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Diverse Vortex Dynamics in Superfluids

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Abstract. The appearance of quantized vortices is a typical signature of superfluidity which has received a lot of attention in the studies of superfluid Helium, superconductivity and more recently the Bose-Einstein condensation. Recent theoretical and computational studies on the vortex dynamics have shed light on the characteristics of the quantized vortices. Here, we discuss several dynamical systems governing the motion and interaction of quantized vortices and give some computational results that illustrate the diverse nature of vortex dynamics under different driven forces.

1. Introduction

The recent experimental confirmation of Bose-Einstein condensation (BEC) has once again drawn spotlight to the phenomena of quantized vortices. BEC, the so called fifth state of matter, rapidly becomes an experimental tool for scientists to uncover the secrets of superfluidity. A particularly interesting signature of superfluids is the ability to support quantized circulation, thus, leading to the nucleation of quantized vortices.

Quantized vortices have a long history that begins with the studies of liquid Helium and superconductors. In recent BEC experiments, vortices have been nucleated with the help of laser stirring and rotating magnetic traps. Remarkably, many of the phenomenological properties of quantized vortices have been well captured by relatively simple mathematical models, for example, the Ginzburg-Landau equations and the Gross-Pitaesvkii equations.

In recent years, there have been many works on the mathematical analysis and numerical simulation of quantized vortices. The vortex structures have been studied through various approaches ranging from asymptotic analysis, numerical simulations and rigorous mathematical analysis. For instance, in the context of superconductivity, the nucleation mechanism of quantized vortices due to the presence of the applied magnetic field has become more and more clear through the study

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of the phenomenological Ginzburg-Landau equations. Moreover, vortices may be set in motion due to vortex interactions, thermal fluctuations and applied voltages and currents. Such vortex motion, unfortunately, induces electrical resistance and cause the loss of superconductivity. Understanding the dynamics of quantized vortices thus bears tremendous importance. When the coherence length of the superconductor is very small, represented by a large Ginzburg-Landau parameter κ, dynamical laws of well separated vortices have been deduced and mathematically justified in the case of a static constant applied magnetic field. In spite of the much progress made in the last decade, it should be pointed out that the rigorous mathematical study of a large part of the subject on vortex dynamics remains nearly non-existent, in particular, much of the studies on the vortex dynamic laws have not incorporated issues like the effect of applied current and thermal fluctuations. On the other hand, numerical simulations have become useful tools that could help providing a more clear picture on the exotic vortex dynamics driven by various forces. In this paper, we present several dynamical systems that have been used to model the vortex motion and interaction. Some computational results as well as open questions are also presented to illustrate the diverse characteristics of the vortex dynamics in superfluids.

2. Ginzburg-Landau Dynamics

The phenomenological model of Ginzburg and Landau [**30, 64**], and its various generalizations have been widely used in the mathematical and numerical studies of the vortex phenomena in superconductivity. Recent mathematical works have laid rigorous foundation to many claims made based on physical intuitions.

Time dependent Ginzburg-Landau equations. Let $\Omega \subset R^d$ be the region occupied by the superconducting sample. The primary variables used in the time dependent GL model are the complex scalar-valued *order parameter* ψ, the real vector-valued *magnetic potential* $\mathbf{A}$, and the real scalar-valued electric potential $\bar{\Phi}$. In a non-dimensional form, these variables are related to the physical variables by:

density of superconducting charge carriers	$\lvert\psi\rvert^2$
induced magnetic field	$\mathrm{curl}\mathbf{A}$
current	$\mathbf{j} = \mathbf{curl}\,\mathrm{curl}\mathbf{A}$
electric field	$\frac{\partial}{\partial t}\mathbf{A} + \nabla\bar{\Phi}\,.$

For an applied magnetic field H, the conventional Ginzburg-Landau free energy is

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(\frac{1}{2} \left| \left(\frac{i}{\kappa}\nabla + \mathbf{A} \right) \psi \right|^2 + \frac{1}{4}(1 - |\psi|^2)^2 + \frac{1}{2}|\mathrm{curl}\ \mathbf{A} - H|^2 \right) d\Omega\,,$$

where κ, the Ginzburg-Landau parameter, is a material constant.

Let η_1 and η_2 be given relaxation parameters, the conventional time-dependent Ginzburg-Landau (TDGL) model is given by

$$\eta_1(\frac{\partial\psi}{\partial t} + i\kappa\bar{\Phi}\psi) + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi \quad = \quad 0\,, \tag{2.1}$$

$$\eta_2 \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\bar{\Phi} \right) + \mathbf{curl}\,\mathrm{curl}\mathbf{A} + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A} \quad = \quad 0\,. \tag{2.2}$$

ψ^* is the complex conjugate of ψ. On $\Gamma \times (0, \infty)$, the boundary conditions are

$$(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi)\cdot\mathbf{n} = -\gamma\psi\,, \tag{2.3}$$

$$\mathrm{curl}\mathbf{A} = H\,, \tag{2.4}$$

$$\eta_2\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\bar{\Phi}\right)\cdot\mathbf{n} = \mathbf{J}\cdot\mathbf{n}\,, \tag{2.5}$$

where $\mathbf{J}$ is an applied current, γ is a parameter accounting for the proximity effect.

The initial conditions are:

$$\psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}) \quad \text{and} \quad \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0(\mathbf{x}) \quad \text{in} \quad \Omega\,.$$

It is convenient to introduce an auxiliary variable $\Phi_a(\mathbf{x}, t) = (\mathbf{J}\cdot\mathbf{x})/\eta$ and define $\Phi = \bar{\Phi} - \Phi_a$. The triple $(\psi, \mathbf{A}, \Phi)$ is used as the primary variables for the TDGL equations. Note that the equations are related to the energy functional by

$$\eta_1(\frac{\partial\psi}{\partial t} + i\kappa\bar{\Phi}\psi) = -\frac{\partial\mathcal{G}}{\partial\psi}(\psi, \mathbf{A})\,, \tag{2.6}$$

$$\eta_2\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\Phi\right) = -\frac{\partial\mathcal{G}}{\partial\mathbf{A}}(\psi, \mathbf{A})\,. \tag{2.7}$$

The TDGL equations enjoys the *gauge invariance* property, see [**27, 33**] for more detailed discussions.

Gradient Dynamics. Let $\mathbf{J} = 0$, $\gamma = 0$, $\eta_1 > 0$ and $\eta_2 > 0$, the TDGL equations can be viewed as the gradient flow of the free energy $\mathcal{G}$. As $t \to \infty$, solutions of the TDGL equations would go to some steady states which are critical points of the free energy functional [**33, 61**].

When the magnetic field H reaches the vicinity of the so-called lower critical field H_{c1}, the solutions start to nucleate the quantized vortices, that is, the order parameter has isolated zeros with non-trivial topological degrees [**33, 58, 64**]. The magnetic field penetrates the material sample through the cores of these quantized vortices. When H becomes larger than H_{c2}, the vortex structures (lattices) would generally be destroyed and the solution becomes one that corresponding to the normal state, see Fig. 1 for an illustration.

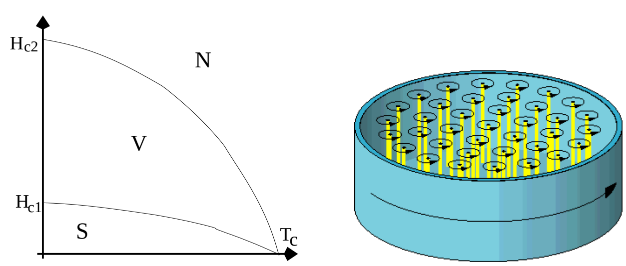

FIGURE 1. Equilibrium phase diagram for superconductors in an applied field and an illustration of quantized vortices.

Typical snapshots of the solutions of TDGL for a two dimensional square domain are given in the Fig. 1. An initial superconducting state is used with an applied field $H > H_{c1}$. In the plots, contour lines for $|\psi|$ are drawn. $|\psi|^2$ is proportional to the density of superconducting charge carriers, thus, $|\psi| = 0$ corresponds to the normal state and, in our nondimensionalization, $|\psi| = 1$ corresponds to the superconducting state. For more numerical simulations, see [**21, 31, 32, 42, 44, 53**].

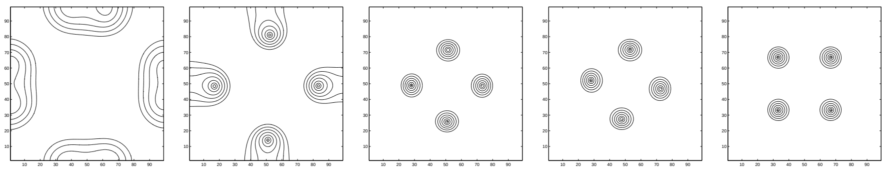

FIGURE 2. Contour plots of $|\psi|$ in a 2-d square at t= $5, 10, 20, 25, \infty$.

For high-T_c superconductors, the G-L parameter value κ is very large, which is a fact that has been utilized to derive various reductions of the original G-L equations. Comprehensive studies of the minimizers of the free energy functional have become available recently [**33, 58, 59, 60**] in the $\kappa \to \infty$ limit. In such a limit, the vortices becomes point defects in an otherwise superconducting state. These studies provided rigorous characterization and justification of the physical behavior on the dependence of critical fields on κ in the high-κ limit.

Dynamics in the Absence of Magnetic Field. The rigorous study of vortex dynamical laws started with the case of $\mathbf{H} = 0$, i.e., the following model problems were first considered:

$$\frac{1}{\lambda_\epsilon}\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\epsilon^2}\, u\,(1-|u|^2) \quad \text{in} \quad \Omega \times R_+ \,, \tag{2.8}$$

$$u(x,0) = u_\epsilon^0(x), \quad x \in \Omega\,, \quad \text{and} \quad u(x,t) = g(x), \quad x \in \partial\Omega\,. \tag{2.9}$$

Here, ϵ is inversely proportional to κ. The dynamics of the vortices in the limit $\epsilon \to 0$ can be considered within the framework of a general program initiated by J. Neu, and later extended, and improved by many others [**14, 56, 57, 7, 37, 38**] using the method of matched asymptotic expansions and rigorously proved in [**50, 51, 45**].

For $\Omega \subset R^2$, assuming that u_ϵ^0 consists of d vortices,

$$u_\epsilon^0(x) \to u_0(x) = \prod_{j=1}^{d} \frac{(x-b_j)}{|x-b_j|} e^{i\,h(x)}\,,$$

it has been shown that [**50**], to leading order, the vortex dynamical laws are:

$$m_i\,\frac{d\,b_i\,(t)}{dt} = -\nabla_{b_i} W_n(b), \qquad i = 1, 2, \ldots d, \tag{2.10}$$

where $b = (b_1, \ldots, b_d)$ are positions of vortices, the constants $\{m_i\}$ are called mobilities of the vortices, and W_n is the so-called renormalized energy [**6, 50**]. In particular, it has been shown that isolated vortices of the same signs tend to repel while isolated vortices of opposite signs attract.

High-κ, high field dynamics. In the high-κ limit considered in [**11, 29**], the applied field is assumed to be high so that it penetrates the sample completely to the leading order, i.e., the induced field is equal to the applied field. With a proper scaling and invoking a gauge choice $\Phi = 0$, and $\mathbf{A} \cdot \mathbf{n} = 0$, the time-independent magnetic potential $\mathbf{A}_0$ can be solved for separately from the Maxwell's equations, i.e., $\nabla \cdot \mathbf{A}_0 = 0$, $\nabla \times \mathbf{A}_0 = H$, with boundary conditions $\mathbf{A}_0 \cdot \mathbf{n} = 0$. Ignoring

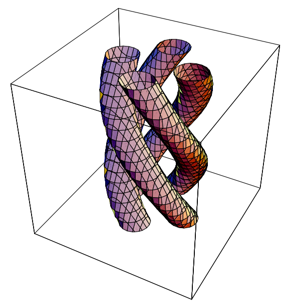

FIGURE 3. Vortex torsion due to the pinning of the normal inclusions in a three dimensional superconductor sample.

the proximity effect, the resulting simplified leading-order system for the order parameter ψ_0 is given by:

$$\frac{\partial \psi_0}{\partial t} + i\Phi_a \psi) + (i\nabla + \mathbf{A}_0)^2 \psi_0 - \psi_0 + |\psi_0|^2 \psi_0 = 0 \quad \text{in } \Omega \tag{2.11}$$

$$(i\nabla + \mathbf{A}_0)\psi_0 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\,, \quad \text{and} \quad \psi_0(\mathbf{x}, 0) = \varphi(\mathbf{x}) \quad \text{in } \Omega\,, \tag{2.12}$$

Interestingly, with the presence of Φ_a and $\mathbf{A}_0$, the above high-κ high-field (HKHF) equation preserves much of the physics described by the original TDGL equations. The dynamics can also be much more diverse than the case $\mathbf{A}_0 = 0, \Phi_a = 0$, illustrated by the examples given later.

Dynamics involving spatial inhomogeneities. When modeling spatial in homogeneities such as thin films of variable thickness [**10**] and SNS junctions [**49, 35**], various modified dynamic equations have been considered. For instance, when modeling normal inclusions, we may consider

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\epsilon^2} u(a(x) - |u|^2), \text{ in } \Omega \times (0, +\infty) \tag{2.13}$$

for some function $a > 0$. Under proper scalings, the dynamic laws would change dramatically with a non-degenerate coefficient a, namely, the equation of motion becomes effectively [**26, 46**]:

$$\frac{db_j}{dt} = -\nabla \log a(b_j)\,, \tag{2.14}$$

in the limit $\epsilon \to 0$. The same dynamic law applies to the model for the variable thickness thin film:

$$\frac{\partial u}{\partial t} = a\left\{\Delta u + \frac{1}{\epsilon^2} u(1 - |u|^2)\right\}, \text{ in } \Omega \times (0, +\infty) \tag{2.15}$$

where $a > 0$ is a function measuring the relative thickness of the three dimensional thin film [**9, 22, 23**]. Physically, this is associated to the phenomena that the vortices turn to move to regions where the film is thin. Extensions to the cases with the presence of the applied magnetic field have also been obtained [**22, 24, 26, 60**].

More general version involving the effect of applied magnetic field can also be considered [**35**]:

$$\frac{\partial u}{\partial t} = (\nabla - i\mathbf{A}_0)^2 u + \frac{1}{\epsilon^2} u(a(x) - |u|^2), \text{ in } \Omega \times (0, +\infty)\,. \tag{2.16}$$

The corresponding free energy also shares remarkable resemblance with the Gross-Pitaevskii free energy used to model the vortex state of a BEC under the rotating magnetic trap. In such case, $a(x)$ is a conventional trapping potential while curl$\mathbf{A}_0$ represents the angular velocity of the rotating trap [**2**].

3. Dynamics driven by applied current

An applied current J generally exerts a Lorentz force $F = J \times B$ on each vortex core, analog to the action of the Magnus force on a spinning tennis ball. The motion of vortices due to the Lorentz force induces an electric field, and thus produces electrical resistance. Thus, in superconductivity, it is important to understand the interaction of the vortices with the applied current and study the critical values of the applied current which will dislodge the vortices from their equilibrium positions.

The TDGL equations may be used as a prototype model for the study of critical current. However, it is equally enlightening to study an even simpler system, the high-κ high-field (HKHF) model. Assume that Ω is the unit square (or disk) in R^2, and the applied current is applied along the x_2 direction, then, we have the following HKHF equation:

$$\frac{\partial \psi}{\partial t} + i\Phi_a \psi + (i\nabla + \mathbf{A}_0)^2 \psi - \psi + |\psi|^2 \psi = 0 \quad \text{in } \Omega \tag{3.1}$$

with boundary condition $(i\nabla + \mathbf{A}_0)\psi \cdot \mathbf{n} = 0$ and initial condition $\psi = \psi_0$. Here, $\Phi_a(x_1, x_2) = c + Jx_2$ for some constant $c > 0$, and $\mathbf{A}_0(x_1, x_2) = H(x_2, -x_1)/2$ for some constant field H.

Beyond the basic well-posedness, there exists very little mathematical analysis on the HKHF equation. Even many questions on its steady state solutions remain largely unanswered. In [**34**], a perturbative technique was used to show that for H near H_{c1}, for sufficiently small ϵ, and for small enough J, the steady state has a solution which possesses a single vortex, see Fig. 4.

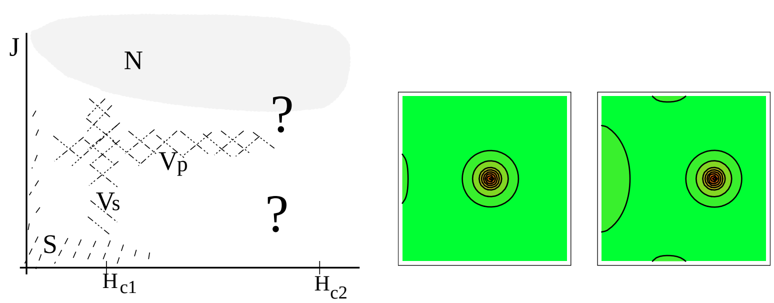

FIGURE 4. Left:an incomplete diagram for the HKHF model driven by current; right: steady state solutions with $J = 0.1, 0.5$.

In Fig. 4, a phase diagram describing the dynamic solutions using H and J as the parameters remains incomplete, though numerical evidence have indicated various possibilities that we now outline. Consider for simplicity the case H is near H_{c1}, there are several possible regimes in which the time-dependent solutions may behave differently as observed in recent numerical experiments. For small J, in accordance with the result of [**34**], due to the hysteresis, the solution starting at the superconducting state (no vortex) would approach a steady state which is vortex-free. Meanwhile, the solution starting with a vortex would undergo a gentle shift

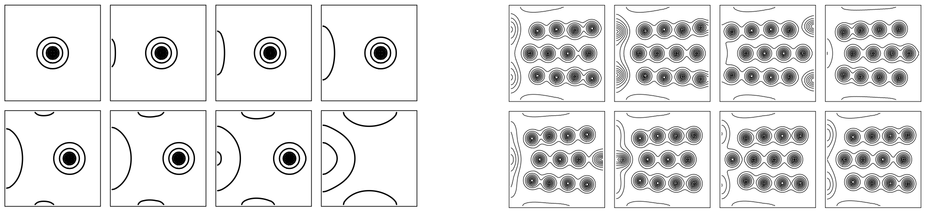

FIGURE 5. Motion of vortices in the presence of an applied current.

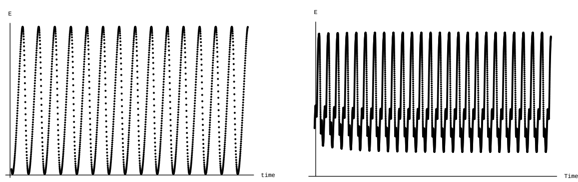

FIGURE 6. Plots of energy in time: evidence of time-periodic solution.

of the vortex position due to the Lorentz force, but will eventually come to a rest. Numerically we find that as the current gets stronger, the more the vortex shifts from their equilibrium positions. The solution beginning with the normal state will remain normal in time. For larger J, however, the superconducting solution would start to nucleate vortices because the Lorentz force is now strong enough to overcome the boundary barrier of the applied magnetic field. These vortices will move through the sample, and lead to possibly time-periodic motion of vortex arrays, see Fig. 5.

If J is exceedingly large, the time dependent solution would eventually collapse to the normal state regardless of its initial state. The plots of the free energy in time for two different values of current are given in Fig. 6.

We note that similar discussions for the full TDGL also remain to be carried out. As the size of the sample domain also affects the dynamics of the equations and their long time behavior, a much more complex and multi-dimensional diagram than that in Fig. 4 should be constructed. Numerical simulations can again be very helpful here. In [**1**], we have provided some computational results of the steady states for two dimensional samples without taking into account the applied current.

4. Langevin dynamics driven by noises

The conventional Ginzburg-Landau theory is applicable only to highly idealized physical contexts that do not take into account factors like inhomogeneities, thermal fluctuations, and random applied fields. For example, it is well known that thermal fluctuations and material defects play a central role in the pinning of vortices in type-II superconductors [**48, 64**]. Thermal fluctuations may be modeled by Langevin-type dynamics [**41, 25, 62, 63**], in the context of the HKHF model, we have [**19, 20**]

$$\frac{\partial \psi}{\partial t} + (i\nabla + \mathbf{A})^2\psi - \psi + |\psi|^2\psi = \eta \quad \text{in } \Omega \tag{4.1}$$

$$(i\nabla + \mathbf{A})\psi \cdot \mathbf{n} = 0 \ \text{ on } \Gamma\,, \quad \text{and} \quad \psi(\mathbf{x}, 0) = \varphi(\mathbf{x}) \ \text{ in } \Omega\,, \tag{4.2}$$

where η is a random, continuous, complex-valued field in time and space. When modeling the effects of thermal fluctuations, the variance of η is on the order of $K(1-T/T_c)^{-2}$, where T and T_c are the temperature and the critical transition temperature, respectively, and K is a constant. One may consider also the stochastic dynamics based on a multiplicative noise

$$\frac{\partial \psi}{\partial t} + (i\nabla + \mathbf{A})^2\psi - \psi + |\psi|^2\psi = \eta\psi \quad \text{in } \Omega \tag{4.3}$$

with similar boundary and initial conditions. In figure 7, snapshots of solutions of the multiplicative model with $\sigma = 4$ are provided, see [**19, 20**] for details.

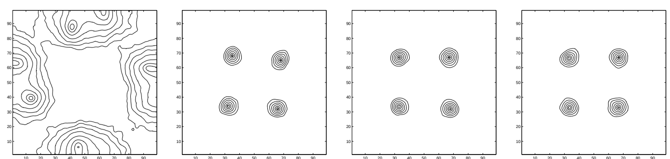

FIGURE 7. *Plots of $|\psi|$ at different time with multiplicative noise.*

5. Macroscopic vortex dynamics

On a macroscopic level, when the number of vortices becomes exceedingly large, the vortex state may be described using a vortex density function [**14, 37**]. A rigorous justification has given in [**52**]. Studies along this direction also include [**7, 8, 17, 14, 37, 38, 39, 40**] on model derivation, mathematical analysis, and numerical simulations

In the absence of the magnetic field, by allowing the number of vortices going to infinity, one set of limiting vortex motion laws becomes

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ u = M\nabla\triangle^{-1}\rho, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{5.1}$$

with ρ_0 being a bounded Radon measure and M being a constant orthogonal matrix. The global existence of measure-valued solution and the classical weak solution to the above equations were studied in [**36**]. The matrix M contains information on the various time relaxation constants such as the η_i's given in the TDGL equations. In particular, $M^2 = I$ gives the two dimensional incompressible Euler equations.

There are also other limits which are of interests when dealing with large number of vortices. For more detailed discussions on the hierarchy of macroscopic models, see [**8**].

6. Nonlinear Schrodinger dynamics

Study of vortices in the BEC often are based on the Gross-Pitaevskii equations:

$$\eta\frac{\partial u}{\partial t} - (\nabla - i\mathbf{A})^2 u + \frac{1}{\epsilon^2}(|u|^2 - a_\epsilon(x))u = 0\ . \tag{6.1}$$

Here, $\eta = i$ is for the real time dynamics.

FIGURE 8. Vortex tubes in a BEC with a rotating magnetic trap.

Vortices in a rotating magnetic trap. In some recent experiments where a rotating magnetic trap is applied to the BEC cloud, vortex nucleation is observed when the angular velocity is above certain critical velocity. We refer to [**2**] for more detailed analysis and additional references. For problems confined by a trapping potential, no particles escape to infinity, we thus can use the boundary condition $u = 0$ on the boundary of a large enough box. Note that the mass conservation condition is automatically satisfied by the above equation.

To compute the ground state solution, one may proceed with the time dependent equation in the imaginary time, that is by taking $\eta = 1$:

$$\frac{\partial u}{\partial t} - (\nabla - i\mathbf{A})^2 u + \frac{1}{\epsilon^2}|u|^2 u - \frac{a_\epsilon(x)}{\epsilon^2} u = \mu_\epsilon(u) u \ , \tag{6.2}$$

where $\mu_\epsilon(u)$ denotes the Lagrange multiplier corresponding to the $\|u\| = 1$. Simple calculation shows that

$$\mu_\epsilon(u) = \int_{\mathcal{D}} \left\{ |(\nabla - i\mathbf{A})\, u|^2 + \frac{1}{\epsilon^2}|u|^4 - \frac{a_\epsilon(x)}{\epsilon^2}|u|^2 \right\} d\,\mathcal{D} \ .$$

Vortex shedding behind a stirring laser beam. With a blue tune laser beam stirring the BEC cloud, energy dissipation has also been observed in recent experiments [**55**]. A time-dependent Gross-Pitaeskii model in a moving frame and the associated Painlevé reduction were used [**3**]:

$$i\frac{\partial u}{\partial t} - \Delta u + iv\frac{\partial u}{\partial x_1} + \frac{1}{\epsilon^2}(|u|^2 - a(x))u = 0 \ . \tag{6.3}$$

The equation is imposed outside an obstacle Ω, and on its boundary, we set $u = 0$. Here, v represents the velocity of the laser which takes up the interior of Ω.

The above equations provided good models for the study of the onset of dissipation, and the vortex-sound interaction. Similar analysis and simulations have been carried out for the liquid Helium, see [**43, 47**]. A typical two-dimensional simulation is given in Fig. 9. Much investigations are currently underway [**2**].

7. Numerical algorithms

The numerical computations of the steady-state and time-dependent Ginzburg-Landau models along with several of their variants are based on a finite element code developed in [**21, 30, 31, 32**]. Discrete gauge invariant approximations have also been studied for TDGLs [**28**]. Other works include, but not limited to, the

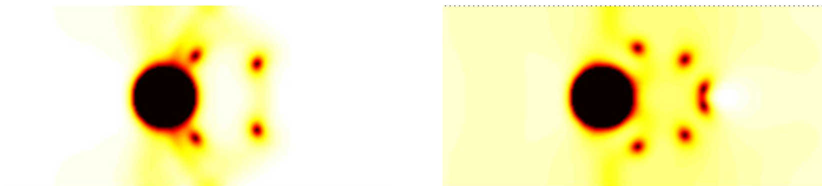

FIGURE 9. Vortex shedding in a superflow around an obstacle.

alternating marching scheme [**54**] and adaptive finite element methods [**16**]. For the time-dependent Gross-Pitaevskii equations, we have developed both finite element and finite difference spatial discretizations as well as explicit and implicit time integration schemes [**3**].

Recently, we have made progress on the analysis of a class of splitting schemes for computing the ground state solutions of the BEC condensate based on the normalized gradient flow [**5**].

8. Conclusion

Due to page limit, we rarely touched upon the subject of vortex tubes and vortex rings in three dimensional spaces. We have not covered other interesting topics such as the relations of vortices and dark solitons in BEC and the vortex-glass and vortex-liquid states in high-T_c superconductivity.

Historically, the story of vortex dynamics in superfluids was first told almost a century ago with the study of liquid helium. For the last 50 years, the center of attention n quantized vortices has been devoted to the issues related to superconductivity. The recent experiments on BECs provided exciting breakthroughs in new techniques to probe the properties of superfluidity, there is no doubt more stories are to be told on the subject of quantized vortex dynamics in years to come.

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