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Asymptotically compatible schemes for the approximation of fractional Laplacian and related nonlocal diffusion problems on bounded domains

Xiaochuan Tian[1] · Qiang Du[1] · Max Gunzburger[2]

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Abstract Approximations of solutions of fractional Laplacian equations on bounded domains are considered. Such equations allow global interactions between points separated by arbitrarily large distances. Two approximations are introduced. First, interactions are localized so that only points less than some specified distance, referred to as the interaction radius, are allowed to interact. The resulting truncated problem is a special case of a more general nonlocal diffusion problem. The second approximation is the spatial discretization of the related nonlocal diffusion problem. A recently developed abstract framework for asymptotically compatible schemes is applied to prove convergence results for solutions of the truncated and discretized problem to the solutions of the fractional Laplacian problems. Intermediate results also provide new convergence results for the nonlocal diffusion problem. Special

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✉ Qiang Du
qd2125@columbia.edu

Xiaochuan Tian
xt2156@columbia.edu

Max Gunzburger
gunzburg@fsu.edu

[1] Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, USA

[2] Department of Scientific Computing, Florida State University, Tallahassee, FL 32306-4120, USA

attention is paid to limiting behaviors as the interaction radius increases and the spatial grid size decreases, regardless of how these parameters may or may not be dependent. In particular, we show that conforming Galerkin finite element approximations of the nonlocal diffusion equation are always asymptotically compatible schemes for the corresponding fractional Laplacian model as the interaction radius increases and the grid size decreases. The results are developed with minimal regularity assumptions on the solution and are applicable to general domains and general geometric meshes with no restriction on the space dimension and with data that are only required to be square integrable. Furthermore, our results also solve an open conjecture given in the literature about the convergence of numerical solutions on a fixed mesh as the interaction radius increases.

Keywords Nonlocal diffusion · Fractional Laplacian · Numerical approximation · Convergence · Asymptotically compatible schemes

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1 Introduction

The fractional Laplacian operator appears in several nonstandard models in situations for which standard models have been found to not always adequately represent the physics of the problem being modeled. Popular examples include anomalous diffusion problems and quantum theory [20]. Analogous to the way that the Laplacian operator is related to Brownian processes and Feymann path integrals, the fractional Laplacian is related to α-stable Lévy processes and Lévy path integrals. More precisely, the fractional Laplacian operator can be the generator of an α-stable Lévy process [28]. Unlike Brownian processes for which paths are continuous, α-stable Lévy processes are jump processes for which paths can include jumps of arbitrarily large length.

In this paper, we consider the fractional Laplacian equation posed on *general* bounded domains $\Omega \subset \mathbb{R}^n$. Unlike the classical Poisson problem for the Laplacian operator for which well posedness requires the specification of values of the solution along only the boundary of Ω, for the fractional Laplacian equation considered in this paper, one needs to specify values of the solution over all of the complement of Ω in $\mathbb{R}^n$. Thus, in designing a discretization algorithm, e.g., a finite element method, for the fractional Laplacian equation one must deal with the unbounded nature over which the constraint in the solution is applied. Here, we precede the spatial discretization of the equation by a truncation step such that interactions between a point in Ω with other points in $\mathbb{R}^n$ is limited to a ball of finite radius λ. This is reminiscent to the introduction of a horizon parameter in the nonlocal peridynamic theory of elasticity [30] that characterizes the finite range of nonlocal interactions. Our focus is on studying the convergence properties of approximations of the solution of the fractional Laplacian equation as the interaction radius $\lambda \to \infty$ and also as $h \to 0$, where h denotes a measure of the grid size of, e.g., a finite element discretization of the truncated problem.

1.1 The fractional Laplacian problem and its approximation through truncation

Let $\Omega \subset \mathbb{R}^n$ denote a bounded, open domain with a piecewise planar boundary. The fractional diffusion problem, as modeled using the fractional Laplacian operator, is defined as

$$\begin{cases} (-\Delta)^\alpha u = f & \text{on } \Omega \\ u = 0 & \text{on } \Omega^c = \mathbb{R}^n \backslash \Omega\,. \end{cases} \tag{1}$$

Here, the fractional Laplacian operator $(-\Delta)^\alpha$ with $0 < \alpha < 1$ is the pseudo-differential operator with symbol $|\xi|^{2\alpha}$, that is [31],

$$\mathcal{F}[(-\Delta)^\alpha u](\xi) = |\xi|^{2\alpha} \mathcal{F}[u](\xi)\,,$$

where $\mathcal{F}[\cdot]$ denotes the Fourier transform. It is well known [26] that an equivalent definition of the fractional Laplacian operator is by given by

$$(-\Delta)^\alpha u(\mathbf{x}) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2\alpha}} d\mathbf{y}\,, \tag{2}$$

where

$$C_{n,\alpha} = \pi^{-(\frac{n}{2}+2\alpha)} \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(-\alpha)}$$

with $\Gamma(\cdot)$ denoting the gamma function. The integral in Eq. 2 and similar integrals in the sequel should be understood in the sense of principal value [18, 23].

The problem (1) involves global interactions in $\mathbb{R}^n$. Computationally, it is convenient to restrict the interaction to a smaller neighborhood. Thus, based on Eq. 2, we approximate, for any $\mathbf{x} \in \Omega$, the fractional Laplacian operator as

$$(-\Delta)^\alpha u \approx \mathcal{L}_\lambda u(\mathbf{x}) := C_{n,\alpha} \int_{\mathbb{R}^n} \mathbb{1}_\lambda(\mathbf{x}, \mathbf{y}) \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2\alpha}} d\mathbf{y}\,, \tag{3}$$

where, for some $0 < \lambda < \infty$ and with $B_\lambda(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \,:\, |\mathbf{y} - \mathbf{x}| < \lambda\}$, $\mathbb{1}_\lambda(\mathbf{x}, \mathbf{y})$ denotes the indicator function

$$\mathbb{1}_\lambda(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \forall\, \mathbf{y} \in B_\lambda(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

Formally speaking, the fractional Laplacian operator $(-\Delta)^\alpha$ is the limit, as $\lambda \to \infty$, of the nonlocal integral operator $\mathcal{L}_\lambda$ defined by the right-hand side of Eq. 3; this notion is made rigorous in this paper.

It is easy to see that if the fractional Laplacian in Eq. 1 is replaced by the approximation (3), we are then led to a finite domain approximate problem given by

$$\begin{cases} \mathcal{L}_\lambda u_\lambda(\mathbf{x}) = C_{n,\alpha} \displaystyle\int_{\Omega \cup \Omega_\lambda} \mathbb{1}_\lambda(\mathbf{x}, \mathbf{y}) \frac{u_\lambda(\mathbf{x}) - u_\lambda(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2\alpha}} d\mathbf{y} = f & \text{for } \mathbf{x} \in \Omega \\ \qquad\qquad\qquad\qquad\qquad u = 0 & \text{for } \mathbf{x} \in \Omega_\lambda\,, \end{cases} \tag{4}$$

where Ω_λ denotes the *shape interaction domain* defined as

$$\Omega_\lambda = \{\mathbf{y} \in \mathbb{R}^n \backslash \Omega \quad \text{such that} \quad \text{dist}(\mathbf{y}, \partial\Omega) \le \lambda\}. \tag{5}$$

Following [12], we refer to the problem (4) as a *nonlocal diffusion problem* with a *volume constraint*, the latter indicating that the constraint is applied on a set Ω_λ having positive measure in $\mathbb{R}^n$, as contrasted to the partial differential equation setting for which constraints are applied on the boundary $\partial\Omega$ of Ω. Of course, the problem (1) is also a nonlocal problem, but we will continue to refer to it as a fractional Laplacian problem.

This paper is devoted to exploring the connections between solutions of Eq. 4 and those of (1), especially as the interaction parameter $\lambda \to \infty$; in this setting, we view (4) as an approximation, effected by domain truncation, of Eq. 1. In practice, for general domains Ω, solutions of the nonlocal diffusion problem (4) cannot be obtained exactly for generic data so that we are also interested in finite-dimensional numerical approximations such as those given by finite element methods. In particular, with h denoting a measure of the grid size, we would like to also obtain a better understanding of the convergence properties of finite element approximations as $h \to 0$. Our main approach is to treat it as approximations to a family of problems parameterized by the parameter λ. While there are various mathematical frameworks on such approximations, see for example [7], we elect to adopt the notion of *asymptotically compatible schemes* and its abstract framework introduced in [35]. The abstract framework, to be briefly reviewed in Section 3, enables us to identify asymptotically compatible finite element methods for the robust discretization of Eq. 4 as well as the limiting problem (1). This also allows us to further advocate nonlocal models as bridges between local and fractional models and asymptotically compatible schemes as robust algorithms for their numerical approximations.

Our two main results are as follows. First, we establish the convergence of conforming numerical approximations of solutions of Eq. 4 to those of Eq. 1 in cases for which the mesh size h decreases and the nonlocal interaction length λ increases, regardless of the dependence or independence of these two parameters. We also solve a conjecture given in [11] on the convergence of numerical solutions on a fixed mesh with increasing nonlocal interactions. In [11], careful a priori error estimates on the solution were derived under additional regularity assumptions and the results of numerical experiments were provided that illustrated the predicted order of convergence with respect to λ or h when one of them is sufficiently small. However, it is not clear from [11] whether the correct solution to the fractional Laplacian is obtained as $\lambda \to \infty$ and $h \to 0$ simultaneously, given that not all dependences of the error terms on λ and h can be given in simple and specific manners. Here, we prove the conjecture by showing rigorously that any conforming Galerkin finite element method for Eq. 4 is asymptotic compatible, regardless of how the nonlocal interaction parameter λ and the spatial discretization parameter h change.

We note that the main results presented here are developed with minimal regularity assumptions on the solution and are applicable to general domains and general geometric meshes with no restriction on the space dimension. The forcing term is only assumed to be in in L^2.

Clearly, Eq. 4 is a special form of the problem

$$\begin{cases} -\mathcal{L}u(\mathbf{x}) = 2\displaystyle\int_{\mathbb{R}^n} \mathbb{1}_\lambda(\mathbf{x},\mathbf{y})\big(u(\mathbf{y}) - u(\mathbf{x})\big)\gamma(\mathbf{x},\mathbf{y})d\mathbf{y} = f(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega \\ \hfill u = 0 & \text{for } \mathbf{x} \in \Omega_\lambda \end{cases} \tag{6}$$

with the kernel $\gamma(\mathbf{x},\mathbf{y})$ given by $\gamma(\mathbf{x},\mathbf{y}) = C_{n,\alpha}C_\lambda 2|\mathbf{x}-\mathbf{y}|^{-n-2\alpha}$ and $C_\lambda = 1$. Related studies can be found in, e.g., [1–3, 5, 6, 8, 9, 11, 12, 15–17, 19, 21–25, 27, 29, 32, 34, 35, 37, 38, 40, 41] and the references cited therein. In these earlier works, several useful kernels, in addition to the type specified above, are considered. In these and other papers (see those cited in [33]), mathematical analyses of Eq. 6 are provided as are numerical methods, including finite difference, finite element, quadrature, and particle-based methods. In particular, with appropriate constraints applied on the volume Ω_λ [12, 23], these problems are known to be well posed. We refer to [11] for numerical examples that demonstrate the competitiveness of the approximations.

Note that in several previous works cited above, especially those related to studies of nonlocal peridynamic models, the symbol δ is used instead of λ; there, δ is often referred to as the horizon parameter; regardless of the notation used, λ (or δ) is a measure of the range of nonlocal interactions. In such works, the attention is focused on the case of a small λ and the local limit $\lambda \to 0$; in such cases, $C_\lambda \neq 1$ is chosen so as to get consistent parameters with the partial differential equations obtained in the local limit. For large λ and for limiting behavior as $\lambda \to \infty$ of interest here, we may simply set $C_\lambda = 1$. We further note that, for convenience, a different notation σ is used in the abstract framework on parametrized problems given in Section 3. While σ could represent different quantities in more general setting, it plays the same role as λ in this work.

The rest of the paper is organized as follows. In Section 2, we formulate a variational formulation of the nonlocal diffusion problem (4) and of its finite-dimensional discretization. We then review, in Section 3, the basic theory of asymptotically compatible schemes given in [35]. That scheme is then applied, in Section 4, to develop a convergence analysis of nonlocal diffusion and fractional Laplacian equations and their conforming Galerkin finite element approximations. Further discussions relating local, nonlocal and fractional models and their numerical approximations are given in Section 5. Some conclusion remarks are given at the end.

2 Variational formulations and Galerkin approximations

For the bilinear form

$$(u,v)_{\mathcal{T}_\lambda} = \int_{\Omega\cup\Omega_\lambda}\int_{\Omega\cup\Omega_\lambda} \frac{C_{n,\alpha}}{2|\mathbf{x}-\mathbf{y}|^{n+2\alpha}}\big(u(\mathbf{x})-u(\mathbf{y})\big)\big(v(\mathbf{x})-v(\mathbf{y})\big)\,d\mathbf{x}d\mathbf{y}\,,$$

we define [12, 23] the natural “energy” space

$$\tilde{\mathcal{T}}_\lambda = \left\{u \in L^2(\Omega\cup\Omega_\lambda) \;:\; \|u\|^2_{L^2(\Omega\cup\Omega_\lambda)} + (u,u)_{\mathcal{T}_\lambda} < \infty, \right\} \subset L^2(\Omega\cup\Omega_\lambda)\,.$$

We let the solution space $\mathcal{T}_\lambda$ be the completion of $C_c^\infty(\Omega)$ in $\tilde{\mathcal{T}}_\lambda$ which contains functions $u \in \tilde{\mathcal{T}}_\lambda$ satisfying $u\mid_{\Omega_\lambda}= 0$. As discussed later, the Poincaré type inequality in Section 4 implies that $\mathcal{T}_\lambda$ is a Hilbert space with $(\cdot,\cdot)_{\mathcal{T}_\lambda}$ being an equivalent inner product and $\|u\|_{\mathcal{T}_\lambda} = (u,u)^{1/2}_{\mathcal{T}_\lambda}$ being the corresponding norm. Similarly, we may let

$$(u,v)_{\mathcal{T}_\infty} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{C_{n,\alpha}}{2|\mathbf{x}-\mathbf{y}|^{n+2\alpha}}(u(\mathbf{x})-u(\mathbf{y}))(v(\mathbf{x})-v(\mathbf{y}))\,d\mathbf{x}d\mathbf{y}\,,$$

$\|u\|_{\mathcal{T}_\infty} = (u,u)^{1/2}_{\mathcal{T}_\infty}$, and $\mathcal{T}_\infty = H^\alpha_\Omega(\mathbb{R}^n)$ be the completion of $C_c^\infty(\Omega)$ in $\tilde{\mathcal{T}}_\infty = H^\alpha(\mathbb{R}^n)$, the fractional Sobolev space of order α. For any $u \in \mathcal{T}_\infty = H^\alpha_\Omega(\mathbb{R}^n)$, we have $u \in H^\alpha(\mathbb{R}^n)$ with $u\mid_{\mathbb{R}\setminus\Omega}= 0$. As shown in Section 4, $\mathcal{T}_\lambda$ and $\mathcal{T}_\infty$ are, in fact, equivalent for any $\lambda > 0$ and their norms are uniformly equivalent for λ bounded uniformly from below by a positive constant.

A weak formulation of Eq. 4 is given as follows:

$$\begin{aligned}&\textit{given } f \in (\mathcal{T}_\infty)^* = (\mathcal{T}_\lambda)^*, \textit{find } u_\lambda \in \mathcal{T}_\lambda\\ &\textit{such that} \quad a_\lambda(u_\lambda, v) = \langle f, v\rangle\,, \quad \forall\, v \in \mathcal{T}_\lambda,\end{aligned} \tag{7}$$

where $a_\lambda(\cdot,\cdot) := (\cdot,\cdot)_{\mathcal{T}_\lambda}$. $\langle\cdot,\cdot\rangle$ denotes the duality pairing for $\mathcal{T}_\infty$ and its dual space $\mathcal{T}_\infty^*$. It also represents the standard inner product in $\mathcal{T}_0 = L^2(\Omega)$, or equivalently in the space of functions in $L^2(\Omega\cup\Omega_\lambda)$ that vanish outside Ω (for any $\lambda \geq 0$). Indeed, we let $\mathcal{T}_0$ serve as the pivot space between $\mathcal{T}_\lambda^*$ and $\mathcal{T}_\lambda$ so that a realization of the duality pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{T}_\lambda^*$ and $\mathcal{T}_\lambda$ is given as the extension of the inner product on $\mathcal{T}_0$, i.e., for any $\lambda \in [0,\infty]$ and for $w \in \mathcal{T}_0 \subseteq \mathcal{T}_\lambda^*$,

$$\langle w, v\rangle = (w,v)_{\mathcal{T}_0}\,, \quad \forall\, v \in \mathcal{T}_\lambda\,. \tag{8}$$

Thus, we do not specify any subscript related to λ to distinguish the duality pairing.

Now, for any $\lambda \geq 0$, we introduce a finite-dimensional approximation space $\{\mathcal{T}_{\lambda,h}\} \subset \mathcal{T}_\lambda$ parameterized by $h \to 0$, usually taken as a measure of a grid size in the case of finite element methods or a quantity that is inversely proportional to the dimension of the discrete space.

For any λ and h, the Galerkin approximation $u_{\lambda,h} \in \mathcal{T}_{\lambda,h}$ of the solution $u \in \mathcal{T}_\lambda$ of Eq. 7 is defined by replacing $\mathcal{T}_\infty$ by $\mathcal{T}_{\lambda,h}$ in Eq. 7:

$$\begin{aligned}&\textit{given } f \in \mathcal{T}_\infty^* = \mathcal{T}_\lambda^*, \textit{find } u_{\lambda,h} \in \mathcal{T}_{\lambda,h}\\ &\textit{such that} \quad a_\lambda(u_{\lambda,h}, v) = \langle f, v\rangle\,, \quad \forall\, v \in \mathcal{T}_{\lambda,h}.\end{aligned} \tag{9}$$

Computational illustrations of the application of the approximation scheme (9) using piecewise linear finite element spaces are given in [11]. One of our objectives here is to provide a theoretical justification for some of the observations gleaned from those numerical experiments and, as mentioned previously, we want to particularly explore, in rigorous terms, behaviors in the limits $\lambda \to \infty$ and $h \to 0$. We stress that our study only assumes minimal regularity on the solutions and the forcing term is only assumed to be in $f \in L^2$. This distinguishes our convergence results from many existing works that require higher regularities on the data.

3 Asymptotically compatible schemes

We now review the notion of asymptotically compatible schemes and an abstract framework for their numerical analysis when they are applied to a special class of parametrized problems. This framework was introduced in [35] where it was applied to nonlocal models, including (4), and their local limits as the extent of nonlocal interactions vanish, i.e., as $\lambda \to 0$. In Section 4, we apply the framework to the $\lambda \to \infty$ limit. The review is limited to the main conclusions only as they are necessary for the new application to the approximation of fractional diffusion equations.

We adopt notations used in [35] that include many already introduced in Sections 1 and 2 but which may be interpreted differently in a more general context. We consider a family of Hilbert spaces $\{\mathcal{T}_\sigma, \sigma \in [0, \infty]\}$ over $\mathbb{R}$, such that $\mathcal{T}_{\sigma_2}$ is a dense subspace of $\mathcal{T}_{\sigma_1}$ for any $0 \le \sigma_1 \le \sigma_2 \le \infty$. Let $(\cdot, \cdot)_{\mathcal{T}_\sigma}$ and $\|\cdot\|_{\mathcal{T}_\sigma}$ respectively denote the corresponding inner product and norm on $\mathcal{T}_\sigma$ and the dual space of $\mathcal{T}_\sigma$ is denoted by $\mathcal{T}_{-\sigma} = \mathcal{T}_\sigma^*$.

One of the distinctions of the framework of [35] from other convergence studies of parameterized variational problems is that the parameterized spaces (and their limit) are allowed to have different topologies. The following assumption on the function spaces is given in [35].

Assumption 1 [35, Assumption 2.1] $\{\mathcal{T}_\sigma\}$ *are assumed to satisfy:*

i) Uniform embedding: *there exist positive constants M_1 and M_2, with values independent of $\sigma \in [0, \infty]$, such that $M_1\|u\|_{\mathcal{T}_0} \le \|u\|_{\mathcal{T}_\sigma}$, $\forall u \in \mathcal{T}_\sigma$ and $\|u\|_{\mathcal{T}_\sigma} \le M_2\|u\|_{\mathcal{T}_\infty}$, $\forall u \in \mathcal{T}_\infty$.*
ii) Asymptotically compact embedding: *for any sequence $\{u_\sigma \in \mathcal{T}_\sigma\}$, if there exists a constant $C > 0$ with value independent of σ such that $\|u_\sigma\|_{\mathcal{T}_\sigma} \le C$, $\forall \sigma$, then, the sequence $\{u_\sigma\}$ is relatively compact in $\mathcal{T}_0$ and each limit point belongs to $\mathcal{T}_\infty$.*

Note that we need only the case where $\{\mathcal{T}_\sigma\}$ are equivalent spaces in order to discuss the special class of variational problems and approximations defined by Eqs. 7 and 9. By the results of Section 2, we can see that the above assumption is automatically satisfied. We now recall that there are main assumptions discussed in [35] on the bilinear forms, induced linear operators, and finally on approximations.

Assumption 2 [35, Assumption 2.2] *Let a_σ: $\mathcal{T}_\sigma \times \mathcal{T}_\sigma \to \mathbb{R}$ be symmetric and C_1 and C_2 be constants independent of σ such that a_σ is:*

i) bounded: $a_\sigma(u, v) \le C_2\|u\|_{\mathcal{T}_\sigma}\|v\|_{\mathcal{T}_\sigma}$, $\forall u, v \in \mathcal{T}_\sigma$*; and*
ii) coercive: $a_\sigma(u, u) \ge C_1\|u\|^2_{\mathcal{T}_\sigma}$, $\forall u \in \mathcal{T}_\sigma$.

The bilinear form $a_\sigma(\cdot,\cdot)$ induces a bounded self-adjoint and positive definite linear operator, denoted by $\mathcal{A}_\sigma$, from $\mathcal{T}_\sigma$ to its dual $\mathcal{T}_\sigma^*$, with a bounded inverse $\mathcal{A}_\sigma^{-1}:\mathcal{T}_\sigma^*\to\mathcal{T}_\sigma$, via $\langle\mathcal{A}_\sigma u, v\rangle = a_\sigma(u,v)$, $\forall\, u, v\in\mathcal{T}_\sigma$.

Assumption 3 [35, Assumption 2.3] *For operators* $\{\mathcal{A}_\sigma\}$, *there is a subspace* $\mathcal{T}_*$ *such that* $\mathcal{T}_*$ *is dense in* $\mathcal{T}_\infty$, *and is also dense in any* $\mathcal{T}_\sigma$ *with* $\sigma\geq 0$, *such that* $\mathcal{A}_\sigma u\in\mathcal{T}_0$, $\forall\, u\in\mathcal{T}_*$ *and as* $\sigma\to\infty$, $\|\mathcal{A}_\sigma u-\mathcal{A}_\infty u\|_{\mathcal{T}_\sigma^*}\to 0$, $\forall\, u\in\mathcal{T}_*$.

Let us now consider a family of closed subspaces $\{\mathcal{T}_{\sigma,h}\subset\mathcal{T}_\sigma\}$ parametrized by the mesh parameter $h\in(0,h_0]$. The fact that $\mathcal{T}_{\sigma,h}$ is taken as a subspace of $\mathcal{T}_\sigma$ implies that we are effectively considering a standard, internal (or equivalently conforming) Galerkin approximations. For the spaces $\{\mathcal{T}_{\sigma,h}\subset\mathcal{T}_\sigma\}$, we make the following assumptions.

Assumption 4 [35, Assumption 2.4] *Assume that the family of subspaces* $\{\mathcal{T}_{\sigma,h}\subset\mathcal{T}_\sigma\}$, *parametrized by* $\sigma\in[0,\infty]$ *and* $h\in(0,h_0]$, *and satisfies:*

i) *For each* $\sigma\in[0,\infty]$, *the family* $\{\mathcal{T}_{\sigma,h}, h\in(0,h_0]\}$ *is dense in* $\mathcal{T}_\sigma$ *in the sense that,* $\forall\, v\in\mathcal{T}_\sigma$, *there exists a sequence* $\{v_j\in\mathcal{T}_{\sigma,h_j}\}$ *with* $h_j\to 0$ *as* $j\to\infty$, *such that* $\|v-v_j\|_{\mathcal{T}_\sigma}\to 0$ *as* $j\to\infty$.
ii) $\{\mathcal{T}_{\sigma,h},\sigma\in[0,\infty), h\in(0,h_0]\}$ *is asymptotically dense in* $\mathcal{T}_\infty$, *i.e.,* $\forall\, v\in\mathcal{T}_\infty$, *there exists a sequence* $\{v_j\in\mathcal{T}_{\sigma_j,h_j}\}_{h_j\to 0,\,\sigma_j\to\infty}$ *as* $j\to\infty$ *such that* $\|v-v_j\|_{\mathcal{T}_\infty}\to 0$ *as* $j\to\infty$.

It has been noted in [35] that Assumption 4-i ensures the convergence of approximations to $\mathcal{T}_\sigma$ as $h\to 0$ for each σ, whereas Assumption 4-ii is used for studying the limit of both $h\to 0$ and $\sigma\to\infty$.

We now consider a family of variational problems parameterized by $\sigma\in[0,\infty]$ and their approximate problems: for $f\in\mathcal{T}_\sigma^*$,

$$\textit{find } u_\sigma\in\mathcal{T}_\sigma \textit{ such that } a_\sigma(u_\sigma,v)=\langle f,v\rangle\quad\forall\, v\in\mathcal{T}_\sigma. \tag{10}$$

$$\textit{find } u_{\sigma,h}\in\mathcal{T}_{\sigma,h} \textit{ such that } a_\sigma(u_{\sigma,h},v)=\langle f,v\rangle\quad\forall\, v\in\mathcal{T}_{\sigma,h}. \tag{11}$$

The existence and uniqueness of u_σ and $u_{\sigma,h}$ follow from Assumption 2.

We may also express (10) and (11) in strong forms as $\mathcal{A}_\sigma u_\sigma = f$, and $\mathcal{A}_\sigma^h u_{\sigma,h} = \pi_\sigma^h f$ respectively, where π_σ^h denotes the $\mathcal{T}_0$ projection operator onto the subspace $\mathcal{T}_{\sigma,h}$ and $\mathcal{A}_\sigma^h:\mathcal{T}_{\sigma,h}\to\mathcal{T}_{\sigma,h}^*$ is the operator induced by the bilinear form a_σ in $\mathcal{T}_{\sigma,h}$ (or the solution operator of Eq. 11 in the specified subspace).

An abstract framework of *asymptotically compatible schemes* for the analysis of above parametrized problems is established in [35] to study the various limits of $\{u_{\sigma,h}\}$ as we take limits in the parameters.

Definition 5 The family of convergent approximations $\{u_{\sigma,h}\}$ defined by Eq. 11 is said to be *asymptotically compatible* to the solution u_∞ defined by Eq. 10 with $\sigma=\infty$ if, for any sequence $\sigma_j\to\infty$ and $h_j\to 0$, we have $\|u_{\sigma_j,h_j}-u_\infty\|_{\mathcal{T}_0}\to 0$.

Note that because u_{σ_j,h_j} and u_∞ may live in different spaces, the space $\mathcal{T}_0$ is the most natural space that contains all the elements involved.

We first recall a convergence result for the solutions of the parametrized variational problem (10) as $\sigma \to \infty$.

Theorem 6 [35, Theorem 2.5] **Convergence of variational solutions as $\sigma \to \infty$.** *Given the Assumptions 1–3 on the family of spaces and the bilinear forms and operators, for $f \in \mathcal{T}_0$, we have $\|u_\sigma - u_\infty\|_{\mathcal{T}_0} \to 0$ as $\sigma \to \infty$.*

The convergence of approximations as $h \to 0$ for a fixed σ is given below.

Theorem 7 [35, Theorem 2.6] **Convergence as $h \to 0$ for fixed $\sigma \in [0, \infty]$.** *For any $\sigma \in [0, \infty]$ and $f \in \mathcal{T}_0$, given the Assumptions 1–4, there exists a constant $C > 0$ with value independent of h such that $\|u_{\sigma,h} - u_\sigma\|_{\mathcal{T}_\sigma} \le C \inf_{v \in \mathcal{T}_{\sigma,h}} \|v - u_\sigma\|_{\mathcal{T}_\sigma} \to 0$ as $h \to 0$.*

To study the limiting behaviors as both $\sigma \to \infty$ and $h \to 0$ simultaneously and possibly independently, we have

Theorem 8 [35, Theorem 2.9] **Asymptotic compatability.** *Under Assumptions 1–4, for $f \in \mathcal{T}_0$, the family of approximations $\{u_{\sigma,h}\}$ is asymptotically compatible to u_∞.*

We now move on to the analog of Theorem 6 for numerical solutions, that is, the convergence as $\sigma \to \infty$ but for a fixed $h > 0$. We recall the following additional assumptions given in [35] on the approximation spaces are required for this purpose.

Assumption 9 i) Limit of approximate spaces: $\mathcal{T}_{\infty,h} = \mathcal{T}_\infty \cap \left(\bigcap_{\sigma \ge 0} \mathcal{T}_{\sigma,h}\right)$.

ii) Limit of bilinear forms as $\sigma \to \infty$: $a_\sigma(u_h, v_h) \to a_\infty(u_h, v_h)$, $\forall\, u_h, v_h \in \mathcal{T}_{\infty,h}$.

iii) A strengthened continuity property: *for any sequence $\{u_{\sigma,h} \in \mathcal{T}_{\sigma,h}\}$ with uniformly bounded $\{\|u_{\sigma,h}\|_{\mathcal{T}_\sigma}\}$ and satisfying $u_{\sigma,h} \to 0$ in $\mathcal{T}_0$ as $\sigma \to \infty$, we have as $\sigma \to \infty$, $a_\sigma(u_{\sigma,h}, v) \to 0$, $\forall\, v \in \mathcal{T}_{\infty,h}$.*

Theorem 10 [35, Theorem 2.7] **Convergence of approximations as $\sigma \to \infty$ for fixed $h > 0$.** *For $u_{\sigma,h}$ satisfying (11) with $\sigma \in [0, \infty]$ and $f \in \mathcal{T}_0$, we have, under Assumptions 1–4 and 9, $\|u_{\sigma,h} - u_{\infty,h}\|_{\mathcal{T}_0} \to 0$ as $\sigma \to \infty$.*

4 Asymptotically compatibility of approximations of fractional Laplacian problems

Following notations introduced in Sections 1–3, we identify the parameter λ used in Sections 1 and 2 with the parameter σ used in Section 3. Let $\mathcal{A}_\sigma = -\mathcal{L}_\lambda$ and $\mathcal{A}_\infty = (-\Delta)^\alpha$. To apply the abstract framework of Section 3 to the fractional Laplacian problem, we need to verify the assumptions made in that section.

First, it is clear that the inequality $\|u\|_{\mathcal{T}_\lambda} \le \|u\|_{\mathcal{T}_\infty}$ for $\lambda > 0$ and any $u \in \mathcal{T}_\infty$ immediately follows from the definition of the norms. Then, for $\lambda = 1$, the Poincaré type inequality [23] $\|u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\Omega)} \leq C\|u\|_{\mathcal{T}_1}$, $\forall u \in \mathcal{T}_1$, implies that $\|u\|_{\mathcal{T}_0} \leq C\|u\|_{\mathcal{T}_\lambda}$ for $\lambda \geq 1$. Because we only consider the asymptotic behavior as $\lambda \to \infty$, we can ignore the case $\lambda < 1$.

Let us establish the particular relationship between the spaces $\mathcal{T}_\lambda$ and $\mathcal{T}_\infty$.

Lemma 11 *$\{\mathcal{T}_\lambda\}_{\lambda\geq 1}$ and $\mathcal{T}_\infty$, as defined in Section 2.1, are equivalent, i.e., $C_1\|u\|_{\mathcal{T}_\infty} \leq \|u\|_{\mathcal{T}_\lambda} \leq C_2\|u\|_{\mathcal{T}_\infty}$ for $\lambda \geq 1$ where C_1 and C_2 are constants with values independent of λ. Moreover, as $\lambda \to \infty$, $\|u\|_{\mathcal{T}_\lambda} \to \|u\|_{\mathcal{T}_\infty}$, $\forall u \in \mathcal{T}_\infty$.*

Proof For the norm equivalence, we only have to prove the left-hand inequality because $\|u\|_{\mathcal{T}_\lambda} \leq \|u\|_{\mathcal{T}_\infty}$ is obvious. For $\lambda = 1$, it is proved in [12] that $\mathcal{T}_1$ is equivalent to the fractional order Sobolev space $H^\alpha_\Omega(\Omega \cup \Omega_1) := \{w \in H^\alpha(\Omega \cup \Omega_1), w|_{\Omega_1} = 0\}$ where for a general domain $\widetilde{\Omega}$, the space $H^\alpha(\widetilde{\Omega})$ is defined by

$$H^\alpha(\widetilde{\Omega}) := \{w \in L^2(\widetilde{\Omega}) \ : \ \int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}} \frac{\big(w(\mathbf{y}) - w(\mathbf{x})\big)^2}{|\mathbf{x}-\mathbf{y}|^{n+2\alpha}} d\mathbf{y}d\mathbf{x} < \infty\}\,.$$

Thus, $\|u\|_{H^\alpha_\Omega(\Omega\cup\Omega_1)} \leq C\|u\|_{\mathcal{T}_1}$.

In addition, it is shown in [39] that $H^\alpha_\Omega(\Omega \cup \Omega_1)$ is equivalent to $\mathcal{T}_\infty := H^\alpha_\Omega(\mathbb{R}^n)$. Thus we have $\|u\|_{\mathcal{T}_\infty} \leq \widetilde{C}\|u\|_{\mathcal{T}_1} \leq \widetilde{C}\|u\|_{\mathcal{T}_\lambda}$ for $\lambda \geq 1$. To complete the proof of Lemma 11, we note that

$$\mathbb{1}_\lambda(\mathbf{x},\mathbf{y})\frac{C_{n,\alpha}}{2|\mathbf{x}-\mathbf{y}|^{n+2\alpha}}\big(u(\mathbf{y}) - u(\mathbf{x})\big)^2 \to \frac{C_{n,\alpha}}{2|\mathbf{x}-\mathbf{y}|^{n+2\alpha}}\big(u(\mathbf{y}) - u(\mathbf{x})\big)^2$$

almost everywhere in $\mathbb{R}^n$ as $\lambda \to \infty$, Thus, we have that $\|u\|_{\mathcal{T}_\lambda} \to \|u\|_{\mathcal{T}_\infty}$ by the dominated convergence theorem. □

Together with the above lemma, we may complete the verification of Assumption 1 with the result below.

Lemma 12 *Suppose $u_j \in \mathcal{T}_{\lambda_j}$ with $\lambda_j \to \infty$. If*

$$\sup_j \|u_j\|_{\mathcal{T}_{\lambda_j}} < \infty\,, \tag{12}$$

then u_j is precompact in $L^2_0(\Omega)$. Moreover, any limit point $u \in \mathcal{T}_\infty$.

Proof By Lemma 11, Eq. 12 implies $\sup_j \|u_j\|_{\mathcal{T}_\infty} < \infty$. Thus the result is true because H^α is compactly embedded in L^2. □

We next verify the Assumption 2 on the bilinear forms. Note that $a_\lambda(\cdot,\cdot)$ is exactly the inner product defined on $\mathcal{T}_\lambda$ so that Assumption 2 is naturally satisfied with $C_1 = C_2 = 1$.

Moving onto the Assumption 3, we need to establish the convergence of the operator $\mathcal{L}_\lambda$ to the fractional Laplacian $(-\Delta)^\alpha$ on a dense subspace of $\mathcal{T}_\infty$; we state this here as a lemma.

Lemma 13 *For $\mathcal{L}_\lambda$ in* Eq. 3 *and $\forall\, w \in C_c^\infty(\Omega)$ with zero extension outside Ω, we have $\mathcal{L}_\lambda w \in L_0^2(\Omega \cup \Omega_\lambda)$ and $\|\mathcal{L}_\lambda w - (-\Delta)^\alpha w\|_{L^2(\mathbb{R}^n)} \to 0$ as $\lambda \to \infty$.*

Proof Fixing any $w \in C_c^\infty(\Omega)$, we know that $(-\Delta)^\alpha w \in L^2(\mathbb{R}^n)$. Also, it is not difficult to see, because $\mathcal{L}_\lambda w$ is bounded, that $\mathcal{L}_\lambda w \in L_0^2(\Omega \cup \Omega_\lambda)$ (which is the space with functions in $L^2(\Omega \cup \Omega_\lambda)$ with zero extension outside $\Omega \cup \Omega_\lambda$). In addition, it is easy to see that $\mathcal{L}_\lambda w(\mathbf{x})$ converges to $(-\Delta)^\alpha w(\mathbf{x})$ pointwise and uniformly because

$$|\mathcal{L}_\lambda w(\mathbf{x}) - (-\Delta)^\alpha w(\mathbf{x})| \le C\|w\|_\infty \int_{\mathbb{R}^n \setminus B_\lambda(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}|^{n+2\alpha}} d\mathbf{y} \to 0\,.$$

Now that both $\mathcal{L}_\lambda w$ and $(-\Delta)^\alpha w$ are in L^2, we conclude by the dominated convergence theorem that $\mathcal{L}_\lambda w$ converges to $(-\Delta)^\alpha w$ in the L^2 norm. □

Before we verify Assumptions 4 and 9 and consider the proof of convergence of discrete solutions, for the sake of completeness we provide the following theorem. It states the existence and uniqueness of solutions of the nonlocal diffusion problem (2) land the fractional Laplacian problem (1) and addresses some technical difficulties encountered in [39]. In addition, it also proves the convergence, as $\lambda \to \infty$, of the solution u_λ of the nonlocal diffusion problem to the solution u of the fractional Laplacian problem under minimal regularity assumptions.

Theorem 14 *For any $\lambda \in (0, \infty]$ and $f \in L^2(\Omega)$, there exists a unique solution u_λ of* Eq. 7. *Moreover, $\|u_\lambda - u_\infty\|_{H^\alpha} \to 0$ as $\lambda \to \infty$.*

Proof The existence and uniqueness follows from the Lax-Milgram theprem. Now, as a direct application of Theorem 6, we know that $\|u_\lambda - u_\infty\|_{L^2} \to 0$ as $\lambda \to \infty$. In addition, we know from Lemma 12 that u_λ converges to u_∞ weakly in $\mathcal{T}_\infty$. Now, because u_λ and u_∞ are solutions of Eq. 7, we have

$$a_\lambda(u_\lambda, u_\lambda) = \langle f, u_\lambda\rangle \to \langle f, u_\infty\rangle = a_\infty(u_\infty, u_\infty)\,,$$

which is equivalent to $\|u_\lambda\|_{\mathcal{T}_\lambda} \to \|u_\infty\|_{\mathcal{T}_\infty}$ as $\lambda \to \infty$. We know that $u_\lambda|_{\Omega^c} = 0$ so that, for λ large enough, Here Ω^c is the complement of Ω.

$$\begin{aligned}
\|u_\lambda\|_{\mathcal{T}_\infty}^2 - \|u_\lambda\|_{\mathcal{T}_\lambda}^2 &= \frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n \setminus B_\lambda(\mathbf{x})} \frac{\big(u_\lambda(\mathbf{y}) - u_\lambda(\mathbf{x})\big)^2}{|\mathbf{x}-\mathbf{y}|^{n+2\alpha}} d\mathbf{y}d\mathbf{x} \\
&= \frac{C_{n,\alpha}}{2}\int_\Omega u_\lambda^2(\mathbf{x}) \int_{\mathbb{R}^n \setminus B_\lambda(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}|^{n+2\alpha}} d\mathbf{y}d\mathbf{x} \\
&= C\|u_\lambda\|_{L^2} \int_{\mathbb{R}^n \setminus B_\lambda(\mathbf{0})} \frac{1}{|\mathbf{z}|^{n+2\alpha}} d\mathbf{z} \to 0\,.
\end{aligned}$$

Thus we have $\|u_\lambda\|_{\mathcal{T}_\infty} \to \|u_\infty\|_{\mathcal{T}_\infty}$. This implies $\|u_\lambda - u_\infty\|_{\mathcal{T}_\infty} \to 0$ because

$$\begin{aligned}\|u_\lambda - u_\infty\|^2_{\mathcal{T}_\infty} &= \|u_\lambda\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2(u_\lambda, u_\infty)_{\mathcal{T}_\infty} \\ &= \|u_\lambda\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2\langle u_\lambda, (-\Delta)^\alpha u_\infty\rangle \\ &\to 2\|u_\infty\|^2_{\mathcal{T}_\infty} - 2\|u_\infty\|^2_{\mathcal{T}_\infty} = 0\,.\end{aligned}$$

□

Next, we consider the numerical solution $u_{\lambda,h}$ of the discretized nonlocal diffusion problem (9) and show that it converges to the solution $u = u_\infty$ of the fractional Laplacian problem.

For $\mathcal{T}_{\lambda,h}$, let us focus on using finite-dimensional spaces associated with a regular triangulation $\tau_h = \{K\}$ of the domain Ω with h denoting the maximum element diameter of the elements in τ_h. For example, we consider standard conforming finite element spaces with piecewise polynomials:

$$\mathcal{T}_{\lambda,h} = \mathcal{T}_h := \{v \in \mathcal{T}_\infty = H^\alpha_\Omega(\mathbb{R}^n),\ v|_K \in P(K) \quad \forall K \in \tau_h\} \tag{13}$$

where $P(K) = \mathcal{P}_p(K)$ is the space of polynomials on $K \in \tau_h$ of degree less or equal than p. We note that functions in $\mathcal{T}_{\lambda,h}$ vanish automatically outside of Ω. For each given α, as $h \to 0$, $\{\mathcal{T}_h\}$ is dense in $\mathcal{T}_\lambda$ and $\mathcal{T}_\infty = H^\alpha_\Omega(\mathbb{R}^n)$, i.e., for any $v \in \mathcal{T}_\lambda$, there exists a sequence $\{v_h \in \mathcal{T}_h\}$ such that

$$\|v_h - v\|_{\mathcal{T}_\lambda} \to 0 \quad \text{as} \quad h \to 0\,, \tag{14}$$

$$\|v_h - v\|_{H^\alpha} \to 0 \quad \text{as} \quad h \to 0\,. \tag{15}$$

Since the solution space is the closure of compactly supported smooth functions in Ω, these properties can be established as in standard finite element approximation theory [10].

A couple of comments on the choices of discrete finite element spaces are in order. For $0 < \alpha < 1/2$, because the space of piecewise constant functions and the space of continuous piecewise linear functions are both dense in H^α, we can apply any finite element space containing either of these two finite element spaces as a subspace to the nonlocal diffusion equation to obtain an asymptotically compatible scheme for the fractional Laplacian, i.e., regardless of how λ and h change, the solution of (9) always approximates the solution of the fractional equation (7) properly as $\lambda \to \infty$ and $h \to 0$. On the other hand, for $\alpha \geq 1/2$, discontinuous piecewise constant functions are no longer dense in H^α, but the space of continuous piecewise linear functions remains dense. Thus, to guarantee the asymptotical compatibility, we require the finite element spaces to contain the space of continuous piecewise linear functions as a subspace whenever $1/2 \leq \alpha < 1$. Naturally, the piecewise linear functions can be replaced by some generalized or extended finite element functions [4] or any other discrete space that are conforming and enjoy the density properties in $H^\alpha_\Omega(\mathbb{R}^n)$ as $h \to 0$.

The standard approximation properties (14) and (15) ensure that $\mathcal{T}_{\lambda,h} = \mathcal{T}_h$ satisfies Assumption 4. For Assumption 4-ii, we note that the zero extension of the elements in $\mathcal{T}_\lambda$ can be identified with those in $\mathcal{T}_\infty$ and vice versa, and the respective norms in those spaces are equivalent uniformly with respect to $\lambda \to \infty$.

By Theorem 7 and the approximation property (14), we obtain the following convergence result for Galerkin approximations for any fixed $\lambda \in (1, \infty]$.

Theorem 15 *For any given $\lambda \in (1, \infty]$, let u_λ and $u_{\lambda,h}$ be defined by* Eqs. 10 *and* 11*, respectively for $f \in L^2(\Omega)$. Given the approximate spaces $\mathcal{T}_{\lambda,h} = \mathcal{T}_h \subset H^\alpha_\Omega(\mathbb{R}^n)$ defined in* (13) *with* (14)*, there then exists a constant $C > 0$ with value independent of h such that $\|u_{\lambda,h} - u_\lambda\|_{\mathcal{T}_\lambda} \le C \inf_{v_{\lambda,h}\in\mathcal{T}_h} \|v_{\lambda,h} - u_\lambda\|_{\mathcal{T}_\lambda} \to 0$, as $h \to 0$.*

With all of Assumptions 1–4 verified, the following theorem provides a criterion for asymptotically compatible schemes for solving the nonlocal diffusion problem as an approximation of the fractional Laplacian problem. The theorem is basically a direct application of Theorem 8. However, for the specific setting we consider here, we can prove convergence in the energy norm and not just in the L^2 norm.

Theorem 16 *Let u_λ and $u_{\lambda,h}$ denote the solutions of* Eqs. 7 *and* 9*, respectively, for $f \in L^2(\Omega)$ with $\mathcal{T}_{\lambda,h} = \mathcal{T}_h \subset H^\alpha_\Omega(\mathbb{R}^n)$ defined in* Eq. 13 *with* Eq. 14*. We then have $\|u_{\lambda,h} - u_\infty\|_{H^\alpha} \to 0$ as $\lambda \to \infty$, $h \to 0$.*

Proof First, as a direct application of Theorem 8, we know that

$$\|u_{\lambda_j,h_j} - u_\infty\|_{L^2} \to 0\,, \qquad \text{as } \; j \to \infty$$

for any sequence $\lambda_j \to \infty$ and $h_j \to 0$. Now, the $\mathcal{T}_\infty$, or more specifically, the H^α convergence is essentially the same as the proof of Theorem 14. In more detail, because u_{λ_j,h_j} converges to u_∞ weakly in $\mathcal{T}_\infty$ and they are solutions of Eqs. 9 and 7, respectively, we have

$$a_{\lambda_j}(u_{\lambda_j,h_j}, u_{\lambda_j,h_j}) = \langle f, u_{\lambda_j,h_j}\rangle \to \langle f, u_\infty\rangle = a_\infty(u_\infty, u_\infty)$$

which is equivalent to $\|u_{\lambda_j,h_j}\|_{\mathcal{T}_{\lambda_j}} \to \|u_\infty\|_{\mathcal{T}_\infty}$ as $j \to \infty$. Then, we can show $\|u_{\lambda_j,h_j}\|_{\mathcal{T}_\infty} \to \|u_\infty\|_{\mathcal{T}_\infty}$ as in the proof of Theorem 14, and this implies $\|u_{\lambda_j,h_j} - u_\infty\|_{\mathcal{T}_\infty} \to 0$ because

$$\begin{aligned} \|u_{\lambda_j,h_j} - u_\infty\|^2_{\mathcal{T}_\infty} &= \|u_{\lambda_j,h_j}\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2(u_{\lambda_j,h_j}, u_\infty)_{\mathcal{T}_\infty} \\ &= \|u_{\lambda_j,h_j}\|^2_{\mathcal{T}_\infty} + \|u_\infty\|^2_{\mathcal{T}_\infty} - 2\langle u_{\lambda_j,h_j}, (-\Delta)^\alpha u_\infty\rangle \\ &\to 2\|u_\infty\|^2_{\mathcal{T}_\infty} - 2\|u_\infty\|^2_{\mathcal{T}_\infty} = 0\,. \end{aligned}$$

□

Finally, we consider the limit of the discrete schemes. To do so, we have to verify the Assumption 9, which is needed for applying the Theorem 10. Note that this additional assumption is not required for Theorem 8 which is used for establishing the asymptotical compatibility property in Theorem 16.

With $\mathcal{T}_{\lambda,h} = \mathcal{T}_{\infty,h}$ for any λ, Assumptions 9-i and 9-ii are straightforward to verify. Moreover, Assumption 9-iii follows because we know that a sequence $\{v_{\lambda_j}\}$ with strong convergence to 0 in $L^2(\Omega)$ and having bounded norms $\{\|v_{\lambda_j}\|_{\mathcal{T}_{\lambda_n}}\}$ is automatically uniformly bounded and weakly convergent to 0 in $\mathcal{T}_\infty = H^\alpha_\Omega(\mathbb{R}^n)$. We now have the final result of this paper.

Theorem 17 *For given spaces $\mathcal{T}_{\lambda,h} = \mathcal{T}_h \subset H^\alpha_\Omega(\mathbb{R}^n)$ defined in* Eq. 13 *with* Eqs. 14 *and* 15*, we have*

$$\|u_{\lambda,h} - u_{\infty,h}\|_{\mathcal{T}_\infty} \to 0 \qquad as\ \ \lambda \to \infty. \tag{16}$$

Proof With all of Assumptions 1–4 and Assumption 9 verified, we obtain the strong convergence of $u_{\lambda,h}$ to $u_{\infty,h}$ in $\mathcal{T}_0$ by Theorem 10. Now, following a similar line of arguments as that in the proofs of Theorems 14 and 16, we can conclude that the convergence holds in $\mathcal{T}_\infty$, which leads to the desired result. □

Let us note that while the above discussion is devoted to the limit $\lambda \to \infty$, the same argument and analysis carry out to any limiting process $\lambda \to \lambda^*$ for some positive constant $\lambda^* \in (0, \infty)$ as well. The case $\lambda^* = 0$ requires special consideration which was the subject studied in [35].

5 Local, nonlocal and fractional models and their approximations

Let us first present a summary of the results obtained on the relations between the nonlocal and fractional diffusion models and their numerical approximations in Fig. 1. The vertical (respectively, horizontal) arrows in the figure refer to convergence with respect to the spatial grid size $h \to 0$ (respectively, the interaction radius $\lambda \to \infty$), the right side (respectively, left side) refers to solutions of the fractional Laplacian problem (respectively, the nonlocal diffusion problem) and its spatial approximation, and the curved path refers to taking simultaneous but independent limits of $h \to 0$ and $\lambda \to \infty$. The diagram is similar to that given in [35] that connects the approximations of the nonlocal models and their local limits (as $\lambda \to \infty$).

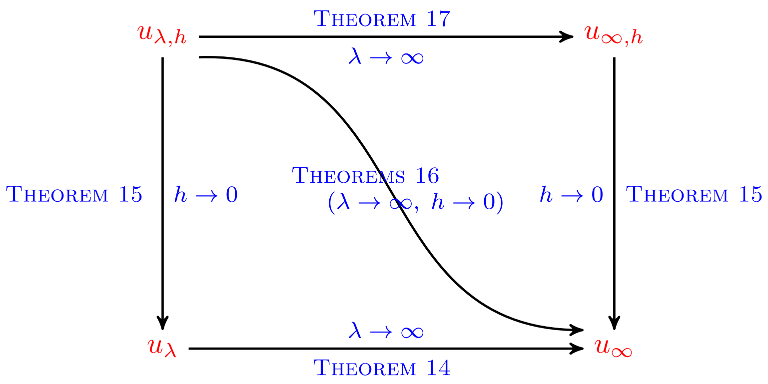

Fig. 1 A diagram for asymptotically compatible schemes and convergence for the fractional Laplacian problem (with solution denoted by u_∞) and the nonlocal diffusion problem with solution denoted by u_λ

Our study took the fractional Laplacian operator as a limit of the linear nonlocal diffusion operator developed in [12, 13], which is similar in spirit to the work presented in [11]. Careful a priori error estimates on the solution were derived in [11] under additional regularity assumptions compared to the ones invoked in this paper; in [11] the results of numerical experiments were provided that illustrated the predicted order of convergence with respect to λ and h when one of them is sufficiently small. However, it is not clear from [11] whether the correct solution to the fractional Laplacian could be ensured when λ and h change simultaneously, given that there is no simple specific dependence of some of the error terms on λ which also involves h. Here, we eliminate such concerns by showing theoretically that any conforming Galerkin finite element method for the nonlocal equation is asymptotically compatible, no matter how the nonlocal interaction parameter λ changes with discretization parameter h. Moreover, the convergence of $u_{\lambda,h}$ to u in the H^α norm is given under minimal regularity assumption (only $\mathcal{L}_\lambda u \in H^{-\alpha}$) as well as general geometric meshes for conforming Galerkin finite element approximation of the nonlocal diffusion equations. This is another major contribution of our work because there are still many open questions concerning the regularity of solutions for fractional differential equations. Furthermore, based on numerical results, it was conjectured in [11] that for fixed h, convergence of solutions is observed as $\lambda \to \infty$. Theorem 17 provides a clean affirmative answer to the conjecture.

The current study can also be viewed as complementary to the studies made in [34, 35] on the approximations of nonlocal diffusion models (associated with the nonlocal operator $\mathcal{L}_\lambda$) and local diffusion models (partial differential equations associated with the Laplacian operator $\mathcal{L}_0 = \Delta$) as the nonlocal interaction horizon $\lambda \to 0$, as illustrated in Fig. 2. With the current paper, we now also have the limit as $\lambda \to \infty$ of the nonlocal diffusion models associated with the fractional Laplacian operator $\mathcal{L}_\infty = (-\Delta)^\alpha$. The fact that we can demonstrate the asymptotic compatibility of numerical schemes for nonlocal diffusion models as the horizon λ increases in size means that one may effectively simulate the fractional models associated with a global nonlocal interaction by a nonlocal model which is less global (i.e., is more localized) resulting in a gain computational efficiency without a loss of fidelity.

As noted at the end of the last section, we in fact have established a complete theory for the asymptotic compatibility of numerical schemes with respect to any limiting process $\lambda \to \lambda^* \in (0, \infty)$. Furthermore, we can advocate asymptotic compatible schemes as robust algorithms for approximating local, nonlocal and fractional models in the sense that the convergence to the correct continuum limit is assured

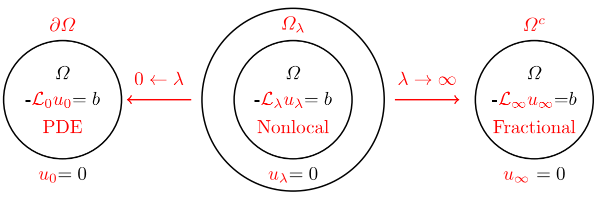

Fig. 2 Different limits of nonlocal diffusion equations: partial differential equations as local limits ($\lambda \to 0$) and fractional Laplacian equations as global limits ($\lambda \to \infty$)

for any values of the parameter λ and in any of its limiting regimes as the numerical resolution is increased.

6 Concluding remarks

In this paper, our central goal was to obtain convergence results for approximations, due to both domain truncation and spatial discretization, of solutions of the fractional Laplacian problem (1). The abstract framework developed in [35] for asymptotically compatible schemes was applied to the specific setting of the fractional Laplacian problem. In meeting our goal, we obtained intermediate results about the convergence of approximations of the nonlocal diffusion problem (4).

In this work, we only considered conforming methods for nonlocal equations which may or may not allow discontinuous finite element methods, depending on the value of α. There may be possible extensions to nonconforming methods even for $\alpha \geq 1/2$, such as those recently developed in [36]. It is also feasible to consider problems involving inhomogeneous nonlocal Dirichlet volumetric constraints or different types of nonlocal boundary conditions.

As for other future works, interesting numerical experiments, especially in two or higher space dimensions may be carried out to illustrate the optimal choices of h and λ for both smooth solutions and solutions with limited regularity. Relations and approximations of the nonlocal diffusion models to fractional diffusion equations in unbounded domains can also be studied. The effect of numerical quadratures may also be of great practical interest and can possibly be analyzed in the same framework. Finally, it is feasible that connections between the general nonlocal diffusion Eq. 6 with kernels other than ones considered here or nonlocal convection diffusion equations [14] and other fractional derivative models can be exploited in much the similar manner as the connection with fractional Laplacian models was exploited in this paper.

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