Dependence and heavy-tailedness in economics, finance and econometrics: Modern approaches to modeling and implications for economic and financial decisions

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Objectives and key results

- **(Sub-)Optimality** of diversification under heavy tails & dependence
- **(Non-)Robustness** of models in economics & finance to heavy tails, heterogeneity & dependence
- General representations for joint cdf’s and copulas of arbitrary r.v.’s
  - Joint cdf’s and copulas of dependent r.v.’s = sums of $U$—statistics in independent r.v.’s
  - Similar results: expectations of arbitrary statistics in dependent r.v.’s
  - New representations for multivariate dependence measures
  - Complete characterizations of classes of dependent r.v.’s
  - Methods for constructing new copulas
  - Modeling different dependence structures
Objectives and key results

- **Copula-based** modeling for time series
- **Characterizations of dependence** in terms of copulas
  - Markovness of arbitrary order
  - Combining Markovness with other dependencies:
    - $m$–dependence, $r$–independence, martingaleness, conditional symmetry
  - **Non-Markovian** processes satisfying Kolmogorov-Chapman SE
Objectives and key results

• New flexible copulas to combine dependencies

• Expansions by linear functions (Eyraud-Fairlie-Gumbel-Morgensten copulas)

• power functions (power copulas); Fourier polynomials (Fourier copulas)

• Impossibility/reduction: Copula-based dependence + specific copulas
  $\iff$ Independence
Objectives & key results

• Long-memory via copulas: various definitions

• Dependence measures & copulas

• Gaussian & EFGM ⇒ short-memory Markov

• Fast exponential decay of dependence between $X_t$ & $X_{t+h}$

• Simulations ⇒ Clayton copula-based Markov $\{X_t\}$: can behave as long memory (copulas) in finite samples
  • High persistence important for finance & economics

• Long memory-like: $X_t$ & $X_{t+h}$: slow decay of dependence for commonly used lages $h$

• Volatility modeling & Nonlinear dependence in finance

• Non-linear CH & long memory-like volatility

• Generalizations of GARCH

• Non-robustness of procedures for detecting long memory in copulas
Stylized Facts of Real-World Returns

Daily % changes in the Dow Jones Industrial Average, Jan. 1980 - Sept. 2007
Dependence vs. margins in economic and financial problems

- Problems in finance, economics & risk management:
  Solution is affected by both
  - Marginal distributions (Heavy-Tailedness, Skewness)
  - Dependence (Positive or Negative, Asymmetry)

- Portfolio choice & value at risk (VaR)
  - Marginal effects under independence: Heavy-Tailedness
    Moderately HT vs. extremely HT $\implies$ Opposite solutions
  - Different solutions: Positive vs. negative dependence
Normal vs. Heavy-tailed Power Laws

Simulated normal and heavy-tailed series

- i.i.d. Student t, 3 d.f.
- i.i.d. Normal N(0, 1)

Extreme events
Heavy-tailed margins

- Many economic & financial time series: power law tails:
  \[ P(|X| > x) \approx \frac{c}{x^\alpha}, \quad \alpha > 0 : \text{tail index} \]

- Moments of order \( p \geq \alpha \): infinite; \( E|X|^p < \infty \) iff \( p < \alpha \)
  - \( \alpha \leq 4 \implies \text{Infinite fourth moments: } EX^4 = \infty \)
  - \( \alpha \leq 2 \implies \text{Infinite variances: } EX^2 = \infty \)
  - \( \alpha \leq 1 \implies \text{Infinite first moments: } E|X| = \infty \)

- Returns on many stocks & stock indices: \( \alpha \in (2, 4) \)
  \[ \implies \text{finite variance, infinite fourth moment} \]
A tale of two tails

Figure: Tails of Cauchy distributions are heavier than those of normal distributions. Tails of Lévy distributions are heavier than those of Cauchy or normal distributions.
A tale of two tails

Simulated data from Normal, Cauchy and Levy distributions, n=25

**Figure:** Heavy-tailed distributions: more extreme observations
Heavy-tailed margins

\[ P(|X| > x) \approx \frac{C}{x^{\alpha}} \]

- **Income**: \( \alpha \in [1.5, 3] \Rightarrow \text{infinite } EX^4 \), possibly **infinite variances**
- **Wealth**: \( \alpha \approx 1.5 \Rightarrow \text{infinite variances!} \)
- **Returns** from **technological** innovations, **Operational risks**: \( \alpha < 1 \Rightarrow \text{infinite means } E|X| = \infty \! \)
- **Firm sizes**, sizes of largest **mutual funds**, **city sizes**: \( \alpha \approx 1 \)
- **Economic losses** from **earthquakes**: \( \alpha \in [0.6, 1.5] \Rightarrow \text{infinite variances, possibly infinite means} \)
- **Economic losses** from **hurricanes**: \( \alpha \approx 1.56; \alpha \approx 2.49 \)
Stable distributions

- $X \sim S_{\alpha}(\sigma)$: symmetric stable distribution, $\alpha \in (0, 2)$
  
  **CF:** $E(e^{ixX}) = \exp\{-\sigma^\alpha |x|^\alpha\}$

- **Normal** $N(0, \sigma)$: $\alpha = 2$

- **Cauchy**: $\alpha = 1$, $f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$

- **Lévy**: $\alpha = 1/2$, support $[0, \infty)$, $f(x) = \frac{\sigma}{\sqrt{2\pi}} x^{-3/2} \exp(-\frac{1}{2x})$

- **Power laws**: $P(|X| > x) \approx \frac{C}{x^{\alpha}}$, $\alpha \in (0, 2)$

- **Moments** $E|X|^p$: finite iff $p < \alpha$

- **Infinite variances** for $\alpha < 2$

- **Portfolio formation**: $\sum_{i=1}^n w_i X_i =_d (\sum_{i=1}^n w_i^\alpha)^{1/\alpha} X_1$
  
  - $\alpha = 2$ (normal): $\frac{1}{\sqrt{n}}(X_1 + \ldots + X_n) =_d X_1$
Value at risk ( VaR )

- **VaR**
  - Risk $X$; positive values = losses
  - Loss probability $q$
  - $\text{VaR}_q(X) = z : P(X > z) = q$

- **Risks** $X_1, \ldots, X_n$

- $Z_w = \sum_{i=1}^{n} w_i X_i$: return on portfolio with weights $w = (w_1, \ldots, w_n)$

- **Problem** of interest:
  
  \[
  \text{Minimize} \text{VaR}_q(Z_w)
  \]

  \[
  \text{s.t. } w_i \geq 0, \sum_{i=1}^{n} w_i = 1
  \]

- When **diversification** $\Rightarrow$ decrease in portfolio riskiness (VaR)?
Diversification & risk

• Most diversified: \( \mathbf{w} = (1/n, 1/n, \ldots, 1/n) \) \( \Rightarrow \) \( Z_{\mathbf{w}} = \frac{1}{n} \sum_{i=1}^{n} X_i \)

• Least diversified: \( \mathbf{\overline{w}} = (1, 0, \ldots, 0) \) \( \Rightarrow \) \( Z_{\mathbf{\overline{w}}} = X_1 \)

• \( X_1, \ldots, X_n \sim \mathcal{N}(0, \sigma) \) (\( \alpha = 2 \))

  • \( Z_{\mathbf{w}} = \frac{1}{n} \sum_{i=1}^{n} X_i =_d \frac{1}{\sqrt{n}} X_1 = \frac{1}{\sqrt{n}} Z_{\mathbf{\overline{w}}} \)

  • \( \text{VaR}_q(Z_{\mathbf{w}}) = \frac{1}{\sqrt{n}} \text{VaR}_q(Z_{\mathbf{\overline{w}}}) \leq \text{VaR}_q(Z_{\mathbf{\overline{w}}}) \)

  • \( \text{VaR}_q(Z_{\mathbf{w}}) \searrow \text{as } n \nearrow \) (Diversification \( \nearrow \))
Diversification & risk

• $X_1, \ldots, X_n \sim S_{1/2}(\sigma)$, $\alpha = 1/2$, Lévy distribution
  • $Z_w = \frac{1}{n} \sum_{i=1}^{n} X_i = d \left[ \sum_{i=1}^{n} \left(\frac{1}{n}\right)^{1/2}\right]^2 X_1 = nX_1 = nZ_w$
  • $\text{VaR}_q(Z_w) = n\text{VaR}_q(Z_w) > \text{VaR}_q(Z_w)$
  • $\text{VaR}_q(Z_w) : \nearrow$ as $n \nearrow$ (Diversification $\nearrow$)

• Heavy tails (margins) matter:
  diversification $\implies$ opposite effects on portfolio riskiness

• Skewness: typically priced
Heavy-tailedness & diversification

- **Moderate** heavy tails $\alpha > 1$: finite first moments

$$\text{VaR}_q(Z_w) < \text{VaR}_q(Z) \quad \forall q > 0$$

**Optimal to diversify for all** loss probabilities $q$

- **Extremely** heavy tails $\alpha < 1$: infinite first moments

$$\text{VaR}_q(Z_w) < \text{VaR}_q(Z) \quad \forall q > 0$$

**Diversification: suboptimal for all** loss probabilities $q$

- **Similar** conclusions: Many other models in economics & finance

- **Firm growth theory, optimal bundling, monotone consistency of sample mean, efficiency of linear estimators**

- **Robust to moderate** heavy tails

- **Properties:** reversed under extremely heavy tails
What happens for intermediate heavy-tails?

- $X_1, ..., X_n$ i.i.d. stable with $\alpha = 1$: Cauchy distribution
  
  - Density $f(x) = \frac{\sigma}{\pi \sigma^2 + x^2}$
  
  - Heavy power law tails: $P(|X| > x) \approx \frac{C}{x}$
  
  - Infinite first moment

- $Z_w = \sum_{i=1}^n w_iX_i =_d X_1 \forall w = (w_1, ..., w_n) : w_i \geq 0$,

- Diversification: no effect at all!
Summary so far: Diversification for heavy-tailed and bounded distributions

VaR of a portfolio of $Z_i$ with equal weights $(1/n, 1/n, ..., 1/n)$

A. Light-tailed i.i.d. $Z_i$ with $\alpha > 1$.
   Example: Traditional situation with normal $Z_i$

B. Extremely heavy-tailed i.i.d. $Z_i$ with $\alpha < 1$.
   Example: Levy distribution with $\alpha = 1/2$

C. Specific boundary case: i.i.d. Cauchy $Z_i$ with $\alpha = 1$

D. Bounded $Z_i$

Number of risks in portfolio, $n$

Figure: $N = 10$ risks/insurer; $M = 7$ insurers

- D: Individual/non-diversification corners vs insurer and reinsurer equilibrium
Diversification & dependence

- Minimize $\text{VaR}_q(w_1 X_1 + w_2 X_2)$ s.t. $w_1, w_2 \geq 0$, $w_1 + w_2 = 1$

- Independence:
  - Optimal portfolio: $(\tilde{w}_1, \tilde{w}_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ (diversified) if $\alpha > 1$ (not extremely heavy-tailed, finite means)
  - $(\tilde{w}_1, \tilde{w}_2) = (1, 0)$ (not diversified, one risk) if $\alpha < 1$ (extremely heavy-tailed, infinite means)
Diversification & dependence

- **Extreme positive dependence**: $X_1 = X_2$ (a.s.) comonotonic risks
  
  - $\text{VaR}_q(w_1 X_1 + w_2 X_2) = \text{VaR}_q(X_1) \forall w$
  
  - **Diversification**: no effect at all (similar to Cauchy) regardless of heavy-tailedness

- **Extreme negative dependence** $X_1 = -X_2$ (a.s.) countermonotonic risks
  
  - $\text{VaR}_q(w_1 X_1 + w_2 X_2) = (w_1 - w_2)\text{VaR}_q(X_1)$
  
  - Optimal portfolio: $w = (1/2, 1/2)$ (most diversified regardless of heavy-tailedness)

- Optimal **portfolio choice**: affected by both dependence & properties of margins
Copulas and dependence

- **Main idea:** separate effects of dependence from effects of margins

- What matters more in portfolio choice: heavy-tailedness & skewness or (positive or negative) dependence?

- **Copulas:** functions that join together marginal cdf’s to form multidimensional cdf
Copulas and dependence

- Sklar’s theorem

- Risks $X, Y$:
  
  - Joint cdf $H_{XY}(x, y) = P(X \leq x, Y \leq y)$: affected by dependence and by marginal cdf’s $F_X(x) = P(X \leq x)$ and $G_Y(x) = P(Y \leq y)$

  - $C_{XY}(u, v)$: copula of $X, Y$:
    
    \[ H_{XY}(x, y) = C_{XY} \left( F_X(x), G_Y(y) \right) \]

    dependence marginals

  - $C_{XY}$: captures all dependence between risks $X$ and $Y$
Copulas and dependence

Advantages:

• **Exists for any risks** (correlation: finiteness of second moments)

• Characterizes all dependence properties

• **Flexibility in dependence modeling**

  • **Asymmetric** dependence: Crashes vs. booms

  • **Positive** vs. negative dependence

  • **Independence**: Nested as a particular case: Product copula, particular values of parameter(s)

  • **Extreme** dependence: $X = Y$ or $X = -Y$ $\iff$ extreme copulas; dependence in $C_{XY}$ varies in between
Copula structures

- Eyraud-Farlie-Gumbel-Morgenstern (EFGM):

\[ C(u, v) = uv[1 + \gamma(1 - u)(1 - v)] \]

\( \gamma \in [-1, 1] : \text{dependence parameter} \)

Tail independent: no contagion

- Heavy-tailed Pareto marginals:

\[ P(X > x) = \frac{1}{x^{\alpha}}, \quad x \geq 1 \]

\[ P(Y > y) = \frac{1}{y^{\alpha}}, \quad y \geq 1 \]

- Power laws, tail index \( \alpha \)
Diversification: EFGM & heavy tails

- **Moderate** heavy tails $\alpha > 1$: finite first moments
  \[
  \text{VaR}_q\left(\frac{X + Y}{2}\right) < \text{VaR}_q(X) \quad \text{for sufficiently small } q
  \]
  Optimal to diversify for sufficiently small loss probabilities $q$

- **Extremely** heavy tails $\alpha < 1$: infinite first moments
  \[
  \text{VaR}_q\left(\frac{X + Y}{2}\right) > \text{VaR}_q(X) \quad \text{for sufficiently small } q
  \]
  Diversification: suboptimal for sufficiently small loss prob. $q$

- **Similar** conclusions: **Multivariate EFGM** copulas

- **Complement** Embrechts et al. (2009): **Archimedean** copulas

- Tail independent EFGM & tail dependent Archimedean (Clayton, Gumbel): same boundary $\alpha = 1$ as in the case of independence
When dependence helps: Student-$t$ copulas

- Conclusions similar to independence: Models with common shocks

\[ X_1 = ZY_1, X_2 = ZY_2, \ldots, X_n = ZY_n \]

- **Common** shock $Z > 0$ affecting all risks $X_1, \ldots, X_n$

- $Y_1, \ldots, Y_n$: i.i.d. normal or heavy-tailed with tail index $\alpha$

  \[
  Z: \text{heavy-tailed with tail index } \beta
  \]

  Then $X_i$: heavy-tailed with tail index $\gamma = \min(\alpha, \beta)$

- Important particular case: (Dependent) Multivariate Student-$t$ $X_1, X_2, \ldots, X_n$ with $\alpha$ d.f. (tail index) $\Rightarrow$ Optimal to diversify for all loss probabilities $q$ regardless of tail index $\alpha$

  - Tail dependent Student-$t$ copula and heavy-tailed margins with arbitrary tail index $\alpha$: diversification pays off

- Contrast: Independent Student-$t$ $X_1, X_2, \ldots, X_n$ with $\alpha$ d.f. (tail index): diversification optimal for $\alpha > 1$; suboptimal for $\alpha < 1$
Diversification: Heavy-tailedness & dependence matter

• **Independence**, **Tail dependent** models with **common shocks** (e.g., Student-\(t\) distr. = Student-\(t\) copula with Student-\(t\) marginals):
  - Diversification always **pays off** for all loss probabilities \(q\)

• **Tail independent** EFGM, possibly **tail dependent** Archimedean copulas (e.g., Clayton & Gumbel):
  - Dividing boundary \(\alpha = 1\) for sufficiently small loss probability \(q\)

• **Numerical** results on interplay of **heavy-tailedness & dependence** (copula) assumptions and **loss probability** \(q\) in **diversification** decisions:
  - Deviations from threshold \(\alpha = 1\) for different **copulas** and **loss probabilities** \(q\)

• **Theoretical** results for **general** copulas = ?

• **(Non-)robustness** of other models in economics & finance
Characterizations of copulas & dependence

- $V_1, ..., V_n$: i.i.d. $\mathcal{U}([0, 1])$

- $C$: $n-$copula iff $\exists \tilde{g}_{i_1}, ..., i_c$ s.t.

  A1 (integrability):
  \[
  \int_0^1 \cdots \int_0^1 |\tilde{g}_{i_1}, ..., i_c(t_{i_1}, ..., t_{i_c})| dt_{i_1} \cdots dt_{i_c} < \infty
  \]

  A2 (degeneracy):
  \[
  E_{V_{i_k}} \left[ \tilde{g}_{i_1}, ..., i_c(V_{i_1}, ..., V_{i_{k-1}}, V_{i_k}, V_{i_{k+1}}, ..., V_{i_c}) \right] = 0
  \]

  A3 (positive definiteness):
  \[
  \tilde{U}_n(V_1, ..., V_n) \equiv \sum_{c=2}^n \sum_{1 \leq i_1 < \cdots < i_c \leq n} \tilde{g}_{i_1}, ..., i_c(V_{i_1}, ..., V_{i_c}) \geq -1
  \]
- Representation for $C$:

$$C(u_1, \ldots, u_n) = \int_0^{u_1} \ldots \int_0^{u_n} (1 + \tilde{U}_n(t_1, \ldots, t_n)) \prod_{i=1}^n dt_i$$

- $\tilde{U}_n$: sum of degenerate $U$–statistics
Device for **constructing** $n$–copulas and cdf’s

- **Bivariate Eyraud-Farlie-Gumbel-Morgenstern copulas & cdf’s:**

  \[
  C_\theta(u, v) = uv (1 + \theta(1 - u)(1 - v))
  \]

  \[
  H_\theta(x, y) = F(x)G(y)\left(1 + \theta(1 - F(x))(1 - G(y))\right)
  \]

  $n = 2$; $\tilde{g}_{1,2}(t_1, t_2) = \theta (1 - 2t_1)(1 - 2t_2), \theta \in [-1, 1]$

- **Multivariate EFGM copulas & cdf’s:**

  \[
  C_\theta(u_1, u_2, ..., u_n) = \prod_{i=1}^{n} u_i \left(1 + \theta \prod_{i=1}^{n} (1 - u_i)\right)
  \]

  \[
  \tilde{g}_{i_1,...,i_c}(t_{i_1}, ..., t_{i_c}) = \theta_{i_1,...,i_c} (1 - 2t_{i_1})(1 - 2t_{i_2})...(1 - 2t_{i_c})
  \]
• **Generalized multivariate EFGM copulas** (Johnson and Kotz, 1975, Cambanis, 1977)

\[
C(u_1, ..., u_n) = \prod_{k=1}^{n} u_k \left(1 + \sum_{c=2}^{n} \sum_{1 \leq i_1 < ... < i_c \leq n} \theta_{i_1, ..., i_c} (1 - u_{i_k}) \right)
\]

\[
\tilde{g}_{i_1, ..., i_c} (t_{i_1}, ..., t_{i_c}) = 0, \ c < n - 1
\]

\[
\tilde{g}_{1, 2, ..., n} (t_1, t_2, ..., t_n) = \theta (1 - 2t_1)(1 - 2t_2)...(1 - 2t_n)
\]

• **Generalized EFGM copulas**: complete **characterization** of joint **cdf**’s of **two-valued r.v.’s** (Sharakhmetov & Ibragimov, 2002)
From dependence to independence through $U$–statistics

$G_n$: sums of $U$–statistics

$$U_n(\xi_1, \ldots, \xi_n) = \sum_{c=2}^{n} \sum_{1 \leq i_1 < \ldots < i_c \leq n} g_{i_1, \ldots, i_c}(\xi_{i_1}, \ldots, \xi_{i_c})$$

$g_{i_1, \ldots, i_c}$: satisfy A1-A3

• Arbitrarily dependent r.v.'s:
  sum of $U$–statistics in independent r.v.'s
  with canonical kernels

• Reduction of problems for dependence to well-studied objects

• Transfer of results for $U$–statistics under independence
From dependence to independence through $U$–statistics

- $X_1, \ldots, X_n$: 1-cdf’s $F_k(x_k)$

- $\xi_1, \ldots, \xi_n$: independent copies (1-cdf’s $F_k(x_k)$)

$\exists U_n \in G_n \text{ s.t. } \forall f : \mathbb{R}^n \to \mathbb{R}$

$Ef(X_1, \ldots, X_n) = Ef(\xi_1, \ldots, \xi_n) \left(1 + U_n(\xi_1, \ldots, \xi_n)\right)$

- Representation for c.f.’s:

$E\exp\left(i \sum_{k=1}^n t_k X_k\right) = E\exp\left(i \sum_{k=1}^n t_k \xi_k\right) +$

$E\exp\left(i \sum_{k=1}^n t_k \xi_k\right) U_n(\xi_1, \ldots, \xi_n)$

† CLT for bivariate r.v.’s
Characterizations of dependence

• **Canonical $g'$s**: complete **characterizations** of dependence properties

• $X_1, ..., X_n$: $r-$**independent** if $\forall$ $r$ jointly independent $\iff$

  \[ g_{i_1, ..., i_c}(V_{i_1}, ..., V_{i_c}) = 0 \text{ (a.s.) } 1 \leq i_1 < ... < i_c \leq n, \ c = 2, ..., r \]

•

\[
\begin{align*}
g_{i_1, ..., i_{r+1}}(u_{i_1}, ..., u_{i_{r+1}}) &= \frac{\alpha_1...\alpha_n}{\alpha_{i_1}...\alpha_{i_{r+1}}} \left( (k + 1)u_{i_1}^k - (k + 2)u_{i_1}^{k+1} \right) \times \cdots \times \left( (k + 1)u_{i_c}^k - (k + 2)u_{i_c}^{k+1} \right) \\
C(u_1, ..., u_n) &= \prod_{i=1}^{n} u_i \left( 1 + \sum_{1 \leq i_1 < ... < i_{r+1} \leq n} \frac{\alpha_1...\alpha_n}{\alpha_{i_1}...\alpha_{i_{r+1}}} \times \left( u_{i_1}^k - u_{i_1}^{k+1} \right) \times \cdots \times \left( u_{i_{r+1}}^k - u_{i_{r+1}}^{k+1} \right) \right)
\end{align*}
\]

Extensions of Wang (1990) ($k = 0$)
Copulas and Markov processes

- Darsow, Nguyen and Olsen, 1992: copulas and first-order Markovness

- \( A, B : [0, 1]^2 \to [0, 1] : \)

\[
(A \ast B)(x, y) = \int_0^1 \frac{\partial A(x, t)}{\partial t} \cdot \frac{\partial B(t, y)}{\partial t} dt
\]

- \( A : [0, 1]^m \to [0, 1], B : [0, 1]^n \to [0, 1] : \ast - product \)

\[
A \ast B(x_1, \ldots, x_{m+n-1}) = \\
\int_0^x \int_0^{x_m} \frac{\partial A(x_1, \ldots, x_{m-1}, \xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1}, \ldots, x_{n+m-1})}{\partial \xi} d\xi
\]
Copulas and Markov processes

- Transition probabilities

\[ P(s, x, t, A) = P(X_t \in A | X_s = x) \text{ satisfy CKE's} \]

iff \( C_{st} = C_{su} \ast C_{ut} \quad \forall s < u < t \)

- \( X_t \): first-order Markov iff

\[ C_{t_1,\ldots,t_n} = C_{t_1t_2} \ast C_{t_2t_3} \ast \cdots \ast C_{t_{n-1}t_n} \]
New results: Higher-order Markovness and copulas

- \( \{X_t\}_{t \in T}: k\text{-order Markov} \iff \)

\[
P(X_t < x_t | X_{t_1}, \ldots, X_{t_{n-k}}, X_{t_{n-k+1}}, \ldots, X_{t_n}) = 
\]

\[
P(X_t < x_t | X_{t_{n-k+1}}, \ldots, X_{t_n}) 
\]

- Complete characterization in terms of \((k + 1)\)-copulas

- \( C_{t_1, \ldots, t_k} \): copulas of \(X_{t_1}, \ldots, X_{t_k}\)

- \( \{X_t\}_{t \in T}: k\text{-order Markov} \iff \forall t_1 < \ldots < t_n, \ n \geq k + 1 \)

\[
C_{t_1, \ldots, t_n} = C_{t_1, \ldots, t_{k+1}}^k \star C_{t_2, \ldots, t_{k+2}}^k \star \ldots \star C_{t_{n-k}, \ldots, t_n}^k 
\]
Stationary case

- $X_t$: stationary $k$-order Markov iff

\[
C_{1,...,n}(u_1, ..., u_n) = C \ast^k C \ast^k ... \ast^k C(u_1, ..., u_n)
\]

\[
= C^{n-k+1}(u_1, ..., u_n) \quad \forall n \geq k + 1
\]

$C$: $(k + 1)$-copula s.t.

\[
C_{i_1+h,...,i_l+h} = C_{i_1,...,i_l}, \quad 1 \leq j_1 < ... < j_l \leq k + 1
\]

- $C^s$: $s$-fold product $\ast^k$ of $C$
Advantages of copula-based approach

- Modeling higher order Markov processes alternative to transition matrices

- Instead of initial distribution & transition probabilities:
  
  Prescribe marginals & \((k + 1)\)–copulas

  Generate copulas of higher order & finite-dimensional cdf’s

- Advantage: separation of properties of marginals (fat-tailedness) & dependence properties (conditional symmetry, \(m\)–dependence, \(r\)–independence, mixing)
Advantages of copula-based approach

- Inversion method:

  New $k$-Markov with dependence similar to a given Markov process

  Different marginals

  † $X_t$: stationary $k$-Markov

  $(k + 1)$-cdf $\tilde{F}(x_1, ..., x_{k+1}), 1$-cdf $F$

  $\Rightarrow (k + 1)$-copula:

  $$C(u_1, ..., u_{k+1}) = \tilde{F}\left(F^{-1}(u_1), ..., F^{-1}(u_{k+1})\right)$$
† Another 1–cdf $G$:

**Stationary $k$–Markov, same** dependence as $\{X_t\}$, **different** 1-marginal $G$:

$(k + 1)$–copula:

$$C(u_1, ..., u_{k+1}) = \tilde{F}\left(G^{-1}(u_1), ..., G^{-1}(u_{k+1})\right)$$

Representation $\Rightarrow$ **Higher-order copulas & cdf’s**

$\{X_t\}$: stationary $C$–based $k$–Markov chain
Advantages of copula-based approach

- **C**: all dependence properties of the time series

  - k-independence, m-dependence, martingaleness, symmetry

- On-going project with Johan Walden: characterizations of time-irreversibility; focus on $C_{t_1, \ldots, t_k} = C_{t_k, \ldots, t_1}$

- Applications: forward-looking vs. backward-looking market participants (“fundamentalists” vs. noise traders or “chartists”)

- “Compass rose” for $P_{t-1}$ and $P_t$: symmetry in copulas
Combining higher-order Markovness with other dependence properties

- A number of studies in dependence modeling: Higher-order Markovness + $m$-dependence & $r$-independence

Lévy (1949): 2nd order Markovness + pairwise independence

Rosenblatt & Slepian (1962): $N$-order $N$-independent stationary Markov

- Impossibility/reduction:

$N$-order Markov + $N$-independence + two-valued $\iff$ joint independence

‡ Testing sensitivity to WD in DGP Rosenblatt & Slepian (1962)
Combining Markovness with other dependencies

† Examples:

Not 1–order Markovian

But 1-st order transition probabilities

\[ P(s, x, t, A) = P(X_t \in A | X_s = x) \] satisfy C-K SE

\[ P(s, x, t, A) = \int_{-\infty}^{\infty} P(u, \xi, t, A) P(s, x, u, d\xi) \]

(Other examples: Feller, 1959, Rosenblatt, 1960)
Combining Markovness with other dependencies

† 1-dependent Markov: Aaronson, Gilat and Keane (1992)
Burton, Goulet and Meester (1993), Matúš (1996)

† Matúš (1998): $m$–dependent
discrete-space Markov

† Impossibility/Reduction:

‡ stationary $m$–dependent Markov if
$\text{card}(\Omega) < m + 2$
Markovness of higher-order and $k$–independence

- Characterization of stationary $k$–independent $k$–Markov processes
- $\{X_t\}$: $C$–based $k$–independent stationary $k$–Markov iff

$$\frac{\partial^{k+1} C(u_1, \ldots, u_{k+1})}{\partial u_1 \cdots \partial u_{k+1}} = 1 + g(u_1, \ldots, u_{k+1})$$

$g : [0, 1]^{k+1} \rightarrow [0, 1]$: canonical $g$–function

(Integrability + more degeneracy + positive definiteness)
Markovness of higher-order and $k$–independence

\[ \int_0^1 \ldots \int_0^1 |g(u_1, \ldots, u_{k+1})| du_1 \ldots du_{k+1} < \infty \]

\[ \int_0^1 \ldots \int_0^1 g(u_1, \ldots, u_{k+1})g(u_2, \ldots, u_{k+2}) \ldots g(u_s, \ldots, u_{k+s}) \, du_1 \ldots du_s = 0 \]

\[ \forall s \leq u_{i_1} < \ldots < u_{i_s} \leq k + 1, \, s = 1, 2, \ldots, \left[ \frac{k+1}{2} \right] \]

\[ g(u_1, \ldots, u_{k+1}) \geq -1 \]

- Integration: w.r. to all $s$ among $u_s, u_{s+1}, \ldots, u_{k+1}$ common to all $g$–functions $g(u_1, \ldots, u_{k+1}), \, g(u_2, \ldots, u_{k+2}), \ldots, \, g(u_s, \ldots, u_{k+s})$

  $k$–marginals: product copulas, independence

  $k$–independence: satisfied
Markovness of higher-order and $m$–independence

- \( \{X_t\} \): C–based $m$–dependent 1-Markov iff

\[
\frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} = 1 + g(u_1, u_2)
\]

$g : [0, 1]^2 \to [0, 1]$: canonical $g$–function:

\[
\int_0^1 \int_0^1 |g(u_1, u_2)| \, du_1 \, du_2 < \infty
\]

\[
\int_0^1 g(u_1, u_2) \, du_i = 0, \quad g(u_1, u_2) \geq -1
\]

\[
\int_0^1 g(u_1, u_2)g(u_2, u_3)\ldots g(u_m, u_{m+1}) \, du_2 \, du_3 \ldots du_m = 0
\]

† Integration: w.r. to $u_2, u_3, \ldots, u_m$ more than once among $g(u_1, u_2), g(u_2, u_3), \ldots, g(u_m, u_{m+1})$

$X_1, X_{m+1}$: independent; Process: $m$–dependent
New examples via existing constructions

• Higher-order Markovness $+$ martingaleness

• Inversion method $+$ existing examples $\Rightarrow$

$k$–independent, $m$–dependent Markov processes

different marginals
Reduction & impossibility for $k$–order Markov processes

- $\{X_t\}$: $C$–based $k$–independent stationary $k$–Markov

\[ \frac{\partial^{k+1} C(u_1, \ldots, u_{k+1})}{\partial u_1 \ldots \partial u_{k+1}} = 1 + g(u_1, \ldots, u_{k+1}) \]

\[ g : \text{product form (EFGM-type)}: \]
\[ g(u_1, u_2, \ldots, u_{k+1}) = \alpha f(u_1)f(u_2)\ldots f(u_{k+1}) \]

$\Leftrightarrow \{X_t\}$: jointly independent
Examples: EFGM and power copulas

- \((k + 1)-\text{EFGM copulas}:

\[
C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \left(1 + \alpha (1 - u_1)(1 - u_2)...(1 - u_{k+1})\right)
\]

\[
g(u_1, u_2, ..., u_{k+1}) = \alpha (1 - 2u_1)(1 - 2u_2)...(1 - 2u_{k+1})
\]

- \((k + 1)-\text{power copulas}

\[
C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \left(1 + \alpha (u_1^l - u_1^{l+1})(u_2^l - u_2^{l+1})...(u_{k+1}^l - u_{k+1}^{l+1})\right)
\]

\(l \geq 0 \) (EFGM: \(l = 0\))
**Impossibility/reduction for \( m \)-dependence**

- \( \{X_t\} \): \( C \)-based \( m \)-dependent Markov

\[ \frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2} = 1 + \alpha f(u_1)f(u_2) \]

(separable product form)

\( \Leftrightarrow X_t \): **jointly independent**

- Representations \( \Rightarrow \)

\[ \int_0^1 \ldots \int_0^1 \alpha^m f(u_1)f^2(u_2)\ldots f^2(u_m)f(u_{m+1}) du_2 \ldots du_m = 0; \]

\[ \alpha^m f(u_1)f(u_{m+1}) \left[ \int_0^1 f^2(u_2) du_2 \right]^{m-1} = 0 \]

\( \Rightarrow f = 0 \Leftrightarrow \text{Independence} \)
Examples, new and old

† EFGM copulas, $k = 1$:

$$C(u_1, u_2) = u_1 u_2 \left( 1 + \alpha (1 - u_1)(1 - u_2) \right)$$

$$g(u_1, u_2) = \alpha (1 - 2u_1)(1 - 2u_2)$$

- Limitations of EFGM copulas,

  separable copulas:

  Complement & generalize existing results
Examples, new and old

‡ Cambanis (1991): common dependencies cannot be exhibited by multivariate EFGM

\[ C_{j_1,\ldots,j_n}(u_{j_1}, \ldots, u_{j_n}) = \prod_{s=1}^{n} u_{j_k} \left( 1 + \sum_{1 \leq l < m \leq n} \alpha_{lm} (1 - u_{j_l})(1 - u_{j_m}) \right) \]

‡ Rosenblatt & Slepian (1962): non-existence of bivariate $\mathcal{N}$–independent $\mathcal{N}$–Markov

Sharakhmetov & Ibragimov (2002):

EFGM copulas for two-valued r.v.'s

‡ Technical difficulties in modeling
Solution: New flexible copula classes

- Copula-based TS with flexible dependencies

† Copulas based on Fourier polynomials

- k-independent k-Markov: Conditions satisfied for

\[
g(u_1, ..., u_{k+1}) = \sum_{j=1}^{N} \left[ \alpha_j \sin(2\pi \sum_{i=1}^{k+1} \beta_i^j u_i) + \gamma_j \cos(2\pi \sum_{i=1}^{k+1} \beta_i^j u_i) \right]
\]

† \( \alpha_j, \gamma_j \in \mathbb{R}, \beta_i^j \in \mathbb{Z}, i = 1, ..., k+1, j = 1, ..., N \):

† \( \beta_1^j + \sum_{l=2}^{s} \epsilon_{l-1} \beta_l^j \neq 0 \)

\( \epsilon_1, ..., \epsilon_{s-1} \in \{-1, 1\}, s = 2, ..., k+1 \)

† \( 1 + \sum_{j=1}^{N} [\alpha_j \epsilon_j + \gamma_j \epsilon_{j+N}] \geq 0, \epsilon_1, ..., \epsilon_{2N} \in \{-1, 1\} \)
Fourier copulas

\[ C(u_1, \ldots, u_{k+1}) = \int_0^{u_1} \cdots \int_0^{u_{k+1}} \left(1 + g(u_1, \ldots, u_{k+1})\right) du_1 \cdots du_{k+1} \]

\((k + 1)-\text{Fourier copulas}\)
Fourier copulas

- $1-$dependent $1-$Markov:

**Conditions satisfied** for Fourier copulas

\[
C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} (1 + g(u_1, u_2)) du_1 du_2
\]

\[
g(u_1, u_2) = \sum_{j=1}^{N} \left[ \alpha_j \sin(2\pi (\beta_{1j}^i u_1 + \beta_{2j}^i u_2)) + \gamma_j \cos(2\pi (\beta_{1j}^i u_1 + \beta_{2j}^i u_2)) \right]
\]

\[\downarrow \alpha_j, \gamma_j \in \mathbb{R}, \beta_{1j}^i, \beta_{2j}^i \in \mathbb{Z}:
\]

\[
\beta_{1j}^i + \beta_{2j}^i \neq 0
\]

\[
\beta_{1j}^i - \beta_{2j}^i \neq 0
\]

\[
1 + \sum_{j=1}^{N} \left[ \alpha_j \epsilon_j + \gamma_j \epsilon_{j+N} \right] \geq 0
\]

\[\forall \epsilon_1, \ldots, \epsilon_{2N} \in \{-1, 1\}\]
Concluding remarks

• (Sub-)Optimality of diversification under heavy tails & dependence

• (Non-)robustness of models in economics & finance to heavy tails, heterogeneity & dependence

• General representations for joint cdf’s and copulas of arbitrary r.v.’s

  • Joint cdf’s and copulas of dependent r.v.’s = sums of $U$—statistics in independent r.v.’s

  • Similar results: expectations of arbitrary statistics in dependent r.v.’s

  • New representations for multivariate dependence measures

  • Complete characterizations of classes of dependent r.v.’s

  • Methods for constructing new copulas

  • Modeling different dependence structures
Concluding remarks

• Copula-based modeling for time series

• Characterizations of dependence in terms of copulas

  • Markovness of arbitrary order

  • Combining Markovness with other dependencies:

    $m$–dependence, $r$–independence, martingaleness, conditional symmetry

Non-Markovian processes satisfying Kolmogorov-Chapman SE
Concluding remarks

- **New flexible copulas** to combine **dependencies**
- Expansions by linear functions (Eyraud-Fairlie-Gumbel-Morgensten copulas)
- Power functions (power copulas); Fourier polynomials (Fourier copulas)
- **Impossibility/reduction:** Copula-based **dependence** + **specific copulas**
  $\iff$ **Independence**
Copula memory

- Long-memory via copulas: various definitions
- Dependence measures & copulas
- Gaussian & EFGM ⇒ short-memory Markov
  - Fast exponential decay of dependence between $X_t$ & $X_{t+h}$
  - Numerical results ⇒ Clayton copula-based Markov \{X_t\} : can behave as long memory (copulas) in finite samples
    - High persistence important for finance & economics
- Long memory-like: $X_t$ & $X_{t+h}$: slow decay of dependence for commonly used lages $h$
- Volatility modeling & Nonlinear dependence in finance
- Non-linear CH & long memory-like volatility
- Generalizations of GARCH
Copula memory


Beare (2008): $\alpha$, $\beta$ & $\phi$–mixing

- $\kappa(h) \leq \alpha(h) \leq \beta(h) \leq 0.5\phi(h)$

- Numerical results $\Rightarrow$ Clayton: exponential decay in $\beta(h) \Rightarrow$ short $\kappa$–memory in copulas

Theoretical results in Chen, Wu & Yi (2008):

- Clayton: weakly dependent & short memory in terms of mixing properties!

- Our numerical results + Chen, Wu & Yi (2008): Non-robustness of procedures for detecting long memory in copulas