Copulas: An Introduction
Part II: Models

Johan Segers

Université catholique de Louvain (BE)
Institut de statistique, biostatistique et sciences actuarielles

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Part II: Models

Archimedean copulas

Extreme-value copulas

Elliptical copulas

Vines
Archimedean copulas

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The (in)famous Archimedean copulas

- By far the most popular (theory & practice) class of copulas
- Plenty of parametric models
- Building block for more complicated constructions:
  - Nested/Hierarchical Archimedean copulas
  - Vine copulas
  - Archimax copulas
  - . . .
- Mindless application of (Archimedean) copulas has drawn many criticisms on the copula ‘hype’
Laplace transform of a positive random variable

Recall the **Laplace transform** of a random variable $Z > 0$:

$$
\psi(s) = E[\exp(-sZ)] = \int_0^\infty e^{-sz} \, dF_Z(z), \quad s \in [0, \infty]
$$

A distribution on $(0, \infty)$ is identified by its Laplace transform.

**Ex.** Show the following properties:

- $0 \leq \psi(s) \leq 1$
- $\psi(0) = 1$ and $\psi(\infty) = 0$.
- $(-1)^k d^k \psi(s) / ds^k \geq 0$ for all integer $k \geq 1$.
- In particular, $\psi$ is nonincreasing ($k = 1$) and convex ($k = 2$).
Survival functions in proportional hazards model: The Laplace transform of the frailty appears

Independent unit exponential random variables $Y_1, \ldots, Y_d$.

Survival times $X_1, \ldots, X_d$ are affected by a common ‘frailty’ $Z > 0$:

$$X_j = Y_j/Z$$

Marginal and joint survival functions:

$$\Pr[X_j > x_j] = \mathbb{E}[e^{-x_jZ}]$$

$$= \psi(x_j)$$

$$\Pr[X_1 > x_1, \ldots, X_d > x_d] = \mathbb{E}[e^{-(x_1+\cdots+x_d)Z}]$$

$$= \psi(x_1 + \cdots + x_d)$$
In proportional hazards models, survival copulas are Archimedean

The survival copula of $X$ is Archimedean with generator $\psi$:

$$\bar{C}(u_1, \ldots, u_d) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$$

**Ex.** Show the above formula.

**Ex.** Show that replacing $Z$ by $\beta Z$ for a constant $\beta > 0$ changes $\psi$ but does not change the copula.

**Ex.** Pick your favourite (discrete/continuous) distribution on $\mathbb{(0, \infty)}$, compute or look up its Laplace transform, and compute the associated Archimedean copula. If it doesn’t exist yet, name it after yourself and publish a paper about it.
A Gamma frailty induces the Clayton copula

If $Z \sim \text{Gamma}(1/\theta, 1)$, with $0 < \theta < \infty$, then

$$\psi(s) = \int_0^\infty e^{-sz} \frac{z^{1/\theta - 1} e^{-z}}{\Gamma(1/\theta)} \, dz = (1 + s)^{-1/\theta}$$

and the resulting survival copula is Clayton:

$$\bar{C}(u) = (u_1^{-\theta} + \cdots + u_d^{-\theta} - d + 1)^{-1/\theta}$$

Ex. Check the above formulas.

Ex. How to use the frailty representation to sample from a Clayton copula?
Generator of the Clayton copula

Generator

\[ w = \psi(s) \]

Inverse generator

\[ s = \psi^{-1}(w) \]

\[ w = \psi(s) = (1 + s)^{-1/\theta} \]

\[ s = \psi^{-1}(w) = w^{-\theta} - 1 \]
Formal definition of an Archimedean copula

A copula $C$ is **Archimedean** if there exists $\psi : [0, \infty] \to [0, 1]$ such that

$$C(u) = \psi\left(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\right)$$

For $C$ to be a copula, it is sufficient and necessary that $\psi$ satisfies

- $\psi(0) = 1$ and $\psi(\infty) = 0$
- $\psi$ is $d$-monotone, i.e.
  - $(−1)^k d^k \psi(s)/ds^k \geq 0$ for $k \in \{0, \ldots, d − 2\}$
  - $(-1)^{d-2} d^{d-2} \psi(s)/ds^{d-2}$ is decreasing and convex

Equivalently, there should exist a random variable $Z > 0$ such that

$$\psi(s) = \mathbb{E}\left[\left(1 - \frac{sZ}{d-1}\right)^{d-1}\right]$$

i.e. $\psi$ is the **Williamson $d$-transform** of the rv $(d − 1)/Z$. 
Standard examples

**Ex.** The independence copula $\Pi(u) = u_1 \cdots u_d$ is Archimedean.
- What is its generator $\psi$?
- What is the frailty variable $Z$?

**Ex.** The Fréchet–Hoeffding lower bound $W(u, v) = \max(u + v - 1, 0)$ is Archimedean too. What is its generator $\psi$?
[This $\psi$ is *not* a Laplace transform; it is 2-monotone but not $d$-monotone for $d \geq 3$.]

**Ex.** One can show that the Fréchet–Hoeffding upper bound $M(u) = \min(u_1, \ldots, u_d)$ is *not* Archimedean. Still, show that the Clayton copula with $\theta \to \infty$ converges to $M$. 
Common generator functions

\[ w = \psi(s) \]

\[ \Pi(u) = u_1 \cdots u_d \]

\[ \psi(s) = e^{-s} \]

\[ W(u, v) = \max(u + v - 1, 0) \]

\[ \psi(s) = \max(1 - s, 0) \]
A bivariate Archimedean copula induces a binary operator

\[ [0, 1] \times [0, 1] \rightarrow [0, 1] : (u, v) \mapsto C(u, v) \]

which is commutative and associative:

\[
C(u, v) = C(v, u),
\]

\[
C(u, C(v, w)) = C(C(u, v), w)
\]

endowing \([0, 1]\) with a semi-group structure.

Link with the theory of associative functions (ABEL, HILBERT).
Derived quantities

Conditional cdf:

\[
\hat{C}_j(u) = \frac{\psi'(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))}{\psi'(\psi^{-1}(u_j))}
\]

Pdf, provided \(\psi\) is \(d\) times continuously differentiable

\[
c(u) = \frac{\psi^{(d)}(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))}{\prod_{j=1}^{d} \psi'(\psi^{-1}(u_j))}
\]

Ex. Show these formulas.
Yet another probability integral transform: Kendall distribution functions

Bivariate cdf $H$, continuous margins $F$ and $G$, copula $C$.

The **Kendall distribution** of a random pair $(X, Y) \sim H$ is the cdf of the rv

$$W = H(X, Y) = C(F(X), G(Y)) = C(U, V)$$

It only depends on $H$ through $C$:

$$K_C(w) = \Pr(W \leq w) = \int_{[0,1]^2} 1\{C(u, v) \leq w\} \, dC(u, v), \quad w \in [0, 1]$$

It is linked to Kendall’s tau via

$$E[W] = \int_{0}^{1} w \, dK_C(w) = \int_{[0,1]^2} C(u, v) \, dC(u, v) = \frac{1 + \tau}{4}$$
Kendall distribution functions:
The $C$-probability below a $C$-level curve

$$K(w) = \int_{[0,1]^2} 1\{C(u,v) \leq w\} \, dC(u,v)$$

contour plot of $C(u, v) = uv$
Bivariate Archimedean copulas are identified by their Kendall distribution function

The Kendall distribution function of a bivariate Archimedean copula with inverse generator $\phi = \psi^{-1} : (0, 1] \to [0, \infty)$ is

$$K(w) = w - \lambda(w),$$

$$\lambda(w) = \frac{\phi(w)}{\phi'(w)} = \frac{1}{d \log \phi(w)/dw} \leq 0$$

Up to a multiplicative constant, $\phi$ and thus $\psi$ can be reconstructed from $\lambda$.

**Ex.** Show the following properties:

- $K_\Pi(w) = w - w \log(w)$ (independence)
- $K_W(w) = 1$ (Fréchet–Hoeffding lower bound)
- $K_M(w) = w$ (Fréchet–Hoeffding upper bound)
- $w \leq K(w) \leq 1$
Kendall distribution functions:
Stochastically smaller than the uniform one
The tail behaviour of a bivariate Archimedean copula can be read off from the inverse generator function

Coefficient of lower tail dependence:

\[
\lambda_L(C) = \lim_{w \downarrow 0} \frac{C(w, w)}{w} = 2^{-1/\theta_0},
\]

where \(\theta_0 = -\lim_{w \downarrow 0} \frac{w \phi'(w)}{\phi(w)} \in [0, \infty]\)

Coefficient of upper tail dependence:

\[
\lambda_U(C) = \lambda_L(\bar{C}) = 2 - 2^{1/\theta_1},
\]

where \(\theta_1 = -\lim_{w \downarrow 0} \frac{w \phi'(1-w)}{\phi(1-w)} \in [1, \infty]\)

⇒ Construction of models with different upper and lower tails
Archimedean copulas enjoy many symmetries

Let \( (U_1, \ldots, U_d) \sim C \) and \( C \) is Archimedean with generator \( \psi \).

- **Permutation symmetry**: For any permutation \( \sigma \) of \( \{1, \ldots, d\} \),

  \[
  (U_{\sigma(1)}, \ldots, U_{\sigma(d)}) \overset{d}{\sim} (U_1, \ldots, U_d)
  \]

- **Closure of margins**: For any subset \( 1 \leq j_1 < \cdots < j_k \leq d \),

  \[
  (U_{j_1}, \ldots, U_{j_k}) \sim k\text{-variate Archimedean, same generator } \psi
  \]

Symmetry is a blessing (simplicity) and a curse (lack of flexibility).

The only radially symmetric Archimedean copula \( (C = \bar{C}) \) is the Frank copula.
Escaping from permutation symmetry: Nested Archimedean copulas

Trivariate copula:

\[ C(u_1, u_2, u_3) = C_{\psi_0}(u_1, C_{\psi_{23}}(u_2, u_3)) \]
\[ = \psi_0(\psi_0^{-1}(u_1) + \psi_23(\psi_{23}^{-1}(u_2) + \psi_{23}^{-1}(u_3))) \]

Bivariate margins:

- \((U_1, U_2)\) Archimedean with generator \(\psi_0\)
- \((U_1, U_3)\) Archimedean with generator \(\psi_0\)
- \((U_2, U_3)\) Archimedean with generator \(\psi_{23}\)
Nested Archimedean copulas:
Hierarchical dependence structure

Dependence at deeper levels must be stronger than at higher levels:
*Sufficient nesting condition* on generator functions
Archimedian copulas: Some literature


Archimedean copulas

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How to define the maximum of a multivariate sample?

Consider iid $X_1, \ldots, X_n$ from $F$ with continuous margins $F_1, \ldots, F_j$ and copula $C$.

Vector of component-wise maxima:

$$M_n = (M_{n,1}, \ldots, M_{n,d})$$
$$M_{n,j} = \max(X_{1,j}, \ldots, X_{n,j}), \quad j \in \{1, \ldots, d\}$$

In general, $M_n \notin \{X_1, \ldots, X_n\}$.

**Ex.** Draw a scatter plot of a bivariate sample and locate the point representing the pair of maxima.
The copula of the vector of sample maxima

The joint and marginal cdfs of $M_n$:

$$\Pr(M_n \leq x) = F^n(x),$$
$$\Pr(M_{n,j} \leq x_j) = F^n_j(x_j)$$

The copula of $M_n$:

$$C_n(u) = C(u_1^{1/n}, \ldots, u_d^{1/n})^n$$

**Ex.** Prove the above equations.

**Ex.** If $d = 2$ and $X_{i,2} = -X_{i,1}$, we find the Clayton copula with $\theta = -1/n$. 
A copula is an extreme-value copula if it can arise in the limit

$$C_\infty(u) = \lim_{n \to \infty} C(u_1^{1/n}, \ldots, u_d^{1/n})^n$$

Extreme-value copulas are max-stable:

$$C_\infty(u_1^{1/k}, \ldots, u_d^{1/k})^k = C_\infty(u)$$

Conversely, max-stable copulas are extreme-value copulas.
The only max-stable Archimedean copula is the Gumbel copula

**Ex.** Show that the *Gumbel* copula is max-stable:

\[
C_\theta(u) = \exp\left\{-\left\{(\log u_1)^\theta + \cdots + (\log u_d)^\theta\right\}^{1/\theta}\right\}, \quad \theta \in [1, \infty)
\]

**Special cases:**

- \( \theta = 1 \) Independence
- \( \theta = \infty \) Fréchet–Hoeffding upper bound
Maxima versus minima: Just switch to survival copulas

Everything can be repeated for minima, but the formulas get unwieldy

- Apply inclusion/exclusion formulas.

Conceptually, just switch to survival copulas:

\[ C_\infty \text{ is max/min-stable } \iff \bar{C}_\infty \text{ is min/max-stable} \]

A solution in practice:
If interest is in minima, change signs and work with maxima.
The domain of attraction of an extreme-value copula

The (max-)domain of attraction of an extreme-value copula $C_\infty$ is the collection of all copulas $C$ such that

$$\lim_{n \to \infty} C(u_1^{1/n}, \ldots, u_d^{1/n})^n = C_\infty(u)$$

(DA)

Clearly, $C_\infty \in DA(C_\infty)$.

Alternative condition for (DA) in terms of behaviour of $C$ near $(1, \ldots, 1)$:

$$\lim_{s \downarrow 0} s^{-1} \{1 - C(1 - sx_1, \ldots, 1 - sx_d)\}$$

$$= \log C_\infty(e^{-x_1}, \ldots, e^{-x_d}) =: \ell(x), \quad x \in [0, \infty)^d$$

The limit is called the stable tail dependence function.

[Proof: In (DA), take logarithms and set $s = 1/n$ and $u_j = e^{-x_j}$.]
Archimedean copulas: Attracted by the Gumbel copula

If $C$ is Archimedean with inverse generator $\phi = \psi^{-1}$ and if

$$\exists \lim_{w \downarrow 0} - \frac{w \phi'(1 - w)}{\phi(1 - w)} = \theta_1 \in [1, \infty]$$

then $C \in \text{DA}(\text{Gumbel copula } C_{\theta_1})$.

**Ex.** Show that the *Joe* copula with inverse generator

$$\phi_\theta(w) = -\log(1 - (1 - w)^\theta), \quad \theta \in [1, \infty),$$

is attracted by the Gumbel copula with parameter $\theta$. 
Archimedean survival copulas: Attracted by the Galambos copula

If $C$ is Archimedean with inverse generator $\phi = \psi^{-1}$ and if

$$\exists \lim_{w \downarrow 0} - \frac{w \phi'(w)}{\phi(w)} = \theta_0 \in [0, \infty]$$

then $\bar{C} \in DA(Galambos \text{ copula } C_{\theta_0})$, with stdf

$$\ell_{\theta}(x) = x_1 + \cdots + x_d - \sum_{I \subset \{1, \ldots, d\}, |I| \geq 2} (-1)^{|I|} \left( \sum_{j \in I} x_j^{-\theta} \right)^{-1/\theta}$$

Ex. Show that the survival Clayton copula with inverse generator

$$\phi_{\theta}(w) = \frac{w^{-\theta} - 1}{\theta}, \quad \theta \in [0, \infty),$$

is attracted by the Galambos copula with the same parameter.
Pickands dependence functions:
A kind of generator function on the unit simplex

If $C_\infty$ is max-stable, the function $A$ on

$$\Delta_{d-1} = \{ t \in [0, 1]^d : t_1 + \cdots + t_d = 1 \}$$

defined by

$$A(t) = \frac{\log C_\infty(w^{t_1}, \ldots, w^{t_d})}{\log w}$$

does not depend on $w \in (0, 1)$. We find the Pickands representation

$$C_\infty(w^{t_1}, \ldots, w^{t_d}) = w^{A(t)}$$
For bivariate extreme-value copulas, Pickands functions are simple objects.

In the bivariate case, identifying \((1 - t, t) \equiv t\) and writing

\[
(u, v) = (w^{1-t}, w^t) \text{ with } w = uv \text{ and } t = \frac{\log(v)}{\log(uv)}
\]

we obtain the representation

\[
C_\infty(u, v) = (uv)^{A(t)}
\]

Necessary and sufficient condition on \(A\) for \(C_\infty\) to be a copula:

- \(\max(t, 1 - t) \leq A(t) \leq 1\)
- \(A\) is convex

**Ex.** Show that if \(C_\infty\) as defined above is a copula, it is max-stable.
Between independence and complete dependence:

\[ uv \leq C(u, v) \leq \min(u, v) \]
\[ 1 \geq A(t) \leq \max(t, 1 - t) \]

The upper and lower bounds are extreme-value copulas too.
Extreme-value copulas:
An abundance of parametric models

Ex. Look up the forms of the following extreme-value copulas and visualize their Pickands dependence functions:

▶ Gumbel aka logistic, and asymmetric extensions
▶ Galambos aka negative logistic, and asymmetric extensions
▶ Marshall–Olkin
▶ Hüsler–Reiss
▶ t-EV
▶ Schlather
▶ ...
Extreme-value copulas: Flexible models for positively associated variables

- Kendall’s tau:
  \[ \tau = \int_0^1 \frac{t(1-t)}{A(t)} \, dA'(t) > 0 \text{ unless independence} \]

- Coefficient of upper tail dependence:
  \[ \lambda_U = 2(1 - A(1/2)) > 0 \text{ unless independence} \]

- Not necessarily symmetric
- Higher dimensions: hierarchical structures possible
- Margins of extreme-value copulas are also extreme-value copulas
Extreme-value copulas: Some literature I


Archimedean copulas

Extreme-value copulas

Elliptical copulas

Vines
Elliptical random vectors:
Affine transformations of spherically symmetric ones

A random vector $X$ has an **elliptical distribution** if it can be written

$$X = \mu + \varrho A V$$

- $\mu \in \mathbb{R}^d$
- $\varrho \geq 0$ random
- $A \in \mathbb{R}^{d \times d}$
- $V$ is uniformly distributed on $\{v \in \mathbb{R}^d : v_1^2 + \cdots + v_d^2 = 1\}$
- $\varrho$ and $V$ are independent
Elliptical distributions:
Elliptically contoured densities

If

- \( \varrho \) has a density \( f_\varrho \)
- \( \Sigma = AA^\top \) is invertible

then \( X \) has a density \( f_X \) too, and \( f_X(x) \) depends on

- \( f_\varrho \) (radial density)
- \( \sqrt{(x - u)^\top \Sigma^{-1} (x - u)} \) (Mahalanobis distance)

Contour sets of \( f_X \) are elliptical.
Most common elliptical distributions: Gaussian and Student

\[
\begin{align*}
\frac{0}{\chi^2_d} & \quad \frac{X}{\text{Gaussian}} \\
\frac{\chi^2_d}{F_{d, \nu}} & \quad \text{Student}
\end{align*}
\]

Link between both: If

- \( Z \sim N_d(0, \Sigma) \)
- \( V \sim \chi^2_\nu \)
- \( Z \) and \( V \) are independent

Then \( X = Z / \sqrt{V / \nu} \) is Student\((0, \Sigma, \nu)\).

If \( \nu \to \infty \), then ‘Student’ tends to ‘Gaussian’.
Meta-elliptical copulas:
Copulas of elliptical distributions

A copula is **meta-elliptical** if it is the copula of an elliptical distribution.

A meta-elliptical copula is itself not an elliptical distribution. Hence ‘meta’; suppressed in practice.

Without loss of generality, we can assume that

- $\mu = 0$
- $\Sigma$ is a correlation matrix, notation $R$

**Ex.** Why?
The Gaussian and Student copulas

Gaussian copula: copula of $Z \sim N_d(0, R)$,

$$C^\text{Gauss}_R(u) = \Pr[\Phi(Z_1) \leq u_1, \ldots, \Phi(Z_d) \leq u_d]$$

Student copula: copula of $T \sim \text{Student}_d(0, R, \nu)$,

$$C^\text{Student}_{R,\nu}(u) = \Pr[t_\nu(T_1) \leq u_1, \ldots, t_\nu(T_d) \leq u_d]$$

with $t_\nu$ the univariate standard Student$(\nu)$ cdf.
Elliptical copula densities:
Contour lines are not elliptical

![Contour lines are not elliptical](image)

\[
c(u, v) = \text{bivariate Student } t \text{ copula, } \nu = 2, \rho = 0.3
\]
Zero correlation implies independence for Gaussian copulas only.
Elliptical copulas are convenient to work with

- Densities are explicitly available.
- Pairwise distributions determine the full distribution.
- Lower-dimensional margins are elliptical copulas again.
- If \( \mathbf{U} \sim \mathcal{C} \) is elliptical, then, whatever the radial distribution,

\[
\tau(U_j, U_k) = \frac{\arcsin(r_{jk})}{\pi/2} \quad \text{(Kendall’s tau)}
\]

- Tail dependence follows from power-law tail of \( \rho \), e.g.
  - Gaussian copula: asymptotic independence
  - Student copula: \( \lambda_L = \lambda_U = 2 \, t_{\nu+1}(-\sqrt{(\nu + 1)(1 - \rho)}/(1 + \rho)) \)
Putting structure on the correlation matrix allows for interpretable models

**Factor models:** for $\Gamma^{k \times d}$ with $k < d$,

$$\Sigma = \Gamma' \Gamma + \sigma^2 I_d$$

**Graphical models:** Gaussian with *sparse* inverse matrix $R^{-1}$

$$(R^{-1})_{jk} = \text{partial correlation of } Z_j \text{ and } Z_k \text{ given the other variables}$$

$\Rightarrow$ Conditional independence graphs.
Elliptical copulas: Some literature


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The simplifying assumption:

The copula of a conditional distribution

For random variables \((X, Y)\) and a random vector \(Z\), assume:

\[
\text{The copula of } (X, Y) \mid Z = z \text{ does not depend on } z.
\]

Equivalently, assume:

\[
(F_{X \mid Z}(X \mid Z), F_{Y \mid Z}(Y \mid Z)) \text{ is independent of } Z.
\]

- True if \((X, Y, Z)\) are jointly Gaussian
  - \((X, Y) \mid Z = z\) is bivariate Gaussian
  - Conditional correlation is partial correlation \(\rho_{XY \mid Z}\), whatever \(z\)
- Simplifying assumption not verified in general
Vine copulas or pair-copula constructions:

Combine $d(d - 1)/2$ arbitrary bivariate copulas into a $d$-variate copula.

- The bivariate copulas are *not* the bivariate margins.
- They rather arise through repeated conditioning.
- Construction made possible by the simplifying assumption.
From bivariate to conditional densities

Random pair \((X, Y)\):

\[
f_{X,Y}(x, y) = c(F_X(x), F_Y(y)) \, f_X(x) \, f_Y(y) \quad \text{bivariate}
\]
\[
f_{X|Y}(x, y) = c(F_X(x), F_Y(y)) \, f_X(x) \quad \text{conditional}
\]

Similarly, but now conditionally on a random vector \(Z\):

\[
f_{X,Y|Z}(x, y \mid z) = c_{X,Y|Z}(F_{X|Z}(x \mid z), F_{Y|Z}(y \mid z)) \, f_{X|Z}(x \mid z) \, f_{Y|Z}(y \mid z)
\]
\[
f_{X|Y,Z}(x \mid y, z) = c_{X|Y,Z}(F_{X|Z}(x \mid z), F_{Y|Z}(y \mid z)) \, f_{X|Z}(x \mid z)
\]

Vines: use this formula iteratively to factorize a multivariate pdf

**Ex.** Where exactly was the simplifying assumption used?
Vines in dimension three

Taking $X_3$ as ‘pivot’ variable:

$$f(x_1, x_2, x_3) = f_3(x_3)$$

$$f_{2|3}(x_2|x_3)$$

$$f_{1|23}(x_1|x_2, x_3)$$

$$= f_3(x_3)$$

$$c_{23}(F_2(x_2), F_3(x_2)) f_2(x_2)$$

$$c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3))$$

$$= c_{13}(F_1(x_1), F_3(x_3)) f_1(x_1)$$

$$= f_1(x_1) f_2(x_2) f_3(x_3)$$

$$c_{13}(F_1(x_1), F_3(x_3)) c_{23}(F_2(x_2), F_3(x_2)) c_{12|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3))$$

$$= ?$$
The conditional cdf’s follow from the pair copulas too

Conditional cdf:

\[
F_{1|3}(x_1|x_3) = \int_{-\infty}^{x_1} f_{1|3}(x_1'|x_3) \, dx_1'
\]

\[
= \int_{-\infty}^{x_1} c_{13}(F_1(x_1'), F_3(x_3)) \, f_1(x_1') \, dx_1'
\]

\[
= \int_{0}^{F_1(x_1)} c_{13}(u_1, F_3(x_3)) \, du_1
\]

\[
= \frac{\partial}{\partial u_3} C_{13}(F_1(x_1), u_3) \bigg|_{u_3=F_3(x_3)}
\]

Depends on \( C_{13}, F_1 \) and \( F_3 \)
Vines in dimension four

Single out one variable:

\[ f(x_1, x_2, x_3, x_4) = f_{234}(x_2, x_3, x_4) \cdot f_{1|234}(x_1 \mid x_2, x_3, x_4) \]

Decompose the conditional density:

\[ f_{1|234}(x_1 \mid x_2, x_3, x_4) = c_{12|34}(F_{1|34}(x_1 \mid x_3, x_4), F_{2|34}(x_2 \mid x_3, x_4)) \]

\[ f_{1|34}(x_1 \mid x_2, x_3) \]

The conditional density \( f_{1|34}(x_1 \mid x_3, x_4) \) was treated above.

By the same argument as on the previous slide, the conditional cdf is

\[ F_{1|34}(x_1 \mid x_3, x_4) = \left. \frac{\partial}{\partial u_3} C_{13|4}(F_{1|4}(x_1 \mid x_4), u_3) \right|_{u_3=F_{3|4}(x_3|x_4)} \]
In dimension four, six pair copulas are needed

Collecting everything, we find a decomposition in terms of six pair copulas:

**Canonical (C) vine**

\[
\begin{align*}
c_{14}, c_{24}, c_{34} & \quad \text{‘ground level’} \\
c_{13|4}, c_{23|4} & \quad \text{‘level 1’} \\
c_{12|34} & \quad \text{‘level 2’}
\end{align*}
\]

With other choices of the conditioning variables, we would have obtained:

**Drawable (D) vine**

\[
\begin{align*}
c_{12}, c_{23}, c_{34} & \quad \text{‘ground level’} \\
c_{13|2}, c_{24|3} & \quad \text{‘level 1’} \\
c_{14|23} & \quad \text{‘level 2’}
\end{align*}
\]

In higher dimensions, even more decompositions are possible: **Regular vines**

And the indices can be permuted too...
A C-vine in dimension five:
At each level, condition on the same variable
A D-vine in dimension five:
Chaining the variables
A non-classified regular vine in dimension five
Vine copulas: Strengths

- Densities are explicit
- Conditioning mechanism also yields simulation algorithms
- Models are easily constructed: any pair copula works
- Highly flexible
  - asymmetries
  - positive/negative dependence
  - tail dependence
Vine copulas: Weaknesses

- Cdf’s not explicitly available
- Taking margins destroys the model
- Meaning of chain of simplifying assumptions is not transparent
- Interpretation becomes difficult
Vine copulas: Some literature

Active and fast-moving field. Check out

http://www-m4.ma.tum.de/forschung/vine-copula-models/


