# Dependence and heavy-tailedness in economics, finance and econometrics: Modern approaches to modeling and implications for economic and financial decisions


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## Objectives and key results

- **(Sub-)Optimality** of diversification under **heavy tails** & **dependence**
- **(Non-)robustness** of models in economics & finance to **heavy tails**, **heterogeneity** & **dependence**
- General **representations** for **joint cdf's** and **copulas** of **arbitrary r.v.'s**
  - Joint cdf's and copulas of **dependent** r.v.'s = sums of $U-$statistics in **independent** r.v.'s
  - Similar results: expectations of **arbitrary statistics** in **dependent** r.v.'s
  - New representations for multivariate **dependence measures**
  - Complete characterizations of **classes of dependent r.v.'s**
  - **Methods** for constructing **new copulas**
  - **Modeling** different **dependence structures**

## Objectives and key results

- **Copula-based** modeling for **time series**
- **Characterizations** of **dependence** in terms of **copulas**
  - **Markovness** of **arbitrary order**
  - Combining **Markovness** with **other dependencies:**

    $m-$**dependence,** $r-$**independence, martingaleness, conditional symmetry**

    **Non-Markovian** processes satisfying **Kolmogorov-Chapman SE**

## Objectives and key results

- **New flexible copulas** to combine **dependencies**
- Expansions by linear functions (Eyraud-Fairlie-Gumbel-Morgensten copulas)
- power functions (power copulas); Fourier polynomials (Fourier copulas)
- **Impossibility/reduction:** Copula-based **dependence** + **specific copulas** $\Leftrightarrow$ **Independence**

# Objectives & key results

- **Long-memory** via **copulas**: various **definitions**
- **Dependence measures** & **copulas**
- **Gaussian** & **EFGM** $\Rightarrow$ **short-memory** Markov
- **Fast** exponential **decay** of **dependence** between $X_t$ & $X_{t+h}$
- Simulations $\Rightarrow$ **Clayton** copula-based Markov $\{Xt\}$ : can behave as **long memory** (copulas) in finite samples
    - **High persistence** important for finance & economics
- **Long memory-like**: $X_t$ & $X_{t+h}$ : **slow decay** of **dependence** for

  commonly used lages $h$
- **Volatility** modeling & **Nonlinear dependence** in finance
- **Non-linear CH** & **long memory**-like **volatility**
- **Generalizations** of **GARCH**
- **Non-robustness** of procedures for **detecting long memory** in copulas

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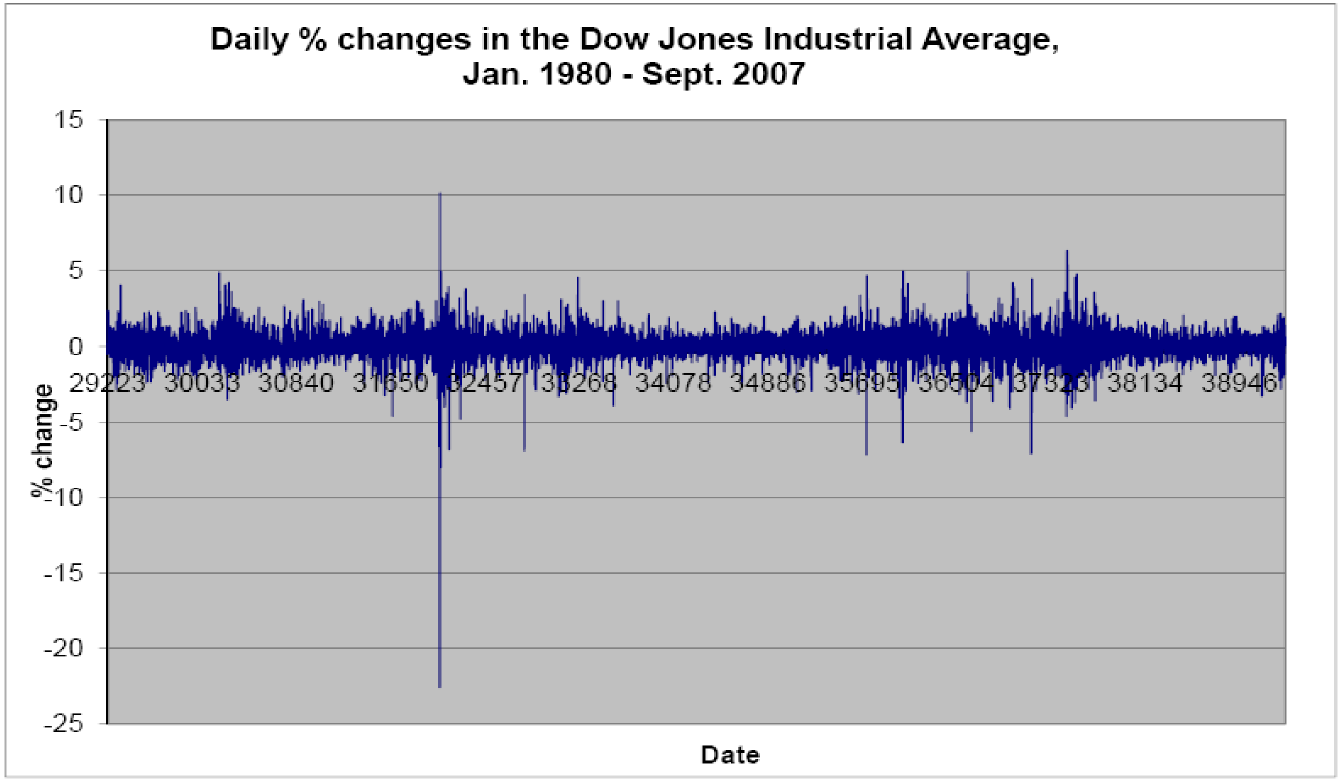
# Stylized Facts of Real-World Returns



## Dependence vs. margins in economic and financial problems

- **Problems** in **finance**, **economics** & **risk management**:

  **Solution** is **affected** by **both**

  - **Marginal** distributions (**Heavy-Tailedness**, **Skewness**)
  - **Dependence** (**Positive** or **Negative**, **Asymmetry**)

- **Portfolio choice** & **value at risk** (VaR)
  - **Marginal** effects under **independence**: **Heavy-Tailedness**

    **Moderately** HT vs. **extremely** HT $\Longrightarrow$ **Opposite solutions**
  - Different solutions: **Positive vs. negative** dependence

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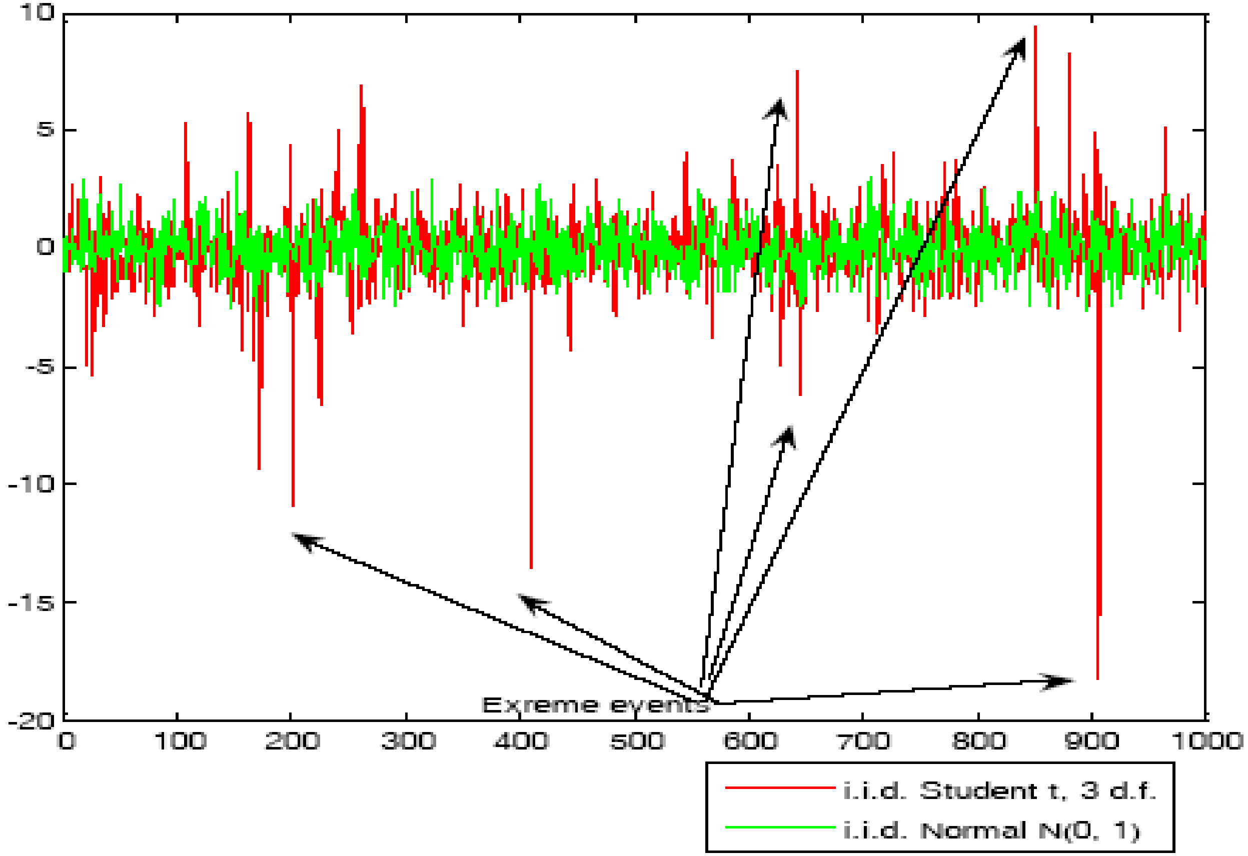
# Normal vs. Heavy-tailed Power Laws

Simulated normal and heavy-tailed series



## Heavy-tailed margins

- Many **economic** & **financial time series**: **power law** tails:

  $P(|X| > x) \approx \frac{C}{x^{\alpha}}$, $\alpha > 0$ : **tail index**

- **Moments** of order $p \geq \alpha$ : **infinite**; $E|X|^p < \infty$ iff $p < \alpha$

  - $\alpha \leq 4 \implies$ **Infinite fourth** moments: $EX^4 = \infty$
  - $\alpha \leq 2 \implies$ **Infinite variances**: $EX^2 = \infty$
  - $\alpha \leq 1 \implies$ **Infinite first moments**: $E|X| = \infty$

- **Returns** on **many stocks** & **stock indices**: $\alpha \in (2, 4)$

  $\Rightarrow$ **finite variance**, **infinite fourth** moment

# A tale of two tails

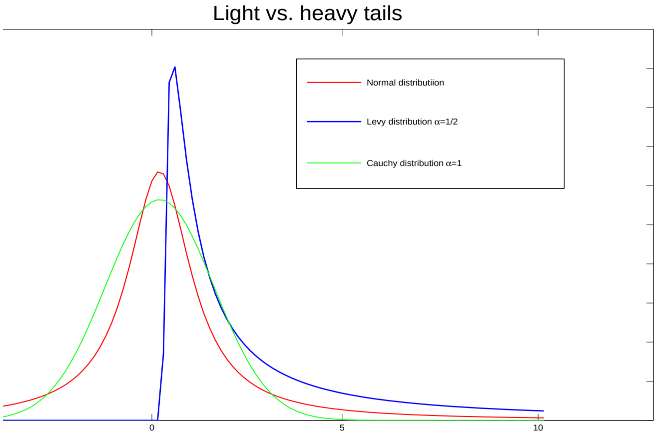



Figure: Tails of Cauchy distributions are heavier than those of normal distributions. Tails of Lévy distributions are heavier than those of Cauchy or normal distributions.

# A tale of two tails

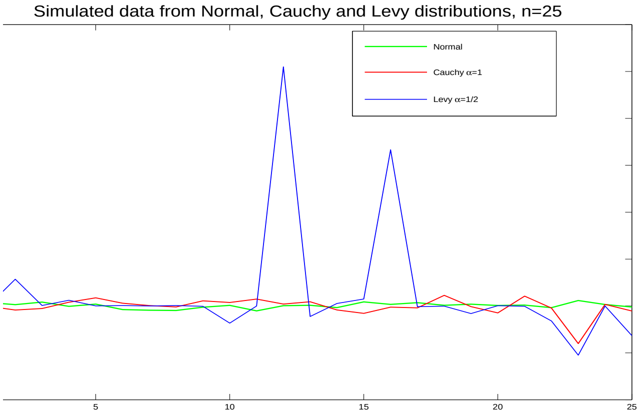



Figure: Heavy-tailed distributions: more extreme observations

## Heavy-tailed margins

$P(|X| > x) \approx \frac{C}{x^\alpha}$

- **Income:** $\alpha \in [1.5, 3] \Rightarrow$ **infinite** $EX^4$, possibly **infinite variances**
- **Wealth:** $\alpha \approx 1.5 \Rightarrow$ **infinite variances**!
- **Returns** from **technological** innovations, **Operational risks**: $\alpha < 1 \Rightarrow$ **infinite means** $E|X| = \infty$!
- **Firm sizes**, **sizes** of largest **mutual funds**, **city sizes**: $\alpha \approx 1$
- **Economic losses** from **earthquakes**: $\alpha \in [0.6, 1.5]$ $\Rightarrow$ **infinite variances**, possibly **infinite means**
- **Economic losses** from **hurricanes**: $\alpha \approx 1.56$; $\alpha \approx 2.49$

# Stable distributions

- $X \sim S_\alpha(\sigma)$ : symmetric **stable** distribution, $\alpha \in (0, 2]$

  **CF:** $E(e^{ixX}) = exp\{ -\sigma^\alpha |x|^\alpha \}$

  - **Normal** $\mathcal{N}(0, \sigma)$: $\alpha = 2$
  - **Cauchy**: $\alpha = 1$, $f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$
  - **Lévy**: $\alpha = 1/2$, support $[0, \infty)$, $f(x) = \frac{\sigma}{\sqrt{2\pi}} x^{-3/2} \exp(-\frac{1}{2x})$

- **Power** laws: $P(|X| > x) \approx \frac{C}{x^\alpha}$, $\alpha \in (0, 2)$

  - **Moments** $E|X|^p$: **finite** iff $p < \alpha$
  - **Infinite variances** for $\alpha < 2$

- **Portfolio** formation: $\sum_{i=1}^n w_i X_i =_d (\sum_{i=1}^n w_i^\alpha)^{1/\alpha} X_1$
  - $\alpha = 2$ (**normal**): $\frac{1}{\sqrt{n}}(X_1 + ... + X_n) =_d X_1$

# Value at risk (VaR)

- **VaR**
  - **Risk** $X$; **positive** values = **losses**
  - **Loss probability** $q$
  - $VaR_q(X) = z : P(X > z) = q$

- **Risks** $X_1, ..., X_n$

- $Z_w = \sum_{i=1}^{n} w_i X_i$: **return on portfolio** with weights $w = (w_1, ..., w_n)$

- **Problem** of interest:

$$\textbf{Minimize}\, VaR_q(Z_w)$$

s.t. $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$

- When **diversification** $\Rightarrow$ **decrease** in **portfolio riskiness** (VaR)?

# Diversification & risk

- **Most diversified:** $\underline{w} = (1/n, 1/n, ..., 1/n) \Rightarrow Z_{\underline{w}} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- **Least diversified:** $\overline{w} = (1, 0, ..., 0) \Rightarrow Z_{\overline{w}} = X_1$

- $X_1, ..., X_n \sim \mathcal{N}(0, \sigma)$ $(\alpha = 2)$
  - $Z_{\underline{w}} = \frac{1}{n}\sum_{i=1}^{n} X_i =_d \frac{1}{\sqrt{n}} X_1 = \frac{1}{\sqrt{n}} Z_{\overline{w}}$
  - $VaR_q(Z_{\underline{w}}) = \frac{1}{\sqrt{n}} VaR_q(Z_{\overline{w}}) < VaR_q(Z_{\overline{w}})$
  - $VaR_q(Z_{\underline{w}}) : \searrow$ as $n \nearrow$ (**Diversification** $\nearrow$)

# Diversification & risk

- $X_1, ..., X_n \sim S_{1/2}(\sigma)$, $\alpha = 1/2$, **Lévy distribution**
  - $Z_{\underline{w}} = \frac{1}{n} \sum_{i=1}^{n} X_i =_d \left[ \sum_{i=1}^{n} (\frac{1}{n})^{1/2} \right]^2 X_1 = nX_1 = nZ_{\overline{w}}$
  - $VaR_q(Z_{\underline{w}}) = nVaR_q(Z_{\overline{w}}) > VaR_q(Z_{\overline{w}})$
  - $VaR_q(Z_{\underline{w}}) : \nearrow$ as $n \nearrow$ (**Diversification** $\nearrow$)

- **Heavy tails** (**margins**) matter:

  **diversification** $\Longrightarrow$ **opposite** effects on **portfolio riskiness**

- **Skewness**: typically **priced**

## Heavy-tailedness & diversification

- **Moderate** heavy tails $\alpha > 1$ : **finite first** moments

$$VaR_q(Z_{\underline{w}}) < VaR_q(Z_{\overline{w}}) \ \forall q > 0$$

**Optimal** to **diversify for all** loss probabilities $q$

- **Extremely** heavy tails $\alpha < 1$ : **infinite first** moments

$$VaR_q(Z_{\underline{w}}) < VaR_q(Z_{\overline{w}}) \ \forall q > 0$$

**Diversification: suboptimal for all** loss probabilities $q$

- **Similar** conclusions: **Many** other **models in economics & finance**
  - **Firm growth** theory, **optimal bundling**, **monotone consistency** of **sample mean**, **efficiency** of **linear estimators**
  - **Robust** to **moderate** heavy tails
  - **Properties:** reversed under **extremely** heavy tails

## What happens for intermediate heavy-tails?

- $X_1, ..., X_n$ i.i.d. **stable with $\alpha = 1$: Cauchy distribution**
  - **Density** $f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$
  - **Heavy power law** tails: $P(|X| > x) \approx \frac{C}{x}$
  - **Infinite first moment**
- $Z_w = \sum_{i=1}^{n} w_i X_i =_d X_1 \; \forall w = (w_1, ..., w_n) : w_i \geq 0,$
- **Diversification**: **no effect** at all!

## Summary so far: Diversification for heavy-tailed and bounded distributions

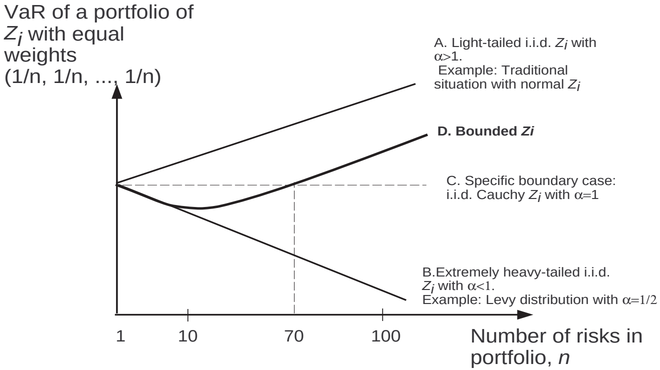



Figure: $N = 10$ risks/insurer; $M = 7$ insurers

- D: Individual/non-diversification corners vs insurer and reinsurer equilibrium

# Diversification & dependence

- **Minimize** $VaR_q(w_1X_1 + w_2X_2)$ s.t. $w_1, w_2 \geq 0,\ w_1 + w_2 = 1$

- **Independence**:

  - **Optimal** portfolio: $(\tilde{w}_1, \tilde{w}_2) = (\frac{1}{2}, \frac{1}{2})$ (**diversified**) if $\alpha > 1$ (**not extremely heavy-tailed**, **finite** means)

  - $(\tilde{w}_1, \tilde{w}_2) = (1, 0)$ (**not diversified**, **one risk**) if $\alpha < 1$ (**extremely heavy-tailed**, **infinite** means)

## Diversification & dependence

- Extreme **positive dependence**: $X_1 = X_2$ (a.s.) **comonotonic risks**
  - $VaR_q(w_1X_1 + w_2X_2) = VaR_q(X_1)\ \forall w$
  - **Diversification**: **no effect** at all (similar to Cauchy) **regardless** of **heavy-tailedness**

- Extreme **negative dependence** $X_1 = -X_2$ (a.s.) **countermonotonic risks**
  - $VaR_q(w_1X_1 + w_2X_2) = (w_1 - w_2)VaR_q(X_1)$
  - Optimal portfolio: $\underline{w} = (1/2, 1/2)$ (**most diversified regardless** of **heavy-tailedness**

- Optimal **portfolio choice**: affected by **both dependence** & properties of **margins**

## Copulas and dependence

- **Main idea**: **separate** effects of **dependence** from effects of **margins**
  - What **matters** more in **portfolio choice**: **heavy-tailedness** & **skewness** or (positive or negative) **dependence**?
- **Copulas**: **functions** that **join together marginal** cdf's to form **multidimensional** cdf

# Copulas and dependence

- **Sklar's theorem**
- **Risks** $X$, $Y$:
  - **Joint cdf** $H_{XY}(x,y) = P(X \leq x, Y \leq y)$: affected by **dependence** and by **marginal** cdf's $F_X(x) = P(X \leq x)$ and $G_Y(x) = P(Y \leq y)$
  - $C_{XY}(u,v)$ : **copula** of $X, Y$:

$$H_{XY}(x,y) = \underbrace{C_{XY}}_{\textbf{dependence}} \quad \Big( \underbrace{F_X(x), G_Y(y)}_{\textbf{marginals}} \Big)$$

- $C_{XY}$: captures **all dependence** between risks $X$ and $Y$

# Copulas and dependence

**Advantages**:

- **Exists** for **any risks** (**correlation**: finiteness of **second moments**)
- Characterizes **all dependence** properties
- **Flexibility** in **dependence modeling**
  - **Asymmetric** dependence: **Crashes** vs. **booms**
  - **Positive** vs. **negative** dependence
  - **Independence**: **Nested** as a particular case: **Product** copula, particular values of **parameter(s)**
  - **Extreme** dependence: $X = Y$ or $X = -Y \Leftrightarrow$ **extreme copulas**; **dependence** in $C_{XY}$ varies in **between**

# Copula structures

- Eyraud-Farlie-Gumbel-Morgenstern (**EFGM**):

$$C(u, v) = uv[1 + \gamma(1 - u)(1 - v)]$$

  $\gamma \in [-1, 1]$ : **dependence** parameter

  Tail **independent: no contagion**

- **Heavy-tailed** Pareto marginals:

$$P(X > x) = \frac{1}{x^{\alpha}}, \;\; x \geq 1$$

$$P(Y > y) = \frac{1}{y^{\alpha}}, \;\; y \geq 1$$

- **Power laws**, **tail index** $\alpha$

## Diversification: EFGM & heavy tails

- **Moderate** heavy tails $\alpha > 1$ : **finite first** moments

$$VaR_q\left(\frac{X+Y}{2}\right) < VaR_q(X) \ \ \textbf{for sufficiently small q}$$

**Optimal** to **diversify for sufficiently small** loss probabilities *q*

- **Extremely** heavy tails $\alpha < 1$ : **infinite first** moments

$$VaR_q\left(\frac{X+Y}{2}\right) > VaR_q(X) \ \ \textbf{for sufficiently small q}$$

**Diversification: suboptimal for sufficiently small** loss prob. *q*

- **Similar** conclusions: **Multivariate EFGM** copulas
- **Complement** Embrechts *et al.* (2009): **Archimedean** copulas
- Tail **independent** EFGM & tail **dependent** Archimedean (Clayton, Gumbel): **same** boundary $\alpha = 1$ as in the case of **independence**

## When dependence helps: Student-$t$ copulas

- Conclusions **similar** to **independence**: Models with **common shocks**

$$X_1 = ZY_1, X_2 = ZY_2, ..., X_n = ZY_n$$

  - **Common** shock $Z > 0$ affecting all risks $X_1, ..., X_n$
  - $Y_1, ..., Y_n$ : **i.i.d. normal** or **heavy-tailed** with **tail index** $\alpha$

    $Z$ : **heavy-tailed** with **tail index** $\beta$

    Then $X_i$ : **heavy-tailed** with **tail index** $\gamma = \min(\alpha, \beta)$

- **Important particular** case: (**Dependent**) **Multivariate Student**-$t$ $X_1, X_2, ..., X_n$ with $\alpha$ d.f. (**tail index**) $\Rightarrow$ **Optimal** to **diversify for all** loss probabilities $q$ **regardless** of **tail index** $\alpha$

  - **Tail dependent Student**-$t$ copula and **heavy-tailed** margins with arbitrary **tail index** $\alpha$ : diversification **pays off**

- **Contrast: Independent** Student-$t$ $X_1, X_2, ..., X_n$ with $\alpha$ d.f. (**tail index**):

  diversification **optimal** for $\alpha > 1$; **suboptimal** for $\alpha < 1$

## Diversification: Heavy-tailedness & dependence matter

- **Independence**, **Tail dependent** models with **common shocks** (e.g., Student-$t$ distr. $=$ Student-$t$ copula with Student-$t$ marginals):
  - Diversification always **pays off** for all loss probabilities $q$
- **Tail independent** EFGM, possibly **tail dependent** Archimedean copulas (e.g., Clayton & Gumbel):
  - **Dividing boundary** $\alpha = 1$ for sufficiently small loss probability $q$
- **Numerical** results on interplay of **heavy-tailedness** & **dependence** (copula) assumptions and **loss probability** $q$ in **diversification** decisions:
  - **Deviations** from threshold $\alpha = 1$ for different **copulas** and **loss probabilities** $q$
- **Theoretical** results for **general** copulas $=$ ?
- **(Non-)robustness** of other models in economics & finance

# Characterizations of copulas & dependence

- $V_1, \ldots, V_n$: i.i.d. $\mathcal{U}\big([0,1]\big)$
- $C$: $n-$**copula** iff $\exists\ \tilde{g}_{i_1,\ldots,i_c}$ s.t.

  **A1 (integrability):**

$$\int_0^1 \ldots \int_0^1 |\tilde{g}_{i_1,\ldots,i_c}(t_{i_1}, \ldots, t_{i_c})| dt_{i_1} \ldots dt_{i_c} < \infty$$

  **A2 (degeneracy):**

$$E_{V_{i_k}}\Big[\tilde{g}_{i_1,\ldots,i_c}(V_{i_1}, \ldots, V_{i_{k-1}}, V_{i_k}, V_{i_{k+1}}, \ldots, V_{i_c})\Big] = 0$$

  **A3 (positive definiteness):**

$$\tilde{U}_n(V_1, \ldots, V_n) \equiv \sum_{c=2}^{n} \sum_{1 \le i_1 < \ldots < i_c \le n} \tilde{g}_{i_1,\ldots,i_c}(V_{i_1}, \ldots, V_{i_c}) \ge -1$$

- **Representation for** $C$ :

$$C(u_1, ..., u_n) = \int_0^{u_1} ... \int_0^{u_n} \left(1 + \tilde{U}_n(t_1, ..., t_n)\right) \prod_{i=1}^{n} dt_i$$

- $\tilde{U}_n$: sum of **degenerate** $U-$**statistics**

Device for **constructing** $n-$**copulas and cdf's**

- **Bivariate Eyraud-Farlie-Gumbel-Morgenstern copulas & cdf's:**

$$C_\theta(u,v) = uv\,(1+\theta(1-u)(1-v))$$

$$H_\theta(x,y) = F(x)G(y)\Big(1+\theta(1-F(x))(1-G(y)\Big)$$

$n=2$; $\tilde{g}_{1,2}(t_1,t_2) = \theta(1-2t_1)(1-2t_2)$, $\theta \in [-1,1]$

- **Multivariate EFGM copulas & cdf's:**

$$C_\theta(u_1,u_2,...,u_n) = \prod_{i=1}^{n} u_i \left(1+\theta\prod_{i=1}^{n}(1-u_i)\right)$$

$$\tilde{g}_{i_1,\ldots,i_c}(t_{i_1},...,t_{i_c}) = \theta_{i_1,\ldots,i_c}(1-2t_{i_1})(1-2t_{i_2})...(1-2t_{i_c})$$

- **Generalized multivariate EFGM copulas** (Johnson and Kotz, 1975, Cambanis, 1977)

$$C(u_1, ..., u_n) = \prod_{k=1}^{n} u_k \Big(1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < \ldots < i_c \le n} \theta_{i_1,\ldots,i_c}(1 - u_{i_k})\Big)$$

$\tilde{g}_{i_1,\ldots,i_c}(t_{i_1}, ..., t_{i_c}) = 0,\ c < n - 1$

$\tilde{g}_{1,2,\ldots,n}(t_1, t_2, ..., t_n) = \theta(1 - 2t_1)(1 - 2t_2)...(1 - 2t_n)$

- **Generalized EFGM copulas:** complete **characterization** of joint **cdf's** of **two-valued r.v.'s** (Sharakhmetov & Ibragimov, 2002)

## From dependence to independence through $U-$statistics

$\mathcal{G}_n$: **sums of $U-$statistics**

$$U_n(\xi_1, ..., \xi_n) = \sum_{c=2}^{n} \sum_{1 \le i_1 < \ldots < i_c \le n} g_{i_1, \ldots, i_c}(\xi_{i_1}, ..., \xi_{i_c})$$

$g_{i_1, \ldots, i_c}$: satisfy A1-A3

- Arbitrarily **dependent r.v.'s**:

  **sum of $U-$statistics** in **independent** r.v.'s

  with canonical kernels

- **Reduction** of problems for **dependence** to **well-studied** objects

- Transfer of results for $U-$**statistics** under

  **independence**

## From dependence to independence through $U-$statistics

- $X_1, ..., X_n$: **1-cdf's** $F_k(x_k)$

- $\xi_1, ..., \xi_n$: independent copies (**1-cdf's** $F_k(x_k)$)

$\exists\ U_n \in \mathcal{G}_n$ s.t. $\forall f : \mathbf{R}^n \to \mathbf{R}$

$$Ef(X_1, ..., X_n) = Ef(\xi_1, ..., \xi_n)\Big(1 + U_n(\xi_1, ..., \xi_n)\Big)$$

- **Representation for c.f.'s:**

$$Eexp\left(i\sum_{k=1}^{n} t_k X_k\right) = Eexp\left(i\sum_{k=1}^{n} t_k \xi_k\right) +$$

$$Eexp\left(i\sum_{k=1}^{n} t_k \xi_k\right) U_n(\xi_1, ..., \xi_n)$$

‡ CLT for bivariate r.v.'s

## Characterizations of dependence

- **Canonical $g'$s:** complete **characterizations** of

**dependence** properties

- $X_1, ..., X_n$: $r-$**independent** if $\forall$ $r$ jointly independent $\Leftrightarrow$

$g_{i_1,...,i_c}(V_{i_1}, ..., V_{i_c}) = 0$ (a.s.) $1 \leq i_1 < ... < i_c \leq n$, $c = 2, ..., r$

- 

$g_{i_1,...,i_{r+1}}(u_{i_1}, ..., u_{i_{r+1}}) =$

$\frac{\alpha_1...\alpha_n}{\alpha_{i_1}...\alpha_{i_r+1}} \left((k+1)u_{i_1}^k - (k+2)u_{i_1}^{k+1}\right) \times ... \times \left((k+1)u_{i_c}^k - (k+2)u_{i_c}^{k+1}\right)$

$$C(u_1, ..., u_n) = \prod_{i=1}^{n} u_i \Big(1 + \sum_{1 \leq i_1 < ... < i_{r+1} \leq n} \frac{\alpha_1 ... \alpha_n}{\alpha_{i_1} ... \alpha_{i_r+1}} \times$$

$$(u_{i_1}^k - u_{i_1}^{k+1}) \times ... \times (u_{i_{r+1}}^k - u_{i_{r+1}}^{k+1})\Big)$$

Extensions of Wang (1990) ($k = 0$)

## Copulas and Markov processes

- Darsow, Nguyen and Olsen, 1992: copulas and first-order Markovness
- $A, B : [0,1]^2 \to [0,1]$ :

$$(A * B)(x,y) = \int_0^1 \frac{\partial A(x,t)}{\partial t} \cdot \frac{\partial B(t,y)}{\partial t} dt$$

- $A : [0,1]^m \to [0,1],\ B : [0,1]^n \to [0,1]$ : $\star-$product

$$A \star B(x_1, ..., x_{m+n-1}) =$$

$$\int_0^{x_m} \frac{\partial A(x_1, ..., x_{m-1}, \xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1}, ..., x_{m+n-1})}{\partial \xi} d\xi$$

## Copulas and Markov processes

- **Transition probabilities**

$P(s, x, t, A) = P(X_t \in A | X_s = x)$ satisfy CKE's

iff $C_{st} = C_{su} * C_{ut}$ $\forall s < u < t$

- $X_t$: **first-order Markov** iff

$$C_{t_1,\dots,t_n} = C_{t_1 t_2} \star C_{t_2 t_3} \star \dots \star C_{t_{n-1} t_n}$$

## New results: Higher-order Markovness and copulas

- $\{X_t\}_{t \in T}$: $k-$**order Markov** $\Leftrightarrow$

$$P\left(X_t < x_t \middle| X_{t_1}, ..., X_{t_{n-k}}, X_{t_{n-k+1}}, ..., X_{t_n}\right) =$$

$$P\left(X_t < x_t \middle| X_{t_{n-k+1}}, ..., X_{t_n}\right)$$

- **Complete characterization** in

terms of $(k+1)-$**copulas**

- $C_{t_1, \dots, t_k}$: copulas of $X_{t_1}, ..., X_{t_k}$

- $\{X_t\}_{t \in T}$: $k-$**order Markov** iff $\forall t_1 < ... < t_n, \quad n \geq k+1$

$$C_{t_1, \dots,\, t_n} = C_{t_1, \dots,\, t_{k+1}} \star^k C_{t_2, \dots,\, t_{k+2}} \star^k \dots \star^k C_{t_{n-k}, \dots,\, t_n}$$

# Stationary case

- $X_t$: **stationary** $k-$**order Markov** iff

$$C_{1,...,n}(u_1, ..., u_n) = C \star^k C \star^k ... \star^k C(u_1, ..., u_n)$$

$$= C^{n-k+1}(u_1, ..., u_n) \;\; \forall n \geq k+1$$

$C$: $(k+1)-$ copula s.t.

$$C_{i_1+h,...,i_l+h} = C_{i_1,...,i_l}, \;\; 1 \leq j_1 < ... < j_l \leq k+1$$

- $C^s$: $s-$fold product $\star^k$ of $C$

# Advantages of copula-based approach

• **Modeling higher order Markov** processes

**alternative** to **transition** matrices

‡ Instead of **initial distribution** & **transition** probabilities:

Prescribe **marginals** & $(k+1)-$**copulas**

Generate **copulas of higher order** & finite-dimensional **cdf's**

‡ **Advantage: separation** of properties of **marginals** (fat-tailedness) & **dependence** properties (conditional symmetry, $m-$dependence, $r-$independence, mixing)

# Advantages of copula-based approach

- Inversion method:

**New** $k-$**Markov** with **dependence similar** to a given Markov process

**Different marginals**

‡ $X_t$: **stationary** $k-$**Markov**

$(k+1)-$cdf $\tilde{F}(x_1, ..., x_{k+1})$, $1-$cdf $F$

$\Rightarrow$ $(k+1)-$copula:

$$C(u_1, ..., u_{k+1}) = \tilde{F}\left(F^{-1}(u_1), ..., F^{-1}(u_{k+1})\right)$$

† Another 1−cdf $G$:

**Stationary** $k-$**Markov**, **same** dependence as $\{X_t\}$, **different** 1-marginal $G$:

$(k+1)-$copula:

$$C(u_1, ..., u_{k+1}) = \tilde{F}\Big(G^{-1}(u_1), ..., G^{-1}(u_{k+1})\Big)$$

Representation $\Rightarrow$ **Higher-order copulas & cdf's**

$\{X_t\}$: stationary $C-$based $k-$Markov chain

## Advantages of copula-based approach

- $C$: **all dependence properties** of the time series

‡ $k-$**independence,** $m-$**dependence, martingaleness, symmetry**

‡ On-going project with Johan Walden: characterizations of **time-irreversibility**; focus on $C_{t_1,\ldots,t_k} = C_{t_k,\ldots,t_1}$

‡ Applications: **forward-looking vs. backward-looking** market participants ("fundamentalists" vs. noise traders or "chartists")

‡ "**Compass rose**" for $P_{t-1}$ and $P_t$: **symmetry in copulas**

## Combining higher-order Markovness with other dependence properties

- A number of studies in **dependence modeling: Higher-order Markovness** + $m-$**dependence** & $r-$**independence**

Lévy (1949): **2nd order Markovness** + **pairwise independence**

Rosenblatt & Slepian (1962): $N-$**order** $N-$**independent stationary Markov**

- **Impossibility/reduction** :

$N-$**order Markov** + $N-$**independence** + **two**-valued $\Leftrightarrow$ joint **independence**

‡ **Testing sensitivity** to WD in DGP Rosenblatt & Slepian (1962)

# Combining Markovness with other dependencies

‡ **Examples:**

**Not** $1-$**order Markovian**

But **1-st order transition** probabilities

$P(s, x, t, A) = P(X_t \in A | X_s = x)$ satisfy **C-K SE**

$$P(s, x, t, A) = \int_{-\infty}^{\infty} P(u, \xi, t, A) P(s, x, u, d\xi)$$

(other examples: Feller, 1959, Rosenblatt, 1960)

# Combining Markovness with other dependencies

‡ 1-dependent Markov: Aaronson, Gilat and Keane (1992)

Burton, Goulet and Meester (1993), Matúš (1996)

‡ Matúš (1998): $m-$dependent

discrete-space Markov

‡ **Impossibility/Reduction:**

$\nexists$ stationary $m-$**dependent Markov** if

$card(\Omega) < m + 2$

# Markovness of higher-order and $k-$independence

- Characterization of stationary

$k-$**independent** $k-$**Markov** processes

- $\{X_t\}$: $C-$based $k-$**independent** stationary

$k-$**Markov** iff

$$\frac{\partial^{k+1} C(u_1, ..., u_{k+1})}{\partial u_1 ... \partial u_{k+1}} = 1 + g(u_1, ..., u_{k+1})$$

$g : [0,1]^{k+1} \to [0,1]$: **canonical** $g-$**function**

(Integrability + more degeneracy + positive definiteness)

# Markovness of higher-order and $k-$independence

$$\int_0^1 \dots \int_0^1 |g(u_1, \dots, u_{k+1})| du_1 \dots du_{k+1} < \infty$$

$$\int_0^1 \dots \int_0^1 g(u_1, \dots, u_{k+1}) g(u_2, \dots, u_{k+2}) \dots g(u_s, \dots, u_{k+s}) du_{i_1} \dots du_{i_s} = 0$$

$$\forall s \leq u_{i_1} < \dots < u_{i_s} \leq k+1, \ s = 1, 2, \dots, \left[\tfrac{k+1}{2}\right]$$

$$g(u_1, \dots, u_{k+1}) \geq -1$$

- Integration: w.r.t. to all $s$ among $u_s, u_{s+1}, \dots, u_{k+1}$ common to all $g-$functions $g(u_1, \dots, u_{k+1})$, $g(u_2, \dots, u_{k+2})$, ..., $g(u_s, \dots, u_{k+s})$

$k-$marginals: product copulas, independence

$k-$independence: satisfied

## Markovness of higher-order and $m-$independence

- $\{X_t\}$: $C-$based $m-$**dependent 1-Markov** iff

$$\frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2} = 1 + g(u_1,u_2)$$

$g : [0,1]^2 \to [0,1]$: canonical $g-$function:

$$\int_0^1 \int_0^1 |g(u_1,u_2)| du_1 du_2 < \infty$$

$$\int_0^1 g(u_1,u_2) du_i = 0, \;\; g(u_1,u_2) \geq -1$$

$$\int_0^1 g(u_1,u_2) g(u_2,u_3) ... g(u_m,u_{m+1}) du_2 du_3 ... du_m = 0$$

‡ Integration: w.r. to $u_2, u_3, ..., u_m$ more than once among $g(u_1,u_2)$, $g(u_2,u_3)$, ..., $g(u_m,u_{m+1})$

$X_1$, $X_{m+1}$: independent; Process: $m-$dependent

## New examples via existing constructions

- **Higher-order Markovness** + **martingaleness**
- **Inversion** method + **existing examples** $\Rightarrow$

$k-$**independent,** $m-$**dependent Markov** processes

**different marginals**

## Reduction & impossibility for $k-$order Markov processes

- $\{X_t\}$: $C-$based $k-$**independent** stationary $k-$**Markov**

‡ $\frac{\partial^{k+1} C(u_1,\ldots,u_{k+1})}{\partial u_1 \ldots \partial u_{k+1}} = 1 + g(u_1, ..., u_{k+1})$

‡ $g$ : **product form** (EFGM-type):

$g(u_1, u_2, ..., u_{k+1}) = \alpha f(u_1) f(u_2) ... f(u_{k+1})$

$\Leftrightarrow$ $\{X_t\}$: **jointly independent**

## Examples: EFGM and power copulas

- $(k+1)-$**EFGM** copulas:

$$C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \Big(1 + \alpha(1-u_1)(1-u_2)...(1-u_{k+1})\Big)$$

$$g(u_1, u_2, ..., u_{k+1}) = \alpha(1-2u_1)(1-2u_2)...(1-2u_{k+1})$$

- $(k+1)-$**power** copulas

$$C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \Big(1 + \alpha(u_1^l - u_1^{l+1})(u_2^l - u_2^{l+1})...(u_{k+1}^l - u_{k+1}^{l+1})\Big)$$

$l \geq 0$ (EFGM: $l = 0$)

# Impossibility/reduction for $m-$dependence

- $\{X_t\}$: $C-$based $m-$dependent Markov

‡ $\frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2} = 1 + \alpha f(u_1) f(u_2)$

(separable product form)

$\Leftrightarrow$ $X_t$: **jointly independent**

- Representations $\Rightarrow$

$$\int_0^1 \dots \int_0^1 \alpha^m f(u_1) f^2(u_2) \dots f^2(u_m) f(u_{m+1}) du_2 \dots du_m = 0;$$

$$\alpha^m f(u_1) f(u_{m+1}) \Big[ \int_0^1 f^2(u_2) du_2 \Big]^{m-1} = 0$$

$\Rightarrow$ $f = 0$ $\Leftrightarrow$ **Independence**

‡ **EFGM copulas**, $k = 1$:

$$C(u_1, u_2) = u_1 u_2 \Big(1 + \alpha(1 - u_1)(1 - u_2)\Big)$$

$$g(u_1, u_2) = \alpha(1 - 2u_1)(1 - 2u_2)$$

- **Limitations** of **EFGM copulas,**

**separable copulas:**

**Complement & generalize** existing results

‡ Cambanis (1991): **common dependencies**

**cannot be exhibited** by multivariate EFGM

$$C_{j_1,\ldots,j_n}(u_{j_1}, \ldots, u_{j_n}) =$$

$$\prod_{s=1}^{n} u_{j_k}\Big(1 + \sum_{1 \le l < m \le n} \alpha_{lm}(1 - u_{j_l})(1 - u_{j_m})\Big)$$

‡ Rosenblatt & Slepian (1962): **non-existence** of **bivariate** $N-$**independent**

$N-$**Markov**

Sharakhmetov & Ibragimov (2002):

**EFGM copulas for two-valued r.v.'s**

‡ **Technical difficulties** in **modeling**

# Solution: New flexible copula classes

- **Copula-based TS** with **flexible dependencies**

‡ Copulas based on **Fourier polynomials**

- $k-$**independent** $k-$**Markov: Conditions satisfied** for

$$g(u_1, ..., u_{k+1}) = \sum_{j=1}^{N} \left[\alpha_j sin(2\pi \sum_{i=1}^{k+1} \beta_i^j u_i) + \gamma_j cos(2\pi \sum_{i=1}^{k+1} \beta_i^j u_i)\right]$$

‡ $\alpha_j, \gamma_j \in \mathbf{R}$, $\beta_i^j \in \mathbf{Z}$, $i = 1, ..., k+1$, $j = 1, ..., N$ :

† $\beta_1^{j_1} + \sum_{l=2}^{s} \epsilon_{l-1} \beta_l^{j_l} \neq 0$

$\epsilon_1, ..., \epsilon_{s-1} \in \{-1, 1\}$, $s = 2, ..., k+1$

† $1 + \sum_{j=1}^{N} \left[\alpha_j \epsilon_j + \gamma_j \epsilon_{j+N}\right] \geq 0$, $\epsilon_1, ..., \epsilon_{2N} \in \{-1, 1\}$

## Fourier copulas

$$C(u_1, ..., u_{k+1}) = \int_0^{u_1} ... \int_0^{u_{k+1}} (1 + g(u_1, ..., u_{k+1})) du_1 ... du_{k+1}$$

$(k+1)-$**Fourier** copulas

- 1−**dependent** 1−**Markov:**

**Conditions satisfied** for Fourier copulas

$$C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} (1 + g(u_1, u_2)) du_1 du_2$$

$$g(u_1, u_2) = \sum_{j=1}^{N} \left[ \alpha_j sin(2\pi(\beta_1^j u_1 + \beta_2^j u_2)) + \gamma_j cos(2\pi(\beta_1^j u_1 + \beta_2^j u_2)) \right]$$

‡ $\alpha_j, \gamma_j \in \mathbf{R}$, $\beta_1^j, \beta_2^j \in \mathbf{Z}$ :

$$\beta_1^{j_1} + \beta_2^{j_2} \neq 0$$

$$\beta_1^{j_1} - \beta_2^{j_2} \neq 0$$

$$1 + \sum_{j=1}^{N} \left[ \alpha_j \epsilon_j + \gamma_j \epsilon_{j+N} \right] \geq 0$$

$\forall \epsilon_1, ..., \epsilon_{2N} \in \{-1, 1\}$

# Concluding remarks

- **(Sub-)Optimality** of diversification under **heavy tails** & **dependence**
- **(Non-)robustness** of models in economics & finance to **heavy tails**, **heterogeneity** & **dependence**
- General **representations** for **joint cdf's** and **copulas** of **arbitrary r.v.'s**
    - Joint cdf's and copulas of **dependent** r.v.'s = sums of $U-$statistics in **independent** r.v.'s
    - Similar results: expectations of **arbitrary statistics** in **dependent** r.v.'s
    - New representations for multivariate **dependence measures**
    - Complete characterizations of **classes of dependent r.v.'s**
    - **Methods** for constructing **new copulas**
    - **Modeling** different **dependence structures**

# Concluding remarks

- **Copula-based** modeling for **time series**
- **Characterizations** of **dependence** in terms of **copulas**
  - **Markovness** of **arbitrary order**
  - Combining **Markovness** with **other dependencies:**

    $m-$**dependence,** $r-$**independence, martingaleness, conditional symmetry**

    **Non-Markovian** processes satisfying **Kolmogorov-Chapman SE**

## Concluding remarks

- **New flexible copulas** to combine **dependencies**
- Expansions by linear functions (Eyraud-Fairlie-Gumbel-Morgensten copulas)
- power functions (power copulas); Fourier polynomials (Fourier copulas)
- **Impossibility/reduction:** Copula-based **dependence** + **specific copulas** $\Leftrightarrow$ **Independence**

# Copula memory

- **Long-memory** via **copulas**: various **definitions**
- **Dependence measures** & **copulas**
- **Gaussian** & **EFGM** $\Rightarrow$ **short-memory** Markov
- **Fast** exponential **decay** of **dependence** between $X_t$ & $X_{t+h}$
- **Numerical** results $\Rightarrow$ **Clayton** copula-based Markov $\{Xt\}$ : can behave as **long memory** (copulas) in finite samples
    - **High persistence** important for finance & economics
- **Long memory-like**: $X_t$ & $X_{t+h}$ : **slow decay** of **dependence** for

  commonly used lages $h$
- **Volatility** modeling & **Nonlinear dependence** in finance
- **Non-linear CH** & **long memory**-like **volatility**
- **Generalizations** of **GARCH**

## Copula memory

Beare (2008) & Chen, Wu & Yi (2008): **numerical** & **theoretical** results on **(short & long) memory** in **copulas**

Beare (2008): $\alpha$, $\beta$ & $\phi-$mixing

- $\kappa(h) \leq \alpha(h) \leq \beta(h) \leq 0.5\phi(h)$
- **Numerical** results $\Rightarrow$ **Clayton**: **exponential decay** in $\beta(h)$ $\Rightarrow$ **short** $\kappa-$**memory** in **copulas**

**Theoretical results** in Chen, Wu & Yi (2008):

- **Clayton**: **weakly dependent** & **short memory** in terms of **mixing** properties!
- Our **numerical** results + Chen, Wu & Yi (2008): **Non-robustness** of procedures for **detecting long memory** in copulas