\gamma_1(|s|)|s|^{2n+2}\mathrm{d}s \geqq Cm_1m_2\varepsilon^{N+2n+2-\beta_1-\beta_2}.$$

As for the condition (ii) in Lemma 6, we show the case for $k = 1$ as an illustration. For $\beta_1 \leqq N$,

$$\begin{aligned}&\left(\int_{|s|<\varepsilon} \gamma_1(|s|)|s|^2\mathrm{d}s\right) \Big/ \left(\int_{\varepsilon<|s|<1} \gamma_1(|s|)\mathrm{d}s\right)\\ &\quad\leqq CM_1\varepsilon^{N+2-\beta_1} \Big/ \left(\int_{\varepsilon<|s|<1} \gamma_1(|s|)\mathrm{d}s\right) \leqq C\varepsilon^2,\end{aligned}$$

and for $N < \beta_1 < N+2$,

$$\begin{aligned}\left(\int_{|s|<\varepsilon} \gamma_1(|s|)|s|^2\mathrm{d}s\right) \Big/ \left(\int_{\varepsilon<|s|<1} \gamma_1(|s|)\mathrm{d}s\right) &\leqq C\frac{M_1}{m_1}\varepsilon^{N+2-\beta_1}/(\varepsilon^{N-\beta_1} - 1)\\ &= C\frac{M_1}{m_1}\varepsilon^2/(1-\varepsilon^{\beta_1-N}) \leqq C\varepsilon^2, \quad \text{for } \varepsilon \leqq 1/2.\end{aligned}$$

We now present the second nonlocal Poincaré inequality whose proof is presented in the next section as it relies on a compactness result given there.

Theorem 3. (The 2nd nonlocal Poincaré inequality) *For $n \in \mathbb{Z}^+$ and kernels $\{\gamma_n\}$ satisfying (K), (12) and (13), there exists $C = C(n, \gamma_n, \gamma_{n+1}, \Omega)$ such that*

$$\|u\|_{\mathcal{S}^n} \leqq C|u|_{\mathcal{S}^{n+1}} \quad \forall\, u \in \mathcal{S}_\Omega^{n+1}.$$

The final result of this section is a Gagliardo–Nirenberg inequality stated below.

Theorem 4. (Nonlocal Gagliardo–Nirenberg inequality) *For nonnegative integers n, k_1 and k_2 with $k_1+k_2>0$, suppose that γ_n satisfies (K), and $\alpha = k_1/(k_1+k_2)$, the following nonlocal Gagliardo–Nirenberg type inequality holds:*

$$|u|_{\mathcal{S}^n} \leqq |u|^{1-\alpha}_{\mathcal{S}^{(n-k_1)}} |u|^{\alpha}_{\mathcal{S}^{(n+k_2)}}, \tag{16}$$

where $\mathcal{S}^{(n-k_1)} = \mathcal{S}^{n-k_1,\gamma_{(n-k_1)}}$ and $\mathcal{S}^{(n+k_2)} = \mathcal{S}^{n+k_2,\gamma_{(n+k_2)}}$ with properly chosen kernels that, in particular, are given by $\gamma_{(n-k_1)} = (\gamma_n)^{1-k_1/n}$ and $\gamma_{(n+k_2)} = (\gamma_n)^{1+k_2/n}$.

We note that for $n \in \mathbb{N}$, if $k_1 = 0$, $k_2 = 1$, then the above reduces to Theorem 3 with the simple case of (15) satisfied.

3.1. Proof of Theorem 2

Variants of the case $n = 1$ can be found in many existing papers, say for example [10, 18]. Our proof follows a similar path.

Suppose that the conclusion of the theorem is false, then we can find a sequence $\{u_k \in \mathcal{S}^n_\Omega\}$ such that $\|u_k\|_{L^2} = 1$ and $|u_k|_{S^n} \to 0$ as $k \to \infty$. This leads to the existence of a weak limit $u \in L^2$ such that $u_k \rightharpoonup u$ in L^2.

Step 1. We show that u is in fact 0. Suppose $\{\phi_\varepsilon\}$ are standard mollifiers, then

$$\begin{aligned}
|\phi_\varepsilon * u_k|^2_{S^n} &= \iint \gamma_n(|s|) \left|D^s_n[\phi_\varepsilon * u_k](x)\right|^2 \mathrm{d}s\mathrm{d}x \\
&= \iint \gamma_n(|s|) \left|\int D^s_n[u_k](x-y)\phi_\varepsilon(y)\mathrm{d}y\right|^2 \mathrm{d}s\mathrm{d}x \\
&\leqq \int \phi_\varepsilon(y) \left(\iint \gamma_n(|s|)|D^s_n[u_k](x-y)|^2 \mathrm{d}s\mathrm{d}x\right) \mathrm{d}y \\
&= \int \phi_\varepsilon(y) \left(\iint \gamma_n(|s|)|D^s_n[u_k](x)|^2 \mathrm{d}s\mathrm{d}x\right) \mathrm{d}y,
\end{aligned}$$

where all integrals are over $\mathbb{R}^N$ or $\mathbb{R}^N \times \mathbb{R}^N$ respectively and the first inequality follows from the Jensen's inequality. This then leads to

$$|\phi_\varepsilon * u_k|_{S^n} \leqq |u_k|_{S^n}. \tag{17}$$

Now $u_k \rightharpoonup u$ in L^2 implies $\phi_\varepsilon * u_k \to \phi_\varepsilon * u$ strongly in L^2. So $\phi_\varepsilon * u_k \to \phi_\varepsilon * u$ almost everywhere as $k \to \infty$. Applying Fatou's lemma to (17), we get, for any fixed $\varepsilon > 0$,

$$\begin{aligned}
\iint \gamma_n(|s|)|D^s_n[\phi_\varepsilon * u](x)|^2 \mathrm{d}s\mathrm{d}x &\leqq \liminf_k \iint \gamma_n(|s|)|D^s_n[\phi_\varepsilon * u_k](x)|^2 \mathrm{d}s\mathrm{d}x \\
&\leqq \liminf_k |u_k|^2_{S^n} = 0.
\end{aligned}$$

With $\phi_\varepsilon * u \to u$ pointwise, by applying Fatou's lemma again, we get

$$\iint \gamma_n(|s|)|D_n^s[u](x)|^2 \mathrm{d}s\mathrm{d}x \leqq \liminf_{\varepsilon} \iint \gamma_n(|s|)|D_n^s[\phi_\varepsilon * u](x)|^2 \mathrm{d}s\mathrm{d}x = 0.$$

In addition, since $u_k|_{\mathbb{R}^N\setminus\overline{\Omega}} = 0$ for any k and $u_k \rightharpoonup u$, we have $u|_{\mathbb{R}^N\setminus\overline{\Omega}} = 0$. This is $u = 0$ by Lemma 5.

Step 2. We next show that $u_k \to u$ strongly in L^2. First, for some $M > 0$, we define $\bar{\gamma}_n(|x|) = \min\{M, \gamma_n(|x|)\}$. Then, with $b_n^j := (-1)^j\binom{n}{j}$, we have

$$\begin{aligned}
|u_k|_{S^n}^2 &\geqq \iint \bar{\gamma}_n(|s|)|D_n^s[u_k](x)|^2 \mathrm{d}s\mathrm{d}x = \iint \bar{\gamma}_n(|s|)\left|\sum_{j=0}^n b_n^j u_k(x + a_n^j s)\right|^2 \mathrm{d}s\mathrm{d}x \\
&= \sum_{j=0}^n (b_n^j)^2 \iint \bar{\gamma}_n(|s|)u_k^2(x + a_n^j s)\mathrm{d}s\mathrm{d}x \\
&\qquad + 2\sum_{i\neq j} b_n^i b_n^j \iint \bar{\gamma}_n(|s|)u_k(x + a_n^i s)u_k(x + a_n^j s)\mathrm{d}s\mathrm{d}x \\
&= \sum_{j=0}^n (b_n^j)^2 \iint \bar{\gamma}_n(|s|)u_k^2(x)\mathrm{d}s\mathrm{d}x \\
&\qquad + 2\sum_{i\neq j} b_n^i b_n^j \int \left(\int \bar{\gamma}_n(|s|)u_k(x + (a_n^i - a_n^j)s)\mathrm{d}s\right) u_k(x)\mathrm{d}x \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Now the first term

$$\mathrm{I} = \sum_{j=0}^n (b_n^j)^2 \int \bar{\gamma}_n(|s|)\mathrm{d}s \int u_k^2(x)\mathrm{d}x \to c\|u_k\|_{L^2}^2, \quad \text{as } k \to \infty$$

for a constant $c > 0$. The second term

$$\begin{aligned}
\mathrm{II} &= 2\sum_{i\neq j} b_n^i b_n^j \int \left(\int \bar{\gamma}_n(|s|)u_k(x + (a_n^i - a_n^j)s)\mathrm{d}s\right) u_k(x)\mathrm{d}x \\
&= 2\sum_{i\neq j} b_n^i b_n^j \int F_k(x)u_k(x)\mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
F_k(x) &= \int_{\mathbb{R}^N} \bar{\gamma}_n(|s|)u_k(x + (a_n^i - a_n^j)s)\mathrm{d}s \\
&= \frac{1}{a_n^i - a_n^j}\int_{\mathbb{R}^N} \bar{\gamma}_n\left(\left|\frac{y - x}{a_n^i - a_n^j}\right|\right)u_k(y)\mathrm{d}y \\
&= \frac{1}{a_n^i - a_n^j}\int_{\Omega} \bar{\gamma}_n\left(\left|\frac{y - x}{a_n^i - a_n^j}\right|\right)u_k(y)\mathrm{d}y,
\end{aligned}$$

which can be seen as the action of a Hilbert-Schmidt operator since $\bar{\gamma}_n \leqq M$. So

$$F_k = F_k(x) \to F = F(x) = \frac{1}{a_n^i - a_n^j} \int_\Omega \bar{\gamma}_n \left(\left| \frac{y-x}{a_n^i - a_n^j} \right| \right) u(y) \mathrm{d}y = 0$$

strongly in L^2. Now that $F_k \to 0$, and $u_k \rightharpoonup u$, we have

$$\lim_k \int F_k(x) u_k(x) \mathrm{d}x = 0,$$

which implies II$\to 0$. Thus $\|u_k\|_{L^2} \to 0$ which is a contradiction to $\|u_k\|_{L^2} = 1$. □

3.2. *Proof of Theorem 4*

First, by applying Lemma 3, we only need to show the following inequality in order to have (16).

$$\begin{aligned} &\iint \gamma_n(|s|)(2-2\cos(\xi \cdot s))^n |\hat{u}(\xi)|^2 \mathrm{d}\xi \mathrm{d}s \\ &\leqq \left(\iint \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi \cdot s))^{n-k_1} |\hat{u}(\xi)|^2 \mathrm{d}\xi \mathrm{d}s \right)^{1-\alpha} \\ &\quad \cdot \left(\iint \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi \cdot s))^{n+k_2} |\hat{u}(\xi)|^2 \mathrm{d}\xi \mathrm{d}s \right)^{\alpha} \end{aligned} \tag{18}$$

where $\hat{u}$ denotes the Fourier transform of u. Now let $n_1 = (1-\alpha)(n-k_1)$ and $n_2 = \alpha(n+k_2)$, then $n = n_1 + n_2$. By applying Hölder's inequality, we have

$$\begin{aligned} \int \gamma_n(|s|)(2-2\cos(\xi \cdot s))^n \mathrm{d}s &= \int (\gamma_n(|s|))^{n_1/n + n_2/n} (2-2\cos(\xi \cdot s))^{n_1+n_2} \mathrm{d}s \\ &\leqq \left(\int \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi \cdot s))^{n-k_1} \mathrm{d}s \right)^{1-\alpha} \\ &\quad \cdot \left(\int \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi \cdot s))^{n+k_2} \mathrm{d}s \right)^{\alpha}. \end{aligned}$$

Then, by splitting $|\hat{u}(\xi)|^2$ and using Hölder's inequality with respect to the integral in ξ, we have

$$\begin{aligned} &\iint \gamma_n(|s|)(2-2\cos(\xi \cdot s))^n |\hat{u}(\xi)|^2 \mathrm{d}s \mathrm{d}\xi \\ &\leqq \int \left(\int \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi \cdot s))^{n-k_1} \mathrm{d}s |\hat{u}(\xi)|^2 \right)^{1-\alpha} \\ &\quad \cdot \left(\int \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi \cdot s))^{n+k_2} \mathrm{d}s |\hat{u}(\xi)|^2 \right)^{\alpha} \mathrm{d}\xi \end{aligned}$$

$$\leqq \left(\iint \gamma_{(n-k_1)}(|s|)(2-2\cos(\xi\cdot s))^{n-k_1}|\hat{u}(\xi)|^2 \mathrm{d}s\mathrm{d}\xi \right)^{1-\alpha} \cdot \left(\iint \gamma_{(n+k_2)}(|s|)(2-2\cos(\xi\cdot s))^{n+k_2}|\hat{u}(\xi)|^2 \mathrm{d}s\mathrm{d}\xi \right)^{\alpha}.$$

Thus (18) is true. □

4. Compact embeddings

The nonlocal Poincaré type inequalities imply continuous embedding between spaces. In many applications, a stronger compact embedding result is necessary. Here we give conditions so that such compactness holds.

Let γ_1 satisfy (K). In addition, as in [5], we assume that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \left(\int_{|x|<\varepsilon} |x|^2 \gamma_1(|x|)\mathrm{d}x \right)^{-1} = 0. \tag{19}$$

We note that if $\gamma_1(|x|)$ has a singularity of the type $1/|x|^{N+2s}$ for x at the origin with exponent $s \in (0, 1)$, which is a typical kernel for the standard Sobolev space H^s, then the assumption (19) is satisfied. The results established here are more general, however, as in the spirit of [5], that are valid for kernels which may not have such singularities at the origin, including $\gamma_1(|x|) \in L^1_{loc}(\mathbb{R}^N)$.

Theorem 5. *Suppose that the kernels $\gamma_1, \gamma_n, \gamma_{n+1}$ satisfy (12), (13) and (19). Let $\mathcal{F}$ be a bounded set in $\mathcal{S}^n_\Omega$. If*

$$|u|_{\mathcal{S}^{n+1}} \leqq C_0 \quad \forall\, u \in \mathcal{F}, \tag{20}$$

then $\mathcal{F}$ is precompact in $\mathcal{S}^n_\Omega$ and any of its limit point is in $\mathcal{S}^{n+1}_\Omega$ with a norm bounded by C_0.

4.1. Proof of Thoerem 5

Step 1. Suppose (ϕ_ε) are standard mollifiers defined as $\phi_\varepsilon(x) = \varepsilon^{-N}\phi(x/\varepsilon)$ with integrals equal to 1, $\|\phi\|_\infty \leqq C$ and $\mathrm{Supp}\{\phi_\varepsilon\} \subset B_\varepsilon(\mathbf{0})$. We claim that for any $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon)$ such that

$$|\phi_\varepsilon * u - u|_{\mathcal{S}^n} \leqq \sigma \quad \forall\, u \in \mathcal{F}.$$

Indeed,

$$\begin{aligned} |\phi_\varepsilon * u - u|_{\mathcal{S}^n} &= \iint \gamma_n(|s|)|D^s_n[\phi_\varepsilon * u - u](x)|^2 \mathrm{d}s\mathrm{d}x \\ &= \iint \gamma_n(|s|)|(\phi_\varepsilon * D^s_n[u] - D^s_n[u])(x)|^2 \mathrm{d}s\mathrm{d}x \\ &= \iint \gamma_n(|s|) \left| \int_{\mathbb{R}^N} (D^s_n[u](x-y) - D^s_n[u](x))\phi_\varepsilon(y)\mathrm{d}y \right|^2 \mathrm{d}s\mathrm{d}x \\ &\leqq \int_{\mathbb{R}^N} \iint \gamma_n(|s|)\phi_\varepsilon(y)|D^s_n[u](x-y) - D^s_n[u](x)|^2 \mathrm{d}y\mathrm{d}s\mathrm{d}x, \end{aligned}$$

by Jensen's inequality. Similarly to the proof of Lemma 3,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\iint \gamma_n(|s|)\phi_\varepsilon(y)|D_n^s[u](x-y)-D_n^s[u](x)|^2\mathrm{d}y\mathrm{d}s\right)\mathrm{d}x\\
&=\iint \left\|\sqrt{\gamma_n(|s|)\phi_\varepsilon(y)}\mathscr{F}\left(D_n^s[u](\cdot-y)-D_n^s[u](\cdot)\right)\right\|^2_{L^2(R^N)}\mathrm{d}y\mathrm{d}s\\
&=\iint\left(\int_{\mathbb{R}^N}\gamma_n(|s|)\phi_\varepsilon(y)(2-2\cos(\xi\cdot y))(2-2\cos(\xi\cdot s))^n|\hat{u}(\xi)|^2\mathrm{d}\xi\right)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Combining the above two equalities and using $\|\phi_\varepsilon\|_\infty \leqq C\varepsilon^{-N}$, we get

$$\begin{aligned}
&|\phi_\varepsilon * u - u|_{\mathcal{S}^n}\\
&\leqq C\varepsilon^{-N}\int_{|y|<\varepsilon}\iint \gamma_n(|s|)(1-\cos(\xi\cdot s))^n|\hat{u}(\xi)|^2(1-\cos(\xi\cdot y))\mathrm{d}\xi\mathrm{d}s\mathrm{d}y\\
&= C\varepsilon^{-N}\int_0^\varepsilon t^{N-1}G(t)\mathrm{d}t, \qquad (21)
\end{aligned}$$

where $G(t)$ is defined as

$$G(t)=\int_{\omega\in S^{N-1}}\iint \gamma_n(|s|)(1-\cos(\xi\cdot s))^n|\hat{u}(\xi)|^2(1-\cos(t\xi\cdot w))\mathrm{d}\xi\mathrm{d}s\mathrm{d}\omega.$$

Using the fact that $1-\cos(2a)\leqq 2^2(1-\cos(a))$ for any $a\in\mathbb{R}$, we have

$$G(2t)\leqq 2^2G(t).$$

Applying Lemma 4 with $g(t)=t^{-2}G(t)$, $h(t)=\gamma_1(t)$, and $d=N+2$, we get

$$\frac{C}{\varepsilon^{N+2}}\int_0^\varepsilon t^{N-1}G(t)\mathrm{d}t \leqq C\left(\int_0^\varepsilon t^{N-1}G(t)\gamma_1(t)\mathrm{d}t\right)\Big/\left(\int_{|x|<\varepsilon}|x|^2\gamma_1(|x|)\mathrm{d}x\right). \qquad (22)$$

For the right hand side, we have that, without loss of generality, for $\varepsilon<1$,

$$\begin{aligned}
\int_0^\varepsilon t^{N-1}G(t)\gamma_1(t)\mathrm{d}t &= \int_{\mathbb{R}^N}|\hat{u}(\xi)|^2\mathcal{I}_n(\xi)\left(\int_{|s|<\varepsilon}\gamma_1(|s|)(1-\cos(\xi\cdot s))\mathrm{d}s\right)\mathrm{d}\xi\\
&\leqq \int_{\mathbb{R}^N}|\hat{u}(\xi)|^2\mathcal{I}_n(\xi)\left(\int_{|s|<1}\gamma_1(|s|)(1-\cos(\xi\cdot s))\mathrm{d}s\right)\mathrm{d}\xi\\
&= \int_{\mathbb{R}^N}|\hat{u}(\xi)|^2\mathcal{I}_n(\xi)\,\mathcal{I}_1(\xi)\mathrm{d}\xi,
\end{aligned}$$

where $\mathcal{I}_n(\xi)$ and $\mathcal{I}_1(\xi)$ are defined as in (14).

With the assumptions (12)–(13), we have by Lemma 3 that

$$\begin{aligned}
\int_0^\varepsilon t^{N-1}G(t)\gamma_1(t)\mathrm{d}t &\leqq C\int_{\mathbb{R}^N}|\hat{u}(\xi)|^2\mathcal{I}_{n+1}(\xi)\mathcal{I}_1(\xi)\mathrm{d}\xi\\
&= \int_{\mathbb{R}^N}|\hat{u}(\xi)|^2\int_{|s|<1}\gamma_{n+1}(|s|)(1-\cos(\xi\cdot s))^{n+1}\mathrm{d}s\mathrm{d}\xi\\
&\leqq C\iint \gamma_{n+1}(|s|)|D_{n+1}^s[u](x)|^2\mathrm{d}s\mathrm{d}x. \qquad (23)
\end{aligned}$$

Combining (21), (22) and (23), we obtain

$$|\phi_\varepsilon * u - u|_{\mathcal{S}^n} \leqq C\varepsilon^2 \left(\int_{|x|<\varepsilon} |x|^2 \gamma_1(|x|) \mathrm{d}x \right)^{-1} \to 0 \quad \text{as} \ \ \varepsilon \to 0$$

by condition (19)

Step 2. In order to use Arzela–Ascoli, we first claim that $\{\phi_\varepsilon * u\}_{u\in\mathcal{F}}$ are uniformly bounded and equicontinuous, that is,

$$\|\phi_\varepsilon * u\|_{L^\infty(R^N)} \leqq C_\varepsilon \|u\|_{L^2(\mathbb{R}^N)}$$

and

$$|\phi_\varepsilon * u(x) - \phi_\varepsilon * u(y)| \leqq C_\varepsilon \|u\|_{L^2(\mathbb{R}^N)} |x-y|,$$

where C_ε only depends on ε. The first inequality follows from Hölder's inequality with $C_\varepsilon = \|\phi_\varepsilon\|_{L^2}$. For the second inequality, since $\|\nabla(\phi_\varepsilon * u)\|_{L^\infty} = \|(\nabla\phi_\varepsilon) * u\|_{L^\infty} \leqq \|\nabla\phi_\varepsilon\|_{L^2}\|u\|_{L^2}$, the inequality is true with $C_\varepsilon = \|\nabla\phi_\varepsilon\|_{L^2}$. Now by Theorem 2 and the bound (20), we see that $\{\phi_\varepsilon * u\}_{u\in\mathcal{F}}$ are uniformly bounded and equicontinuous as claimed. So $\phi_\varepsilon * u$ has a uniformly convergent subsequence by Arzela–Ascoli. Moreover, since $|\phi_\varepsilon * u|_{\mathcal{S}^n} \leqq |u|_{\mathcal{S}^n} \leqq C$ by (17), the convergence is also true in $\mathcal{S}^n_\Omega$ by dominated convergence theorem. So $\{\phi_\varepsilon * u\}_{u\in\mathcal{F}}$ is precompact in $\mathcal{S}^n_\Omega$ for any $\varepsilon > 0$.

Step 3. We now combine Steps 1 and 2 to show that $\mathcal{F}$ is also precompact in $\mathcal{S}^n_\Omega$.

Indeed, $\forall\varepsilon > 0$, by Step 1, there exists $\sigma > 0$, such that

$$|\phi_\varepsilon * u - u|_{\mathcal{S}^n} \leqq \sigma \quad \forall\, u \in \mathcal{F}.$$

So for $u, g \in \mathcal{F}$ with $|\phi_\varepsilon * u - \phi_\varepsilon * g|_{\mathcal{S}^n} \leqq \varepsilon$, we have

$$|u - g|_{\mathcal{S}^n} \leqq 2\sigma + \varepsilon,$$

by triangle inequality. Now for any $\lambda > 0$, we could choose ε small enough such that $\varepsilon + 2\sigma < \lambda$. Since $\{\phi_\varepsilon * u\}_{u\in\mathcal{F}}$ is precompact in $\mathcal{S}^n_\Omega$, there exists a finite ε-cover $\{\phi_\varepsilon * u_1, \phi_\varepsilon * u_2, \ldots, \phi_\varepsilon * u_k\}$ of $\{\phi_\varepsilon * u\}_{u\in\mathcal{F}}$. Then it immediately follows that $\{u_1, u_2, \ldots, u_k\}$ is a λ-cover of $\mathcal{F}$, which means that $\mathcal{F}$ is precompact in $\mathcal{S}^n_\Omega$.

Step 4. Let us verify that the limit point of $\mathcal{F}$ is in $\mathcal{S}^{n+1}_\Omega$. Suppose without loss of generality that $\{u_k\} \subset \mathcal{F}$ and $u_k \to u$ in $\mathcal{S}^n_\Omega$, then we need to show $|u|_{\mathcal{S}^{n+1}} \leqq C_0$. By Theorem 2 we know that u_k converges to u strongly in L^2. So $u_k(x) \to u(x)$ pointwise up to a set of measure zero. Then by Fatou's lemma

$$|u|^2_{\mathcal{S}^{n+1}} \leqq \liminf_k |u_k|^2_{\mathcal{S}^{n+1}} \leqq C_0^2.$$

□

4.2. Another compactness result

To satisfy the assumption (19), the kernel γ_1 has to have certain singularity at zero in general, in particular, γ_1 cannot be integrable. In the latter case, we have the following variant of Theorem 5.

Theorem 6. (A variant of Thoerem 5) *Suppose that the kernels $\gamma_1, \gamma_n, \gamma_{n+1}$ satisfy assumptions (12)–(13). If (u_k) is a bounded sequence in $\mathcal{S}^n_\Omega$, and*

$$|u_k|_{\mathcal{S}^{n+1}} \to 0 \quad as \quad k \to 0, \tag{24}$$

then (u_k) is relatively compact in $\mathcal{S}^n_\Omega$ and any of its limit point u is in $\mathcal{S}^{n+1}_\Omega$ with $|u|_{\mathcal{S}^{n+1}} = 0$.

Proof. Following Step 1 of proof of Thoerem 5, we have

$$\|\phi_\varepsilon * u_k - u_k\|_{\mathcal{S}^n} \leqq C\varepsilon^2 \left(\int_{|x|<\varepsilon} |x|^2 \gamma_1(|x|)\mathrm{d}x \right)^{-1} \|u_k\|_{\mathcal{S}^{n+1}} \quad \forall\, k. \tag{25}$$

Now since $|u_k|_{\mathcal{S}^{n+1}} \to 0$ as $k \to \infty$, (25) reduces to

$$|\phi_\varepsilon * u_k - u_k|_{\mathcal{S}^n} \to 0 \quad \text{as} \quad k \to \infty \quad \forall \varepsilon > 0. \tag{26}$$

Then similarly as Step 2 of proof of Thoerem 5, we can show $(\phi_\varepsilon * u_k)_k$ is relatively compact in $\mathcal{S}^n_\Omega$ for any $\varepsilon > 0$. Therefore (u_k) is also relatively compact in $\mathcal{S}^n_\Omega$ by (26).

Finally, suppose $u_k \to u$ in $\mathcal{S}^n_\Omega$ without loss of generality. Similarly as Step 4 of Theorem 5, we have

$$|u|^2_{\mathcal{S}^{n+1}} \leqq \liminf_k |u_k|^2_{\mathcal{S}^{n+1}} = 0.$$

4.3. Proof of Theorem 3

The nonlocal Poincaré type inequality in Theorem 3 is a corollary of Theorem 6.

Assume the opposite, then there exists a sequence (u_k) such that $|u_k|_{S^n} = 1$ and $|u_k|_{\mathcal{S}^{n+1}} \to 0$. Then by Theorem 6, (u_k) is relatively compact in $\mathcal{S}^n_\Omega$. Suppose the limit of u_k (up to a subsequence) in $\mathcal{S}^n_\Omega$ is u. Then on one hand, $|u|_{S^n} = 1$. On the other hand, $|u|_{\mathcal{S}^{n+1}} = 0$ by Theorem 6, which implies $u = 0$ by Lemma 5, so it is in contradiction to $|u|_{S^n} = 1$. □

5. Limiting properties for vanishing nonlocality

In this section, we consider a fixed integer n, and study the a family of kernels γ_n^δ parametrized by δ that characterizes the nonlocal interaction length. Suppose that γ_n satisfies (K) and $\mathrm{Supp}\{\gamma_n\} \subset B_1(\mathbf{0})$. Then the rescaled kernel γ_n^δ is defined by

$$\gamma_n^\delta(|s|) = \frac{1}{|\delta|^{N+2n}} \gamma_n \left(\frac{|s|}{\delta} \right) \quad \text{for } s \in B_\delta(\mathbf{0}), \tag{27}$$

which satisfies

$$\int |s|^{2n}\gamma_n^\delta(|s|)\mathrm{d}s = \int |s|^{2n}\gamma_n(|s|)\mathrm{d}s.$$

To study the limiting behavior as $\delta \to 0$, we first give a continuous embedding property.

Lemma 7. *Assume the kernel γ_n satisfies (K) with Supp$\{\gamma_n\} \subset B_1(\mathbf{0})$ and $u \in H^n(\Omega\cup\Omega_1)$. Then*

$$\int_{\Omega\cup\Omega_1}\int_{B_1(\mathbf{0})} \gamma_n(|s|)|D_n^s[u](x)|^2\mathrm{d}s\mathrm{d}x \leqq C\left(\int |s|^{2n}\gamma_n(|s|)\mathrm{d}s\right)\|u\|_{H^n}^2,$$

where C depends only on n and $\Omega\cup\Omega_1$ and H^n is the standard Sobolev space.

Proof. First, by standard extension we may always assume that $u \in H^n(\mathbb{R}^N)$. Then, by the multivariate version of Talyor's theorem, we have

$$D_n^s[u](x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{|\alpha|=n}\frac{|\alpha|}{\alpha!}(a_n^j s)^\alpha\int_0^1(1-t)^{n-1}D^\alpha u(x+ta_n^j s)\mathrm{d}t,$$

where the multi-index notation is used, namely, for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$

$$|\alpha| = \alpha_1+\cdots+\alpha_N, \quad \alpha! = \alpha_1!\cdots\alpha_N!, \quad s^\alpha = s_1^{\alpha_1}\cdots s_N^{\alpha_N}.$$

Therefore we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}|D_n^s[u](x)|^2\mathrm{d}x\\
&\leqq c_1(n)\int_{\mathbb{R}^N}|s|^{2n}\sum_{j=0}^{n}\sum_{|\alpha|=n}\left(\int_0^1(1-t)^{n-1}D^\alpha u(x+ta_n^j s)\mathrm{d}t\right)^2\mathrm{d}x\\
&\leqq c_1(n)\int_{\mathbb{R}^N}|s|^{2n}\sum_{j=0}^{n}\sum_{|\alpha|=n}\int_0^1\left(D^\alpha u(x+ta_n^j s)\right)^2\mathrm{d}t\mathrm{d}x\\
&= c_1(n)|s|^{2n}\sum_{j=0}^{n}\sum_{|\alpha|=n}\int_0^1\int_{\mathbb{R}^N}\left(D^\alpha u(x+ta_n^j s)\right)^2\mathrm{d}x\mathrm{d}t\\
&= c_1(n)(n+1)|s|^{2n}\sum_{|\alpha|=n}\int_{\mathbb{R}^N}\left(D^\alpha u(x)\right)^2\mathrm{d}x\\
&\leqq c_1(n)(n+1)|s|^{2n}\|u\|_{H^n(\mathbb{R})} \leqq C(n,\Omega\cup\Omega_\delta)|s|^{2n}\|u\|_{H^n(\Omega\cup\Omega_\delta)}^2
\end{aligned}$$

where $c_1(n) = \sum_{j=0}^n\sum_{|\alpha|=n}\left((-1)^j\binom{n}{j}(a_n^j)^n\frac{|\alpha|}{\alpha!}\right)^2$. This implies that

$$\iint \gamma_n(|s|)|D_n^s[u](x)|^2\mathrm{d}s\mathrm{d}x \leqq C\left(\int |s|^{2n}\gamma_n(|s|)\mathrm{d}s\right)\|u\|_{H^n}^2.$$

Lemma 8. *Suppose that γ_n^δ are rescaled kernels defined as (27), then for $u \in H_0^n(\Omega \cup \Omega_1)$,*

$$\lim_{\delta\to 0}\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})}\gamma_n^\delta(|s|)|D_n^s[u](x)|^2\mathrm{d}s\mathrm{d}x = \sum_{|\alpha|=n} M(n,\alpha)\int_\Omega (D^\alpha u(x))^2\mathrm{d}x \tag{28}$$

where

$$0 < M(n,\alpha) = \int s^{2\alpha}\gamma_n(|s|)\mathrm{d}s \sum_{\substack{|\beta|=|\theta|=n,\\ \beta+\theta=2\alpha}} \frac{(n!)^2}{\beta!\theta!}, \quad D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}$$

with multi-indices $\alpha = (\alpha_1,\ldots,\alpha_N)$, $\beta = (\beta_1,\ldots,\beta_N)$, and $\theta = (\theta_1,\ldots,\theta_N)$ used.

Proof. First let $U_\delta(x,s) = (\gamma_n^\delta(|s|))^{1/2}|D_n^s[u](x)|$, then we have to prove that

$$\lim_{\delta\to 0}\|U_\delta\|_{L^2}^2 = \sum_{|\alpha|=n} M(n,\alpha)\int_\Omega (D^\alpha u(x))^2\mathrm{d}x.$$

By Lemma 7, we have for any $u, v \in H_0^n(\Omega\cup\Omega_1)$,

$$|\|U_\delta\|_{L^2} - \|V_\delta\|_{L^2}| \leqq \|U_\delta - V_\delta\|_{L^2} \leqq C\|u-v\|_{H^n}.$$

Therefore it suffices to prove the result for u in the dense subset $C_c^\infty(\Omega\cup\Omega_1)$. In this case, we obtain by Taylor expansion that

$$D_n^s[u](x) = \sum_{j=0}^n (-1)^j\binom{n}{j}\sum_{|\beta|=n}\frac{(a_n^j s)^\beta}{\beta!}D^\beta u(x) + O(|s|^{n+1}).$$

Then the higher order terms can be dropped since

$$\int \gamma_n^\delta(|s|)|s|^{2n+1} \to 0.$$

Thus,

$$\begin{aligned}&\lim_{\delta\to 0}\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})}\gamma_n^\delta(|s|)|D_n^s[u](x)|^2\mathrm{d}s\mathrm{d}x \\ &= \left(\sum_{j=0}^n(-1)^j\binom{n}{j}(a_n^j)^n\right)^2 \lim_{\delta\to 0}\int_{\Omega\cup\Omega_\delta}\int_{B_\delta(\mathbf{0})}\gamma_n^\delta(|s|)\left(\sum_{|\beta|=n}\frac{s^\beta}{\beta!}D^\beta u(x)\right)^2\mathrm{d}s\mathrm{d}x\end{aligned}$$

$$= (n!)^2 \lim_{\delta\to 0} \sum_{|\beta|=|\theta|=n} \frac{1}{\beta!\theta!} \int_{\Omega\cup\Omega_\delta} \int_{B_\delta(\mathbf{0})} \gamma_n^\delta(|s|) s^{\beta+\theta} D^\beta u(x) D^\theta u(x) \mathrm{d}s\mathrm{d}x$$
$$= \sum_{|\beta|=|\theta|=n} \frac{(n!)^2}{\beta!\theta!} \int s^{\beta+\theta}\gamma_n(|s|)\mathrm{d}s \int_\Omega D^\beta u(x) D^\theta u(x) \mathrm{d}x,$$

where Equation (6) is used.

Now, for the summation in the last term, if there exists some index i such that $\beta_i + \theta_i$ is odd, then the integral of $s^{\beta+\theta}\gamma_n(|s|)$ becomes zero, which implies that the summation is over all β, θ with $|\beta| = |\theta| = n$ and $\beta + \theta = 2\alpha$, for some $|\alpha| = n$. In addition, since $u \in C_c^\infty(\Omega \cup \Omega_1)$, using integration by parts, we have

$$\int_\Omega D^\beta u(x) D^\theta u(x)\mathrm{d}x = \int_\Omega (D^{\frac{\beta+\theta}{2}} u(x))^2 \mathrm{d}x.$$

Combining the above results we get the expression as the righthand side of (28).

In the following we choose a sequence of kernels $\{\gamma_n^{\delta_k}\}$, where $\delta_k \to 0$, and study the compactness property as $k \to \infty$.

Theorem 7. (Compactness) *Suppose that* $\{u_k\}$ *is a bounded sequence in* $L^2(\Omega)$ *with zero extension outside* Ω. *If*

$$\sup_k \int_{\Omega\cup\Omega_{\delta_k}} \int_{B_{\delta_k}(\mathbf{0})} \gamma_n^{\delta_k}(|s|) |D_n^s[u_k](x)|^2 \mathrm{d}s\mathrm{d}x < \infty, \tag{29}$$

then $\{u_k\}$ *is relatively compact in* $L^2(\Omega)$. *Moreover, any limit point* $u \in H_0^n(\Omega)$.

Proof. We follow the proof of Theorem 5 but with a slight modification. Instead of comparing u_k with $\phi_\varepsilon * u_k$, where ϕ_ε is the standard mollifier, we compare u_k with a combination of mollifications of u_k, in the L^2 norm, in order to get an upper bound in the form of (29).

As D_n^s is defined differently for n odd or even, different estimates are sought after for the two cases. Without any loss of generality, we will only prove the case n where n is an even number. The other case is essentially the same. Now we claim that

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \left\| \binom{n}{\frac{n}{2}} u_k - \sum_{j=0}^{\frac{n}{2}-1} 2(-1)^{\frac{n}{2}-1-j} \binom{n}{j} \phi_{a_n^j \varepsilon} * u_k \right\|_{L^2} = 0. \tag{30}$$

Indeed, we can write $u_k = \int u_k(x)\phi_\varepsilon(s)\mathrm{d}s$ and $\phi_{a_n^j\varepsilon} * u_k = \int u(x + a_n^j s)\phi_\varepsilon(s)\mathrm{d}s$. By equating $\binom{n}{j}$ with $\binom{n}{n-j}$ for $j = 0, 1, \cdots, \frac{n}{2} - 1$, we can see that

$$\left|\binom{n}{\frac{n}{2}}u_k - \sum_{j=0}^{\frac{n}{2}-1} 2(-1)^{\frac{n}{2}-1-j}\binom{n}{j}\phi_{a_n^j\varepsilon} * u_k\right|$$
$$= \left|\int \sum_{j=0}^{n}(-1)^j\binom{n}{j}u_k(x+a_n^j s)\phi_\varepsilon(s)\mathrm{d}s\right|$$
$$= \left|\int D_n^s[u_k](x)\phi_\varepsilon(s)\mathrm{d}s\right|.$$

Then by Jensen's inequality we have

$$\left\|\binom{n}{\frac{n}{2}}u_k - \sum_{j=0}^{\frac{n}{2}-1} 2(-1)^{\frac{n}{2}-1-j}\binom{n}{j}\phi_{a_n^j\varepsilon} * u_k\right\|_{L^2} \leqq \frac{C}{\varepsilon^N}\int_{|s|<\varepsilon}\int |D_n^s[u_k](x)|^2\mathrm{d}x\mathrm{d}s$$
$$= \frac{C}{\varepsilon^N}\int_0^\varepsilon t^{N-1}G(t)\mathrm{d}t,$$

where

$$G(t) = \int_{\omega\in S^{N-1}}\int |D_n^{t\omega}[u_k](x)|^2\mathrm{d}x\mathrm{d}\omega.$$

Notice that $G(2t) \leqq 2^{2n}G(t)$. By applying Lemma 4 with $g(t) = t^{-2n}G(t)$, and $h(t) = \gamma_n^{\delta_k}(t)$, and $d = N + 2n$, we get

$$\int_0^\varepsilon t^{N-1}G(t)\mathrm{d}t \leqq \frac{C\varepsilon^{N+2n}}{\int_{|s|<\varepsilon}|s|^{2n}\gamma_n^{\delta_k}(|s|)\mathrm{d}s}\left(\int_{|s|<\varepsilon}\int \gamma_n^{\delta_k}(|s|)|D_n^s[u_k](x)|^2\mathrm{d}x\mathrm{d}s\right).$$

Now we see that, by (29) and the fact

$$\lim_{k\to\infty}\int_{|s|<\varepsilon}|s|^{2n}\gamma_n^{\delta_k}(|s|)\mathrm{d}s = \int |s|^{2n}\gamma_n(|s|)\mathrm{d}s,$$

we conclude that (30) is true.

Similar to Step 2 of Theorem 5, we can show that the sequence

$$\left\{\sum_{j=0}^{\frac{n}{2}-1} 2(-1)^{\frac{n}{2}-1-j}\binom{n}{j}\phi_{a_n^j\varepsilon} * u_k\right\}_k$$

is uniformly bounded and equicontinuous thus relatively compact in L^2. And (30) implies that $\{u_k\}_k$ is also relatively compact in L^2.

Finally, suppose that $u_k \to u$ in L^2, we will show that $u \in H_0^n(\Omega)$. Suppose that

$$\int_{\Omega\cup\Omega_{\delta_k}}\int_{B_{\delta_k}(\mathbf{0})}\gamma_n^{\delta_k}(|s|)|D_n^s[u_k](x)|^2\mathrm{d}s\mathrm{d}x \leqq C_0^2.$$

Then consider the mollification $\phi_\varepsilon * u_k$, we also have

$$\iint \gamma_n^{\delta_k}(|s|)|D_n^s[\phi_\varepsilon * u_k](x)|^2 \mathrm{d}s\mathrm{d}x \leqq C_0^2,$$

by Jensen's inequality. Observe that u_k vanishes outside Ω, so for each fixed ε, $\phi_\varepsilon * u_k \to \phi_\varepsilon * u$ in $C_c^\infty(\Omega \cup \Omega_\varepsilon)$. So

$$\begin{aligned}
&\sum_{|\alpha|=n} M(n,\alpha) \int_{\Omega\cup\Omega_\varepsilon} (D^\alpha(\phi_\varepsilon * u)(x))^2 \mathrm{d}x \\
&\quad = \lim_{k\to\infty} \iint \gamma_n^{\delta_k}(|s|)|D_n^s[\phi_\varepsilon * u](x)|^2 \mathrm{d}s\mathrm{d}x \\
&\quad = \lim_{k\to\infty} \iint \gamma_n^{\delta_k}(|s|)|D_n^s[\phi_\varepsilon * u_k](x)|^2 \mathrm{d}s\mathrm{d}x \leqq C_0^2.
\end{aligned}$$

where the first and second equalities are obtained by applying Lemmas 8 and 7 respectively. Now let $\varepsilon \to 0$, we have

$$\sum_{|\alpha|=n} M(n,\alpha) \int_\Omega (D^\alpha u(x))^2 \mathrm{d}x \leqq C_0^2,$$

which implies that

$$\int_\Omega (D^\alpha u(x))^2 \mathrm{d}x \quad \text{is bounded} \quad \forall\, \alpha = (\alpha_1, \dots, \alpha_N) \quad \text{with } |\alpha| = n.$$

Then by Gagliardo–Nirenberg interpolation inequalities, we conclude that all lower order derivatives are also bounded. Observe that the above estimates also work for any $\tilde{\Omega}$ that contains Ω (by viewing u_k and u as functions defined on $\tilde{\Omega}$ but vanish outside Ω, that is, $u\,|_{\Omega^c} = 0$). Therefore $u \in H_0^n(\Omega)$.

Finally, by applying the above results, we have a sharper version of the 1st nonlocal Poincaré inequality.

Theorem 8. *There exists δ_0 and $C(\delta_0)$ such that for all $\delta \in (0, \delta_0]$,*

$$\|u\|_{L^2} \leqq C(\delta_0)|u|_{\mathcal{S}^{n,\gamma_n^\delta}}, \quad \forall\, v \in \mathcal{S}_\Omega^{n,\gamma_n^\delta}$$

Proof. Let

$$\frac{1}{A} = \inf\left\{\sum_{|\alpha|=n} M(n,\alpha)\int_\Omega (D^\alpha u(x))^2 \mathrm{d}x : u \in H_0^n(\Omega), \|u\|_{L^2} = 1\right\}.$$

By standard local Poincaré inequalities, $0 < A < \infty$. We claim that for given ε, there exists some $\delta_0(\varepsilon)$ such that for all $\delta < \delta_0$ the lemma holds with $C(\delta_0) = A+\varepsilon$.

We prove it by contradiction. Suppose there exists a $C > A$, such that for all n, there exist $\delta_k \to 0$ and u_k with the property that

$$\|u_k\|_L^2 = 1 \text{ and } \int_{\Omega\cup\Omega_{\delta_k}} \int_{B_{\delta_k}(\mathbf{0})} \gamma_n^{\delta_k}(|s|)|D_n^s[u_k](x)|^2 \mathrm{d}s\mathrm{d}x \leqq \frac{1}{C},$$

then by lemma 8, u_k is relatively compact in L^2. Moreover, any limit point $u \in H_0^n(\Omega)$ and satisfies

$$\sum_{|\alpha|=n} M(n,\alpha) \int_\Omega (D^\alpha u(x))^2 \mathrm{d}x \leqq \frac{1}{C}.$$

This contradicts the assumption that A is the best Poincaré constant.

6. Application

We consider an example to illustrate the application of our analytic framework. In [24,25], nonlocal peridynamic models for beams and plates were developed which in the local limit recover the classical Euler-Bernoulli beam and Kirchhoff-Love plate. The well-posedness for the linear peridynamic beams and plates bending elasticity models, along with rigorous connections to their local limits, can be established using the theoretical results established above.

6.1. Peridynamic beams and plates

Consider $u : \mathbb{R}^N \to \mathbb{R}$ to be the vertical displacement of a beam or plate, with $N = 1$ and $N = 2$ respectively. The total nonlocal bending energy proposed in [24,25] is defined by

$$W_\delta(u) = \frac{1}{2} \int_{\Omega \cup \Omega_\delta} \int_{B_\delta(\mathbf{0})} \omega_\delta(|\xi|) \left(\frac{u(x+\xi) - 2u(x) + u(x-\xi)}{|\xi|} \right)^2 \mathrm{d}\xi \mathrm{d}x, \tag{31}$$

where for consistency with discussions in the earlier sections, the notation ω_δ is used instead of the original notation appeared in [24,25].

6.2. The variational problems

Notice that the nonlocal bending energy (31) is exactly one half of the functional (1) with $n = 2$ and $\gamma_2(|\xi|)$ being replaced by $\frac{\omega_\delta(|\xi|)}{|\xi|^2}$. The existence of the minimizer of the energy (31) can be seen through the following formulation of the variational problem.

We associate it with a bilinear form $b_\delta(\cdot,\cdot) : \mathcal{S}_\Omega^{2,\gamma^\delta} \times \mathcal{S}_\Omega^{2,\gamma^\delta} \to \mathbb{R}$ through

$$b_\delta(u,v) = ((u,v))_{\mathcal{S}^{2,\gamma^\delta}},$$

for any given $\delta > 0$, where $\gamma^\delta(|\xi|) = \omega_\delta(|\xi|)/|\xi|^2$ is supported on $B_\delta(\mathbf{0})$. We then consider the variational problem defined by: given $f \in L^2(\Omega)$,

$$\text{find } \ u_\delta \in \mathcal{S}_\Omega^{2,\gamma^\delta} \quad \text{such that } \ b_\delta(u_\delta, v) = (f,v)_{L^2} \quad \forall\, v \in \mathcal{S}_\Omega^{2,\gamma^\delta}. \tag{32}$$

As before, $b_\delta(\cdot,\cdot)$ induces a natural linear operator $\mathcal{L}^\delta : \mathcal{S}_\Omega^{2,\gamma^\delta} \to (\mathcal{S}_\Omega^{2,\gamma^\delta})^*$ via

$$\langle \mathcal{L}^\delta u, v\rangle = b_\delta(u,v) \quad \text{for } \ u, v \in \mathcal{S}_\Omega^{2,\gamma^\delta},$$

which, based on Section 2.3, corresponds to Equation (11) with $n = 2$. Therefore,

$$\mathcal{L}^\delta u(x) = (\text{P.V.}) \int_{B_\delta(\mathbf{0})} \omega_\delta(|\xi|)/|\xi|^2 D_4^\xi[u](x)\mathrm{d}\xi \quad \text{for } \ x \in \Omega, \tag{33}$$

with

$$D_4^\xi[u](x) = u(x+2\xi) - 4u(x+\xi) + 6v(x) - 4u(x-\xi) + u(x+2\xi).$$

6.3. The limiting behavior for vanishing nonlocality

To study the limit as $\delta \to 0$, we first, similar to the practice in [24,25], assume that

$$m = \int_{B_\delta(0)} \omega_\delta(|\xi|)|\xi|^2.$$

Now suppose that $u \in H^n$, by applying Lemma 8 to the functional (31), namely, with $n = 2$ and $N = 1, 2$, we can get the limit energy functional. Notice that for $N = 1$, we have only one case $\alpha = \beta = \theta = 2$. For $N = 2$, the values of α, β and θ we can take are listed as follows:

- $\alpha = (2,0)$, $\beta = \theta = (2,0)$,
- $\alpha = (0,2)$, $\beta = \theta = (0,2)$,
- $\alpha = (1,1)$, $\beta = \theta = (1,1)$,
- $\alpha = (1,1)$, $\beta = (2,0)$, $\theta = (0,2)$,
- $\alpha = (1,1)$, $\beta = (0,2)$, $\theta = (2,0)$.

Hence, after collecting like terms, the limit local energy functional of (31) is of the form

$$W_0(u) = \frac{am}{2}\int_\Omega (u''(x))^2 \mathrm{d}x$$

for $N = 1$, or, we have for $N = 2$,

$$W_0(u) = \frac{3am}{16}\int_\Omega (\Delta u(x))^2 \mathrm{d}x.$$

Now define the associated bilinear form $b_0(\cdot,\cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$ as

$$b_0(u,v) = \begin{cases} m\int_\Omega u''(x)v''(x)\mathrm{d}x, & \text{for } \ N = 1, \\ \frac{3m}{8}\int_\Omega (\Delta u(x)\Delta v(x))\,\mathrm{d}x, & \text{for } \ N = 2. \end{cases}$$

Then variational problem is defined by: given $f \in L^2(\Omega)$,

$$\text{find } u_0 \in H_0^2(\Omega) \quad \text{such that } \ b_0(u_0, v) = (f, v)_{L^2} \quad \forall\, v \in H_0^2(\Omega). \tag{34}$$

Similarly, we can associate a linear operator $\mathcal{L}^0 : H_0^n(\Omega) \to (H_0^n(\Omega))^*$ through

$$\langle \mathcal{L}^0 u, v\rangle = b_0(u, v) \quad \text{for } u, v \in H_0^2(\Omega).$$

Now from integration by part, we know that for $u \in C^\infty$, $\mathcal{L}^0 u$ can be written as

$$\mathcal{L}^0 u(x) = \begin{cases} mu^{(4)}(x) & \text{for } N = 1; \\ \frac{3m}{8}\Delta^2 u(x) & \text{for } N = 2. \end{cases}$$

Lemma 9. *For all $u \in C_c^\infty(\Omega)$, and $x \in \Omega$, we have*

$$\mathcal{L}^\delta u(x) \to \mathcal{L}^0 u(x) \quad \textit{as } \delta \to 0.$$

Proof. Through Taylor expansion, it is straightforward to check the convergence. Here we only prove for $N = 2$. In this case,

$$\begin{aligned}
&\int_{B_\delta(\mathbf{0})} \omega_\delta(|\xi|)/|\xi|^2 D_4^\xi[u](x)\mathrm{d}\xi \\
&= \sum_{j=0}^{4}(-1)^j \binom{4}{j} (a_4^j)^4 \sum_{|\alpha|=4} \frac{D^\alpha u(x)}{\alpha!} \\
&\quad \times \left(\int_{B_\delta(\mathbf{0})} \frac{\omega(|\xi|)}{|\xi|^2}\xi^\alpha \mathrm{d}\xi + O\left(\int_{B_\delta(\mathbf{0})} \omega(|\xi|)|\xi|^3 \mathrm{d}\xi\right)\right) \\
&= 24 \sum_{|\alpha|=4} \frac{D^\alpha u(x)}{\alpha!} \left(\int_{B_\delta(\mathbf{0})} \frac{\omega(|\xi|)}{|\xi|^2}\xi^\alpha \mathrm{d}\xi \right.\\
&\quad \left. + O\left(\int_{B_\delta(\mathbf{0})} \omega(|\xi|)|\xi|^3 \mathrm{d}\xi\right)\right).
\end{aligned}$$

Notice that for $|\alpha| = 4$, the integrals of $\xi^\alpha \omega(|\xi|)/|\xi|^2$ over $B_\delta(\mathbf{0})$ are not zero only for $\alpha = (4, 0), (0, 4), (2, 2)$. Then

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{B_\delta(\mathbf{0})} \omega(|\xi|)/|\xi|^2 D_4^\xi[u](x)\mathrm{d}\xi \\
&= \int_0^\delta \omega(r) r^3 dr \left(\frac{3\pi}{4}\frac{\partial^4 u}{\partial x_1^4} + \frac{3\pi}{4}\frac{\partial^4 u}{\partial x_2^4} + \frac{6\pi}{4}\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}\right) \\
&= \frac{3}{8}\left(\int_{B_\delta(0)} \omega_\delta(|\xi|)|\xi|^2 \mathrm{d}\xi\right)\Delta^2 u(x) = \frac{3m}{8}\Delta^2 u(x).
\end{aligned}$$

From the proof the above result, we see that the convergence is not only pointwise, but also uniform in Ω. So it is easy to see also that $\mathcal{L}^\delta u \to \mathcal{L}^0 u$ in $L^2(\Omega)$.

6.4. Nonlocal variational problems and local limit

Theorem 9. *The variational problem* (32) *is well posed with a unique solution $u_\delta \in \mathcal{S}_\Omega^{2,\gamma^\delta}$ with a uniformly bounded norm $\|u_\delta|_{\mathcal{S}_\Omega^{2,\gamma^\delta}}$, independent of $\delta > 0$. Moreover,*

$$\|u_\delta - u_0\|_{L^2} \to 0 \quad as \ \ \delta \to 0,$$

where u_0 is the solution of the local variational problems(34).

Proof. First, for each given $\delta > 0$, by Lax–Milgram via the 1st nonlocal Poincaré inequality in Theorem 2, we have the existence of a unique solution $u_\delta \in \mathcal{S}_\Omega^{2,\gamma^\delta}$ to (32) and with a uniformly bounded norm $\|u_\delta|_{\mathcal{S}_\Omega^{2,\gamma^\delta}}$, independent of $\delta > 0$.

As for the local limit as $\delta \to 0$, we first have estimates for $|u_\delta|_{\mathcal{S}^{2,\gamma^\delta}}$ as follows.

$$|u_\delta|^2_{\mathcal{S}^{2,\gamma^\delta}} = b_\delta(u_\delta, u_\delta) = (f, u_\delta)_{L^2} \leqq \|f\|_{L^2}\|u\|_{L^2} \leqq C\|f\|_{L^2}|u_\delta|_{\mathcal{S}^{2,\gamma^\delta}},$$

by the sharp Poincaré inequality, which implies that $|u_\delta|_{\mathcal{S}^{2,\gamma^\delta}}$ is uniformly bounded. Then $\{u_\delta\}$ is relatively compact in L^2 by Theorem 7, and each limit point $u \in H_0^n(\Omega)$. Now we only need to show that $u = u_0$. This is true because for any $v \in C_c^\infty(\Omega)$,

$$\begin{aligned}(f - \mathcal{L}^0 u, v)_{L^2} &= (\mathcal{L}^\delta u_\delta - \mathcal{L}^0 u, v)_{L^2} = (u_\delta, \mathcal{L}^\delta v)_{L^2} - (u_0, \mathcal{L}^0 v)_{L^2} \\ &= (u_\delta, \mathcal{L}^\delta v - \mathcal{L}^0 v)_{L^2} + (u_\delta - u, \mathcal{L}^0 v)_{L^2} \\ &\leqq \|u_\delta\|_{L^2}\|\mathcal{L}^\delta v - \mathcal{L}^0 v\|_{L^2} + \|u_\delta - u\|_{L^2}\|\mathcal{L}^0 v\|_{L^2} \to 0\end{aligned}$$

as $\delta \to 0$.

7. Conclusion

Our study has focused on generalizing analytical properties associated with the nonlocal diffusion operator to higher order nonlocal operators corresponding to, in the local limit, high order elliptic differential operators. Naturally, nonlocal extensions of local differential operators can be defined in various fashions that are different from the way given in this work. For example, one may take compositions of low order nonlocal operators directly to get high order ones, though such a formulation involves integrations in higher and higher dimensional spaces. Indeed, we note with much interest a recent work [27] which has developed a nonlocal biharmonic operator as a square of the nonlocal diffusion (Laplacian) operator. In this wrok, the well-posedness and the local limit to the conventional biharmonic operators are subjected to various types of boundary conditions. Furthermore, one may also consider combinations of nonlocal operators in these different forms to model various physical systems; a practice that was used in [11,20,30]. Moreover, we have limited the study to the case of scalar fields. In the future, it is natural to further study the extension of the nonlocal calculus of variations to high order operators and functionals defined for more general vector fields. Time-dependent and nonlinear problems can also be studied. Such studies may find more applications in the analysis of nonlocal, nonlinear systems, and serve as the rigorous foundation to algorithm development [31] and computational simulations of physical problems based on nonlocal mathematical models.

Appendix 1: Proof of Lemma 1

For the first result, we prove the case with n being even. Similar proof works for n odd. For any $u : \mathbb{R}^N \to \mathbb{R}$, by definition,

$$D_n^s[u](x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} u\left(x + \left(\frac{n}{2} - j\right) s\right) \quad \text{and}$$

$$D_{n-1}^s[u](x) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} u\left(x + \left(\frac{n}{2} - j\right) s\right),$$

$$-D_1^{-s}[u](x) = u(x) - u(x-s).$$

Therefore,

$$\begin{aligned}
&-D_1^{-s} \circ D_{n-1}^s[u](x) \\
&= \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} u\left(x + \left(\frac{n}{2} - j\right) s\right) \\
&\quad - \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} u\left(x + \left(\frac{n}{2} - j - 1\right) s\right) \\
&= \sum_{j=1}^{n-1} (-1)^j \left(\binom{n-1}{j} + \binom{n-1}{j-1}\right) u\left(x + \left(\frac{n}{2} - j\right) s\right) \\
&\quad + u\left(x + \frac{n}{2} s\right) + u\left(x - \frac{n}{2} s\right) \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} u\left(x + \left(\frac{n}{2} - j\right) s\right) = D_n^s[u](x).
\end{aligned}$$

For the second result, we prove it by induction. For $n = 1$, the equality is true because $|e^{i\xi\cdot s} - 1|^2 = 2 - 2\cos(\xi \cdot s)$. Note also that $|e^{i\xi\cdot s} - 1|^2 = |e^{-i\xi\cdot s} - 1|^2$.

Assume the equality is true for $n - 1$, that is,

$$\left| \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} e^{i\xi\cdot a_{n-1}^j s} \right|^2 = (2 - 2\cos(\xi \cdot s))^{n-1}.$$

We consider two cases, where n is even or odd. For n even, we have $a_{n-1}^j = n/2 - j = a_n^j$ for $j = 0, 1, \ldots n$. Then

$$\begin{aligned}
&\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} e^{i\xi\cdot a_{n-1}^j s})(e^{-i\xi\cdot s} - 1) \\
&= \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} e^{i\xi\cdot(a_{n-1}^j - 1)s} + \sum_{j=0}^{n-1} (-1)^{j+1} \binom{n-1}{j} e^{i\xi\cdot a_{n-1}^j s}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1}(-1)^{j+2}\binom{n-1}{j}e^{i\xi\cdot a_{n-1}^{j+1}s} + \sum_{j=0}^{n-1}(-1)^{j+1}\binom{n-1}{j}e^{i\xi\cdot a_{n-1}^{j}s}\\
&= \sum_{j=1}^{n}(-1)^{j+1}\binom{n-1}{j-1}e^{i\xi\cdot a_{n-1}^{j}s} + \sum_{j=0}^{n-1}(-1)^{j+1}\binom{n-1}{j}e^{i\xi\cdot a_{n-1}^{j}s}\\
&= -e^{i\xi\cdot a_{n-1}^{0}s} + \sum_{j=1}^{n-1}(-1)^{j+1}\left(\binom{n-1}{j-1}+\binom{n-1}{j}\right)e^{i\xi\cdot a_{n-1}^{j}s}\\
&\quad +(-1)^{n+1}e^{i\xi\cdot a_{n-1}^{n}s}\\
&= -\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}e^{i\xi\cdot a_{n-1}^{j}s} = -1\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}e^{i\xi\cdot a_{n}^{j}s}.
\end{aligned}
$$

Thus we see that

$$
\begin{aligned}
&\left|\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}e^{i\xi\cdot a_{n}^{j}s}\right|^2 = \left|\sum_{j=0}^{n-1}(-1)^{j}\binom{n-1}{j}e^{i\xi\cdot a_{n-1}^{j}s}\right|^2 |e^{-i\xi\cdot s}-1|^2\\
&= (2-2\cos(\xi\cdot s))^{n-1}\cdot(2-2\cos(\xi\cdot s)) = (2-2\cos(\xi\cdot s))^{n}.
\end{aligned}
$$

For n odd, we can similarly prove the equality by changing the multiplication factor used above with $e^{i\xi\cdot s}-1$. Thus we complete the induction.

Now to show the last equality in the lemma, we notice that

$$
\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}(a_n^j)^n = D_n^1[x^n](0).
$$

Since the nth order difference approximation is exact for a nth degree polynomial, we know that $D_n^1[x^n](0) = n!$. □

Appendix 2: Proof of Lemma 6

First, it is not hard to see that the inequality in assumption (13) is true for $|\xi| \leqq 1$. Indeed, since $a^2/4 \leqq 1-\cos(a) \leqq a^2/2$ for any real number $|a| < 1$, we have

$$
\begin{aligned}
\mathcal{I}_n(\xi)\,\mathcal{I}_1(\xi) &\leqq \frac{|\xi|^{2n+2}}{2^{n+1}}\int_{|s|<1}\gamma_n(|s|)|s|^{2n}\mathrm{d}s\int_{|s|<1}\gamma_1(|s|)|s|^2\mathrm{d}s\\
&\leqq C\frac{|\xi|^{2n+2}}{2^{n+1}}\int_{|s|<1}\gamma_{n+1}(|s|)|s|^{2n+2}\mathrm{d}s,
\end{aligned}
$$

where the last inequality is implied by assumption of the Lemma. Also since we have

$$
\frac{1}{4^{n+1}}\int_{|s|<1}\gamma_{n+1}(|s|)|\xi\cdot s|^{2n+2}\mathrm{d}s \leqq \int_{|s|<1}\gamma_{n+1}(|s|)(1-\cos(\xi\cdot s))^{n+1}\mathrm{d}s,
$$

then the case $|\xi| \leqq 1$ is true only if

$$\tilde{C}|\xi|^{2n+2} \int_{|s|<1} \gamma_{n+1}(|s|)|s|^{2n+2}\mathrm{d}s \leqq \int_{|s|<1} \gamma_{n+1}(|s|)|\xi \cdot s|^{2n+2}\mathrm{d}s$$

or equivalently if for any $\tilde{\xi} \in S^{N-1}$, the unit sphere in $\mathbb{R}^N$,

$$\tilde{C} \int_{|s|<1} \gamma_{n+1}(|s|)|s|^{2n+2}\mathrm{d}s \leqq \int_{|s|<1} \gamma_{n+1}(|s|)|\tilde{\xi} \cdot s|^{2n+2}\mathrm{d}s$$

for some constant $\tilde{C}$. The last can be established by taking note of the rotational symmetry of the integral on the right hand side with respect to $\tilde{\xi}$ so that, with $\{e_k\}$ being the canonical orthonormal basis in $\mathbb{R}^N$, we have

$$\begin{aligned}\int_{|s|<1} \gamma_{n+1}(|s|)|\tilde{\xi} \cdot s|^{2n+2}\mathrm{d}s &= \frac{1}{N}\sum_{k=1}^{N} \int_{|s|<1} \gamma_{n+1}(|s|)|e_k \cdot s|^{2n+2}\mathrm{d}s \\ &\geqq \tilde{C}|\xi|^{2n+2} \int_{|s|<1} \gamma_{n+1}(|s|)|s|^{2n+2}\mathrm{d}s,\end{aligned}$$

where $\tilde{C}$ depends only on N and n.

Now, we consider the case where $|\xi| > 1$. We split the integrals $\mathcal{I}_n(\xi)$ and $\mathcal{I}_1(\xi)$ over the domain $\{|s| < 1\}$ into two parts $\{|s| < t\}$ and $\{t < |s| < 1\}$, where the positive parameter t is sufficiently small such that $0 < t \leqq 1/|\xi|$. Then using the estimates $1-\cos(a) \leqq a^2/2$ for any $|a| < 1$, and $1-\cos(a) \leqq 2$ for any $a \in \mathbb{R}$ respectively, we have

$$\begin{aligned}&\int_{|s|<1} \gamma_k(|s|)(1-\cos(\xi \cdot s))^k\mathrm{d}s \\ &\leqq \frac{|\xi|^{2k}}{2^k} \int_{|s|<t} \gamma_k(|s|)|s|^{2k}\mathrm{d}s + 2^k \int_{t<|s|<1} \gamma_k(|s|)\mathrm{d}s\end{aligned}$$

for $k = 1$ and $k = n$. Now since γ_1 and γ_n are non-increasing, we get that

$$\begin{aligned}&\int_{t<|s|<1} \gamma_1(|s|)\mathrm{d}s \int_{t<|s|<1} \gamma_n(|s|)\mathrm{d}s \leqq (1-t) \int_{t<|s|<1} \gamma_1(|s|)\gamma_n(|s|)\mathrm{d}s \\ &\leqq \int_{t<|s|<1} \gamma_1(|s|)\gamma_n(|s|)\mathrm{d}s = \int_{t<|s|<1} \gamma_{n+1}(|s|)\mathrm{d}s.\end{aligned}$$

Also by assumption (12), we have

$$\int_{|s|<t} \gamma_1(|s|)|s|^2\mathrm{d}s \int_{|s|<t} \gamma_n(|s|)|s|^{2n}\mathrm{d}s \leqq C \int_{|s|<t} \gamma_{n+1}(|s|)|s|^{2n+2}\mathrm{d}s.$$

Denoting, for $k = 1, n$ and $n+1$,

$$A_k^t = C_k|\xi|^{2k} \int_{|s|<t} \gamma_k(|s|)|s|^{2k}\mathrm{d}s, \quad B_k^t = \tilde{C}_k \int_{t<|s|<1} \gamma_k(|s|)\mathrm{d}s.$$

Then the above results show that $A_1^t A_n^t \leqq C A_{n+1}^t$ and $B_1^t B_n^t \leqq B_{n+1}^t$.

Meanwhile, we see from Lemma 6 that for $k = 1, n$ and

$$A_k^t/B_k^t = \frac{C_k|\xi|^{2k}\displaystyle\int_{|s|<t}\gamma_k(|s|)|s|^{2k}\mathrm{d}s}{\tilde{C}_k\displaystyle\int_{t<|s|<1}\gamma_k(|s|)\mathrm{d}s},$$

we have A_k^τ/B_k^τ is bounded for $\tau = \zeta/|\xi|$, where ζ is a small number less than η which is a parameter to be specified later. Together, we get

$$(A_1^\tau + B_1^\tau)(A_n^\tau + B_n^\tau) \leqq \tilde{C}(A_{n+1}^\tau + B_{n+1}^\tau).$$

Now to complete the proof, we only need to show that there exists a constant C_0 independent of ξ such that

$$C_0(A_{n+1}^\tau + B_{n+1}^\tau) \leqq \int_{|s|<1}\gamma_{n+1}(|s|)(1-\cos(\xi\cdot s))^{n+1}\mathrm{d}s.$$

To show the above inequality, we split the integral on the right hand side into two parts $\{|s| < \tau\}$ and $\{\tau < |s| < 1\}$. Similarly as before, we can see that the first part

$$\begin{aligned}\int_{|s|<\tau}\gamma_{n+1}(|s|)(1-\cos(\xi\cdot s))^{n+1}\mathrm{d}s &\geqq \frac{1}{4^{n+1}}\int_{|s|<\tau}\gamma_{n+1}(|s|)|\xi\cdot s|^{2n+2}\mathrm{d}s\\ &\geqq \frac{1}{4^{n+1}}|\xi|^{2n+2}\int_{|s|<\tau}\gamma_{n+1}(|s|)|s|^{2n+2}\mathrm{d}s = \frac{1}{4^{n+1}}A_{n+1}^\tau.\end{aligned}$$

For the second part, define,

$$\mathcal{I}(\xi) = \int_{\tau<|s|<1}\gamma_{n+1}(|s|)(1-\cos(\xi\cdot s))^{n+1}\mathrm{d}s$$

then $\mathcal{I}(\xi)$ is rotationally invariant with respect to ξ, that is $\mathcal{I}(\xi) = \mathcal{I}(|\xi|e)$ for any unit vector $e \in \mathbb{R}^N$. Now define the set

$$E^{|\xi|} = \bigcup_{k=-\infty}^{\infty}\left(\frac{2k\pi - \zeta/2}{|\xi|}, \frac{2k\pi + \zeta/2}{|\xi|}\right),$$

then

$$\begin{aligned}\mathcal{I}(|\xi|e) &\geqq \int_{\{s:\tau<|s|<1, e\cdot s\notin E^{|\xi|}\}}\gamma_{n+1}(|s|)(1-\cos(|\xi|e\cdot s))^{n+1}\mathrm{d}s\\ &\geqq C(\zeta)\int_{\{s:\tau<|s|<1, e\cdot s\notin E^{|\xi|}\}}\gamma_{n+1}(|s|)\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{I}(\xi) &\geqq \frac{C(\zeta)}{|S^{N-1}|}\int_{S^{N-1}}\int_{\{s:\tau<|s|<1, e\cdot s\notin E^{|\xi|}\}}\gamma_{n+1}(|s|)\mathrm{d}s\mathrm{d}e\\ &= \frac{C(\zeta)}{|S^{N-1}|}\int_{\tau<|s|<1}\gamma_{n+1}(|s|)\int_{S^{N-1}\cap\{e:e\cdot s\notin E^{|\xi|}\}}\mathrm{d}e\mathrm{d}s\\ &= C(\zeta)\int_{\tau<|s|<1}\gamma_{n+1}(|s|)\tilde{\mathcal{I}}(s, |\xi|)\mathrm{d}s,\end{aligned}$$

where

$$\tilde{\mathcal{I}}(s,|\xi|) := \frac{1}{|S^{N-1}|}\int_{S^{N-1}\cap\{e:e\cdot s\notin E^{|\xi|}\}} \mathrm{d}e.$$

Notice that $\tilde{\mathcal{I}}(s,|\xi|)$ is rotationally invariant with respect to s, namely, $\tilde{\mathcal{I}}(s,|\xi|) = \tilde{\mathcal{I}}(|s|e',|\xi|)$ for any $e' \in S^{N-1}$. We claim that $\tilde{\mathcal{I}}(s,|\xi|) \geqq C_0$, for $\tau < |s| < 1$ and C_0 independent of ξ. If this is true, then $\mathcal{I}(\xi) \geqq C(\zeta)C_0 B^{\tau}_{n+1}$, which makes the proof complete.

To establish the claim on $\tilde{\mathcal{I}}(s,|\xi|) = \tilde{\mathcal{I}}(|s|e',|\xi|)$, we take e' to be the first axis direction vector in $\mathbb{R}^N$ and define $m = m(r)$ to be the measure of $\{(e^1, e^2, \ldots, e^N) \in S^{N-1} : e^1 \notin E^r\}$ for $r = |\xi||s|$ and $\tau < |s| < 1$. The claim then follows if we can show that $m(r)$ is bounded below for $\zeta \leqq r \leqq |\xi|$ by some constant $C_0 > 0$ independent of $|\xi|$. It is easy to see that $m = m(r)$ is positive and continuous for any finite r, then we only need to show that $\liminf_{r\to\infty} m(r) \neq 0$. Indeed, for $(e^1, e^2, \ldots, e^N) \in S^{N-1}$ with $e^1 \in E^r \cap (-\sqrt{2}/2, \sqrt{2}/2)$, the interval $((4k\pi - \zeta)/(2r), (4k\pi + \zeta)/(2r))$ may be a portion of $(-\sqrt{2}/2, \sqrt{2}/2)$ only if

$$-\frac{\sqrt{2}r}{4\pi} - \frac{\zeta}{4\pi} < k < \frac{\sqrt{2}r}{4\pi} + \frac{\zeta}{4\pi}$$

and the number of such k's is no greater than $\sqrt{2}r/(2\pi) + \zeta/(2\pi) + 1$. In addition,

$$\mathrm{meas}(\{(e^1, \ldots, e^N) \in S^{N-1} : e^1 \in \left(\frac{4k\pi - \zeta}{2r}, \frac{4k\pi + \zeta}{2r}\right) \cap \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}\right)$$

is bounded above by $|S^{N-2}|\sqrt{2}\zeta/r$. Taking $\zeta \leqq \frac{1}{2}\pi|S^{N-1}|/|S^{N-2}|$, we get

$$\begin{aligned}\liminf_{r\to\infty} m(r) &\geqq \liminf_{r\to\infty} \frac{1}{2} - \frac{|S^{N-2}|}{|S^{N-1}|}\sqrt{2}\frac{\zeta}{r}\left(\frac{\sqrt{2}r}{2\pi} + \frac{\zeta}{2\pi} + 1\right) \\ &= \frac{1}{2} - \frac{|S^{N-2}|}{|S^{N-1}|}\frac{\zeta}{\pi} > 0,\end{aligned}$$

which is always positive. This verifies the claim on $\tilde{\mathcal{I}}(s,|\xi|)$. Therefore the proof is completed.

References

1. Andreu, F., Mazón, J.M., Rossi, J.D., Toledo, J.: Nonlocal Diffusion Problems, vol. 165 of Mathematical Surveys and Monographs. *Am. Math. Soc.* (2010)
2. Aubert, G., Kornprobst, P.: Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems?. *SIAM J. Numer. Anal.* **47**, 844–860 (2009)
3. Bass, R., Kassmann, M., Kumagai, T.: Symmetric jump processes: localization, heat kernels and convergence. *Ann. de l'Inst. Henri Poincaré Probab. et Stat.* **46**, 59–71, (2010)
4. Borghol, R.: Some properties of sobolev spaces. *Asymptot. Anal.* **51**, 303–318 (2007)

5. Bourgain, J., Brezis, H., Mironescu, P.: *Another Look at Sobolev Spaces*, pp. 439–455. IOS Press, Amsterdam 2001
6. Buades, A., Coll, B., Morel, J.M.: Image denoising methods. A new nonlocal principle. *SIAM Rev.* **52**, 113–147 (2010)
7. Burch, N., Lehoucq, R.: Classical, nonlocal, and fractional diffusion equations on bounded domains. *Int. J. Multiscale Comput. Eng.* **9**, 661 (2011)
8. Caffarelli, L., Silvestre, L.: An extension problem related to the fractional laplacian. *Commun. Partial Differ. Equ.* **32**, 1245–1260 (2007)
9. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. des Sci. Math.* **136**, 521–573 (2012)
10. Du, Q., Gunzburger, M., Lehoucq, R., Zhou, K.: Analysis and approximation of nonlocal diffusion problems with volume constraints. *SIAM Rev.* **56**, 676–696 (2012)
11. Du, Q., Gunzburger, M., Lehoucq, R., Zhou, K.: A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. *Math. Model. Methods Appl. Sci.* **23**, 493–540 (2013)
12. Felsinger, M., Kassmann, M., Voigt, P.: The Dirichlet problem for nonlocal operators. *Math. Z.* **279**, 1–31 (2013)
13. Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. *Multiscale Model. Simul.* **7**, 1005–1028 (2008)
14. Kao, C., Lou, Y., Shen, W.: Random dispersal vs. nonlocal dispersal. *Discret. Contin. Dyn. Syst. A* **26**, 551–596 (2010)
15. Klafler, J., Sokolov, I.: Anomalous diffusion spreads its wings. *Phys. World*, **18**:29 (2005)
16. Leoni, G., Spector, D.: Characterization of Sobolev and BV spaces. *J. Funct. Anal.* **261**, 2926–2958 (2011)
17. Lou, Y., Zhang, X., Osher, S., Bertozzi, A.: Image recovery via nonlocal operators. *J. Sci. Comput.* **42**, 185–197 (2010)
18. Mengesha, T., Du, Q.: Analysis of a scalar nonlocal peridynamic model with a sign changing kernel. *Discret. Contin. Dyn. Syst, B*, **18**:1415–1437, (2013)
19. Mengesha, T., Du, Q.: The bond-based peridynamic system with Dirichlet type volume constraint. *Proc. R. Soc. Edinb. A* **144**:161–186 (2014)
20. Mengesha, T., Du, Q.: Nonlocal constrained value problems for a linear peridynamic Navier equation. *J. Elast.* **116**:27–51 (2014)
21. Mengesha, T., Du, Q.: Characterization of function spaces of vector fields and an application in nonlinear peridynamics. *Nonlinear Analysis: Theory, Methods & Applications*, **140**, 82–111, (2016)
22. Mengesha, T., Spector, D.: Localization of nonlocal gradients in various topologies. *Calc. Var. Partial Differ. Equ.* **52**, 253–279 (2013)
23. Metzler, R., Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339**, 1–77 (2000)
24. O'Grady, J., Foster, J.: Peridynamic beams: a non-ordinary, state-based model. *Int. J. Solids Struct.* **51**, 3177–3183 (2014)
25. O'Grady, J., Foster, J.: Peridynamic plates and flat shells: A non-ordinary, state-based model. *Int. J. Solids Struct.* **51**, 4572–4579 (2014)
26. Ponce, A.C.: An estimate in the spirit of Poincaré's inequality. *J. Eur. Math. Soc* **6**, 1–15 (2004)
27. Radu, P., Toundykov, D., Trageser, J.: A nonlocal biharmonic operator and its connection with the classical analogue. (2016) (preprint)
28. Ros-Oton, X.: Nonlocal elliptic equations in bounded domains: a survey. arXiv:1504.04099 (2015) (arXiv preprint)
29. Ros-Oton, X., Serra, J.: Local integration by parts and Pohozaev identities for higher order fractional Laplacians. arXiv:1406.1107 (2014) (arXiv preprint)
30. Silling, S.: Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids* **48**, 175–209 (2000)

31. Tian, X., Du, Q.: Nonconforming discontinuous Galerkin methods for nonlocal variational problems. SIAM J. Numer. Anal. **53**, 762–781 (2015)
32. Yang, R.: On higher order extensions for the fractional Laplacian. arXiv:1302.4413 (2013) (arXiv preprint)

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