Fair Allocation Over Time

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Abstract

We consider the problem of allocating an over-demanded, divisible resource over a fixed time horizon. We assume a very high number of agents, competing for the same infinitesimal fraction of the resource in each time period. An agent makes the decision whether or not to compete after observing a randomly distributed utility that will be derived if the resource is actually granted. In the interest of fairness, priority classes at each time period are determined taking into consideration an agent's past behavior. To agents within a certain class the resource is randomly distributed. We provide a thorough treatment of 2-period models and comment on efficiency, monotonicity, and uniqueness properties that equilibrium strategies exhibit. We describe the form of the Nash equilibrium strategy for the multi-period game and compute the strategy explicitly in some special cases. When aggregate demand is high enough, the game admits an equilibrium strategy that is insensitive to it. We provide a simple recursive procedure to efficiently compute this strategy. We close with a brief comment on generalizations of our priority framework.

1 Introduction

Motivation. The need to allocate a scarce good in an efficient and fair manner arises in several contexts. In this paper, we propose a mechanism for allocating repeatedly over time an over-demanded, divisible resource to a continuum of identical agents, each of which has the same demand for the resource and derives an independent and identically distributed random utility from its consumption. Over the course of a multi-period time horizon an agent will have the opportunity to consume the resource multiple times. Thus, if the history of an agent is not taken into account when deciding present allocation decisions, inequities may well build and magnify

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over time. The primary aim of this paper is to design a dynamic allocation procedure that restores fairness over time without placing an undue burden on the system's efficiency.

Our work is motivated by the management of scarce divisible resources such as Internet bandwidth. In the case of bandwidth, there exist a multitude of users who each wish to satisfy their individual demand for access, but the available supply is not able to cover aggregate demand. The intensity of this demand may vary across users according to their potential online activity. Furthermore, it is reasonable to assume that users cannot observe the actions of others, since the scale of the overall population is so huge. In this environment a central planner would like to institute a procedure that reduces congestion to the system and ensures a fair allocation of bandwidth over time without sacrificing too much economic efficiency.

An additional real-life application of this work could be found in the large-scale dynamic management of natural resources. For example, imagine large fisheries, where the government has restricted the total catch to a fixed upper bound. We may imagine individual fishermen as agents competing for access to the ocean (the resource). How much an individual fisherman values going out to sea on a particular day will vary according to a host of unpredictable considerations. In the interest of fairness, the government would like to ensure access to the ocean to every fisherman while at the same time making sure the ocean is not overfished. It would also wish to do so in a manner that is not completely arbitrary, in that it takes into account the intensity of a fisherman's desire to go to work. Another example of such an application of our work could be found in the management of forest logging. In this setting, the relevant economic output which needs to be capped is total timber production and individual loggers face a similar kind of constraint as the fishermen do.

Contributions. In this paper we devise a priority-based framework for the repeated allocation of a divisible resource to a continuum of agents. We focus on two intuitive priority mechanisms in which agents need to weigh the desire to compete for immediate access against the resultant loss in priority such a decision would cause. In Mechanism 1, an agent loses priority if she *requests* the resource, whereas in Mechanism 2 if she *receives* it. We provide a thorough treatment of both mechanisms for the 2-period case. First, we describe the Nash equilibrium, discuss its uniqueness features, and derive it explicitly when utilities are Uniform [0,1]. Restricting ourselves to market-clearing equilibria, we are able to meaningfully compare our priority mechanisms to generic random and efficient allocations. In this context, we show that they are ex-ante more efficient than the random allocation and afford a greater ex-ante probability of access than both the random and efficient allocations. We further show that arbitrary equilibrium strategies may fail to be monotonic in aggregate demand. In the multiple-period case, we provide conditions for the Nash equilibrium and investigate its structural properties. We carry out our analysis for Mechanism 1, but note that similar results can be established for Mechanism 2. When aggregate demand is high enough we show that the game admits an optimal strategy that is insensitive to it and we provide a simple recursive procedure to efficiently compute the equilibrium thresholds. Similarly to 2-period models, market-clearing equilibria are shown to be ex-ante more efficient than the random allocation.

Organization of the Paper. The rest of the paper is organized as follows. We discuss the model more formally in Section 2 and provide an overview of the related work in Section 3. In Section 4 we provide a thorough treatment of the two-period model. In Section 5 we discuss the general multi-period mode and in Section 6 we briefly extend the scope of our mechanisms to a more general priority framework. We conclude in Section 7.

2 Model Description

2.1 Preliminaries

We assume a single resource with a total supply normalized to 1 unit. We further assume a continuum of agents, each requesting an infinitesimal amount of the resource, resulting in a cumulative market demand indexed by x, which exceeds supply (i.e. such that $x > 1$). Demand and supply are present and unchanging for the duration of a multi-period market with N time periods. In each time period, agents are endowed with random utilities associated with the fulfillment of their demand. In particular, each agent's utility is represented by an identical non-negative random variable $U \sim F(\cdot)$, which we assume to be independent and identically distributed (iid) over all time periods and other agents. In each time period an agent observes her utility realization and decides whether she should attempt to fulfil that particular demand. Once this decision has been made, the resource is allocated to all or a subset of those agents who requested it in accordance to some predetermined scheme.

We assume that the utility of an agent is private information, and that her past utility realizations and behavior are also not publicly knowable. Instead, the best agents can do is make ex-ante expected-value judgments about the behavior of their peers. This assumption is not a trivial one, but it can be justified on the grounds that we have a continuum of agents. In such a setting, it may be unreasonable to assume that a particular agent has the logistical capability to have detailed information about the actions of a non-trivial segment of the population.

2.2 Allocation Mechanisms

A *mechanism* is a mapping that determines which agents will have their demand met at a given time period, as a function of their past and present reported utilities and decisions. Concentrating on a particular agent at time t, the relevant information from the mechanism's point of view is:[1]

- Her stated utility realization and decision whether or not to compete for the resource at time t.
- Her stated utility realizations for time periods 1 through $t-1$
- Her decisions on whether or not to compete for the resource in time periods 1 through $t-1$.
- Her prior history of receiving the resource in time periods 1 through $t-1$.

Aggregating all this information across all users, a mechanism then decides to whom to allocate in the current time period. Obviously, the resource is allocated to an agent only if she actually requested it. A mechanism induces a multi-period non-cooperative game in which each agent tries to maximize her expected profit.

Two Canonical Mechanisms. Before we introduce priority based allocation mechanisms, it is worth discussing two straightforward ways of tackling this problem. In what we will refer to as the *efficient* mechanism, the agents with the highest utilities in each time period are awarded the good. Provided that agents declare their utilities truthfully, this mechanism has the advantage of being a priori fair (recall that utilities are iid) and of maximizing social welfare. But, these advantages are compromised by the fact that in the underlying revelation game truth-telling is a dominated strategy: Agents have an incentive to lie and state that their utility is the maximum possible.[2] Thus, the theoretical welfare gains will not be be realized. In addition, while this mechanism is a priori fair, it is not fair in the interim since it fails to take into account who has received the good in the past when deciding how to allocate in the present.

By contrast, the *random* mechanism allocates the resource arbitrarily among all agents in each time period. This mechanism is clearly a priori fair since all agents have an equal shot at the resource. Furthermore, there is no incentive for agents to lie and announce anything else

[1] We avoid more formal notation because it would complicate the exposition without providing much additional insight.

[2] Recall that utilities are private information and must be elicited from the agents.

but their true utility realizations, since the utility reports are ignored anyway. However, this mechanism is inefficient since it completely fails to consider the magnitude of agents' utilities when it arbitrarily decides which agents' demands to satisfy. In addition to this undesirable characteristic, and similarly to the efficient mechanism, it too fails to address inequities over time.

Priority Mechanisms. A reasonable way of reducing congestion, encouraging truthfulness, and restoring fairness to the allocation process over time is by introducing priority classes of agents in each time period. The priority class to which an agent belongs would depend in some fashion on her behavior in previous time periods. Consider the case where agents may compete for the resource in periods 1 and 2. In this context, we may think of granting priority in the second time period to all those agents who chose not to participate in the first. Or, alternatively, priority classes could be determined according to how many times an agent has competed for and actually been granted the resource in the past. Thus, an agent who chooses not to compete for the resource in the present time period is rewarded for her "restraint" with a higher priority in later time periods.

An important feature of priority mechanisms is that they give rise to intuitive *threshold* strategies. Indeed, it would be natural for agents to defer their consumption of the good in cases where the utility they would derive falls below a certain threshold. This is because after a certain utility level, an agent would prefer to not even try to obtain the resource in the current time period. This makes intuitive sense: If an agent's utility is close to zero, why should she choose to compete for the resource and potentially lose priority in later rounds?

3 Related Work

Priority schemes arise naturally in the analysis of stochastic queueing systems. Modeling queueing processes as non-cooperative games and examining the equilibrium behavior that a particular process induces has been the subject of much research. In this context, the idea of using priority classes as a way of regulating the outcome of such a game has been frequently employed. Adiri and Yechiali [1] introduce a queueing game in which entering customers have to choose between purchasing distinct priority levels when wishing to optimize a certain objective. In their setting, and along the same vein as our work, Hassin and Haviv [11] derive Nash equilibrium threshold strategies that determine which of two priority levels an agent should purchase, given the current length and class make-up of the queue. Mendelson and Whang [15] analyze an M/M/1 queue and derive the optimal incentive compatible mechanism that assigns prices to different priority levels. Optimality in this context is taken to mean the maximization of the system's profits as a whole.

Similar efforts regarding priority pricing schemes include papers by Balachandran [2], Dewan and Mendelson [8], and Glazer and Hassin [9]. For an excellent general survey of this literature the reader is referred to Hassin and Haviv [12].

An additional line of research that is relevant to the work presented in this paper has been pursued in the economics literature on sequential voting. In a notable paper, Casella [3] introduced the concept of *storable* votes as a way of improving a multi-period voting procedure. Inspired in part by European Union decision-making institutions, she proposed a process whereby, as individual binary voting decisions arise, countries are not required to vote instantly but are able to store their votes for future use. So at any given point in time, a country may either store its vote and abstain or cast any number of votes that it may have accumulated in previous rounds. This allows it to assert its voting power with greater intensity when it cares the most about the outcome of a particular vote. In [3] Casella showed that storable votes yield efficiency gains over the regular "one decision-one vote" voting procedure. However, due to the complexity of the dynamic game, she was unable to quantify these gains or to get an analytical solution for the equilibrium strategies for all but the simplest case of two countries and two time periods.[3] In light of these analytical difficulties, Casella et al [4], in a follow-up paper, performed an experimental study of the storable votes procedure. They were able to show that theoretical predictions about efficiency gains closely matched what was actually observed in the laboratory, even though the strategies that agents adopted were not consistent with the theoretical equilibrium. They concluded that concerns about the intractability of computing equilibrium strategies can be tempered by the procedure's apparent practical benefits (which accrue even in the absence of equilibrium behavior). In an extension of this work, Casella et al [5] study storable votes and their effect on minority stakeholders. They provide theoretical as well as experimental justification for the assertion that storable votes strengthen the voice of minorities, allowing them to have greater influence on issues that they feel especially strongly about.

Jackson and Sonnenschein [13], inspired in part by the work of Casella [3] and its efficiency implications, examine in a more general framework the effect of linking decisions over time as a way of improving on static procedures. In particular, in traditional non-dynamic models, incentive constraints often preclude certain outcomes from ever being achieved.[4] A famous example of such a model is the bilateral bargaining game introduced by Chatterjee and Samuelson [6], for which, as Myerson and Satterthwaite [16] demonstrated, there is no Pareto efficient, individually rational and incentive compatible mechanism. In the face of these kinds of challenges, Jacskon and

[3] We run into a similar kind of intractability with our model.

[4] In economic theory terms, not all social choice functions are implementable.

Sonnenschein were able to circumvent impossibility results present in static models by examining independent copies of individual decision problems and requiring agents to participate in all of them. In this augmented problem space, they introduced a mechanism that compels agents to declare their types in a way that mirrors the underlying type distribution.[5] Assuming that the type distributions are independent, they showed that this linking mechanism truthfully implements any ex-ante efficient social choice function.

4 2-Period Model

4.1 Priority Mechanism 1

First, we focus our attention on the priority mechanism where an agent loses her priority when she decides to compete for the resource. Thus, in a 2-period context, the only decision that needs to be made is in period 1: after an agent has seen her utility she must decide whether or not to compete for the resource. If she does, then she loses priority in period 2 to all agents who chose to delay their participation.

To fix matters and derive a Nash equilibrium for this game, let us suppose that in this context agents will employ a symmetric threshold strategy denoted by $\bar{u}$. That is, let us assume that an agent will participate in the mechanism if and only if she observes a utility of $\bar{u}$ or higher. Let us also assume that utilities for each agent are iid and distributed according to a continuous distribution F having non-negative support $[u_{min}, u_{max}]$[6]. We allow for the case $u_{max} = \infty$ but impose $E[U] < \infty$.

To estimate the probability that an agent obtains the resource in the first time period we consider the ratio of the available supply (which we have assumed to be 1) over the average demand. The demand reflects the agents who observe a utility level that is higher than $\bar{u}$. In the second time period, we need to distinguish between two groups of agents: those who participated in period 1 and those who did not. The second group is granted priority over the first. Thus, the available supply for these agents ("class 1") will be equal to 1, whereas the available supply for the other group of agents ("class 2") will be whatever is left after the high-priority group has gotten its share. In the first case, the demand will reflect those agents who chose not to participate in period 1, while in the second those agents who chose to participate in the first time period.

[5] For example, if we have K copies of the original problem and the probability of being type u is $P(u)$, then an agent should declare that she is of type u approximately $K \times P(u)$ times.

[6] For simplicity set $u_{min} = 0$

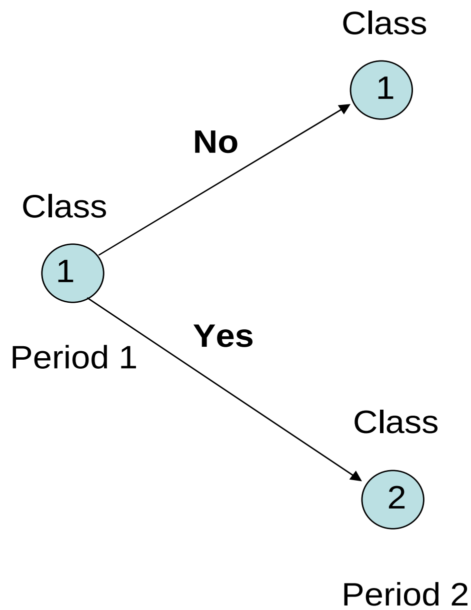

Figure 1: 2-period Priority Model I. "Yes" and "No" denote an agent's decision to request the resource or not.

In this context, we can approximate the probability that an agent receives the resource in period 1, given that every other agent is using the same strategy, by

$$\min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\} \equiv \alpha(1,1,\bar{u}).$$

Similarly we can estimate the probability that a class 1 and class 2 agent receive the resource in period 2. In particular, given a threshold strategy $\bar{u}$ in round 1, the probability that a class 1 agent is granted the resource in period 2 is given by

$$\min\left\{1, \frac{1}{xF(\bar{u})}\right\} \equiv \alpha(1,2,\bar{u}),$$

while the probability that the resource is granted to a class 2 agent is

$$\min\left\{1, \frac{(1-xF(\bar{u}))^{+}}{x\bar{F}(\bar{u})}\right\} = \frac{(1-xF(\bar{u}))^{+}}{x\bar{F}(\bar{u})} \equiv \alpha(2,2,\bar{u})$$

In order to derive a Nash equilibrium threshold strategy we need to find the utility level at which an agent is indifferent between participating in period 1 or delaying her participation for period 2, provided that every other agent is behaving in the same way. Such a utility level would by definition satisfy the following equation:

$$\bar{u} \cdot \alpha(1,1,\bar{u}) + E[U]\alpha(2,2,\bar{u}) = E[U]\alpha(1,2,\bar{u}) \tag{1}$$

The left hand side of Equation (1) denotes the expected utility an agent derives upon observing $\bar{u}$ and deciding to participate in period 1. The right hand side denotes the expected utility an agent gets if she decides not to participate in period 1.

Proposition 1 *Equation (1) has a solution. This solution is unique only if $x \geq 2$.*

Proof. Proving existence is straightforward. Consider the function:

$$f(\bar{u}) = \bar{u} - E[U]\frac{\min\left\{1, \frac{1}{xF(\bar{u})}\right\} - \min\left\{1, \frac{(1-xF(\bar{u}))^+}{x\bar{F}(\bar{u})}\right\}}{\min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\}} = \bar{u} - E[U]g(\bar{u})$$

This function is continuous. Furthermore, we have $f(0) = -E[U](1-1/x)x < 0$, while $f(u_{max}) = u_{max} - E[U]/x > 0$ since $x > 1$. So, this establishes the existence of a zero for $f(\cdot)$ in the interval $[0, u_{max}]$ which in turn implies the existence of $\bar{u}$.

Now we turn to uniqueness. Consider the function

$$g(\bar{u}) = \frac{\min\left\{1, \frac{1}{xF(\bar{u})}\right\} - \min\left\{1, \frac{(1-xF(\bar{u}))^+}{x\bar{F}(\bar{u})}\right\}}{\min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\}}$$

Let us first assume $x \geq 2$. This implies that $1/x \leq (x-1)/x$. The function $g(\bar{u})$ assumes different forms in the consecutive intervals $[0, F^{-1}(1/x)], (F^{-1}(1/x), F^{-1}((x-1)/x)], (F^{-1}((x-1)/x), F^{-1}(1)]$. However, in each case, $g(\bar{u})$ is monotone decreasing, so we can conclude that $g(\cdot)$ as a whole is monotone decreasing. This implies that $f(\bar{u}) = u - E[U]g(\bar{u})$ is monotone increasing and therefore can have only one zero.

When, $x < 2$, the uniqueness result no longer holds. The function $g(\bar{u})$ once again assumes different values in consecutive intervals $[0, F^{-1}((x-1)/x)], (F^{-1}((x-1)/x), F^{-1}(1/x)]$, $(F^{-1}(1/x), F^{-1}(1)]$. But when $\bar{u} \in (F^{-1}((x-1)/x), F^{-1}(1/x)]$ we have:

$$g(\bar{u}) = \frac{x-1}{x\bar{F}(\bar{u})},$$

which is monotone increasing, so we cannot use the previous reasoning. In fact, we will exhibit a distribution for which all $\bar{u} \in (F^{-1}((x-1)/x), F^{-1}(1/x)]$ will be Nash equilibria. To wit, recall that $\bar{u}$ is a Nash equilibrium if and only if

$$\bar{u} - E[U]g(\bar{u}) = 0$$

which implies that for $x < 2$ a threshold policy $\bar{u} \in (F^{-1}((x-1)/x), F^{-1}(1/x)]$ will be a Nash equilibrium if and only if

$$\bar{u} - E[U]\frac{x-1}{x\bar{F}(\bar{u})} = 0 \Rightarrow \bar{u}\bar{F}(\bar{u}) = \frac{x-1}{x}E[U]$$

Now, consider the distribution of U having support $[b_1, b_2]$ and such that $E[U] = 1$ and

$$\bar{F}(\bar{u}) = \frac{x-1}{x}E[U]\frac{1}{\bar{u}} = \frac{x-1}{x}\frac{1}{\bar{u}}$$

To make sure we avoid contradictions we pick $b_1 < 1$, $b_2 > 1$ so that indeed the mean will equal 1, i.e.

$$E[U] = \int_0^{b_2} \bar{F}(\bar{u})du = b_1 + \int_{b_1}^{b_2} \frac{x-1}{x}\frac{1}{\bar{u}}du = 1$$

This implies, $b_1 + (x-1)/x \ln(b_2/b_1) = 1$. This dictates that b_1 and b_2 will have to satisfy the following relation,

$$b_2 = b_1 \exp[(\frac{x}{x-1})(1-b_1)]$$

Given this choice of distribution we have the following holding,

$$\bar{F}(\bar{u}) = \frac{x-1}{x}E[U]\frac{1}{\bar{u}}$$

which implies that for all $u \in [F^{-1}(\frac{x-1}{x}), F^{-1}(\frac{1}{x})]$ we will have:

$$\bar{u}\bar{F}(\bar{u}) = \frac{x-1}{x}E[U]$$

So, for this distribution of U, all $\bar{u} \in [F^{-1}(1-1/x), F^{-1}(1/x)]$ will be Nash equilibria. ■

4.2 Priority Mechanism 2

Now suppose that we have a mechanism where agents lose priority only if they actually *receive* the resource in period 1. The analysis is quite similar to the preceding model. Given a threshold strategy $\bar{u}$ we can once again estimate the probability of being granted the resource in period 1 by $\min\{1, 1/(x\bar{F}(\bar{u}))\} \equiv \alpha(1,1,\bar{u})$. The only change is in the probabilities of getting the resource in period 2. An agent becomes class 2 only if she participates in period 1 and is granted the resource.

In this context, given a threshold strategy $\bar{u}$, the probability that a class 1 agent is granted the resource is given by

$$\min\left\{1, \frac{1}{xF(u) + x\bar{F}(\bar{u})(1 - \min\{1, \frac{1}{x\bar{F}(\bar{u})}\})}\right\} \equiv \alpha(1,2,\bar{u}),$$

while the probability that the resource is granted to a class 2 agent is

$$\min\left\{1, \frac{\left[1 - xF(\bar{u}) - x\bar{F}(\bar{u})\left(1 - \min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\}\right)\right]^+}{x\bar{F}(\bar{u})\min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\}}\right\} \equiv \alpha(2,2,\bar{u})$$

Now, let us consider the threshold level at which an agent is indifferent between demanding the resource in period 1 and delaying his participation until period 2. Once again, equalizing the payoffs from participating and not participating in the first round we obtain the following equation:

$$[\bar{u} + E[U]\alpha(2,2,\bar{u})]\alpha(1,1,\bar{u}) + E[U]\alpha(1,2,\bar{u})(1 - \alpha(1,1,\bar{u})) = E[U]\alpha(1,2,\bar{u}) \Leftrightarrow$$
$$\bar{u} + E[U]\alpha(2,2,\bar{u}) = E[U]\alpha(1,2,\bar{u}) \Leftrightarrow \bar{u} = E[U][\alpha(1,2,\bar{u}) - \alpha(2,2,\bar{u})]$$

Thus we obtain the following equilibrium condition:

$$\begin{aligned}\bar{u} &= E[U]\Bigg[\min\left\{1, \frac{1}{xF(u) + x\bar{F}(\bar{u})\left(1 - \min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\}\right)}\right\} \\ &- \min\left\{1, \frac{\left[1 - xF(\bar{u}) - x\bar{F}(\bar{u})\left(1 - \min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\}\right)\right]^+}{x\bar{F}(\bar{u})\min\left\{1, \frac{1}{x\bar{F}(\bar{u})}\right\}}\right\}\Bigg] \qquad (2)\end{aligned}$$

In a manner similar as before we may note the following proposition.

Proposition 2 *Equation (2) has a solution. This solution is unique only if $x \geq 2$.*

Proof. Identical to Proposition 1. In fact, the same infinite equilibria counterexample will apply. ■

4.3 The Uniform [0,1] Nash equilibrium

In this section we explicitly derive the Nash equilibria for the two models when utilities are distributed according to a Uniform [0,1] random variable.

Proposition 3 *Fix $U \sim \mathrm{Unif}[0,1]$. The unique Nash equilibrium for the two stage game induced by priority mechanism 1 is given by*

$$\bar{u} = \min\left\{\frac{x-1}{2}, \frac{1}{2}\right\}$$

Proof. Here Equation (1) reduces to the following:

$$\bar{u} = \frac{1}{2}\frac{\min\left\{1, \frac{1}{x\bar{u}}\right\} - \min\left\{1, \frac{(1-x\bar{u})^+}{x(1-\bar{u})}\right\}}{\min\left\{1, \frac{1}{x(1-\bar{u})}\right\}}$$

Let us solve for the Nash equilibria. We distinguish between 2 cases, according to whether $x < 2$ or $x \geq 2$.

Case 1 - $x \geq 2$. Here we have $1/x \leq 1 - 1/x$. Now there are potentially three cases to consider:

(a) $0 \leq \bar{u} < 1/x$. In this case, if we solve for $\bar{u}$ we obtain $\bar{u} = (x-1)/2$. Unless $x = 2$, this is a contradiction since for $x \geq 2$ we have $1/x \leq (x-1)/2$ and $\bar{u} \leq 1/x$.

(b) $1/x \leq \bar{u} \leq 1 - 1/x$. In this case, if we solve for $\bar{u}$ we obtain $\bar{u} = 1/2$. This is a valid solution since $1/x \leq 1/2 \leq 1 - 1/x$ for $x \geq 2$.

Since we know that the solution is unique for $x \geq 2$, we do not need to examine the third interval and can conclude that for $x \geq 2$ the only Nash equilibrium is $\bar{u} = 1/2$.[7]

Case 2 - $x < 2$. Here we have $1/x > 1 - 1/x$. Now there are three cases to consider:

(a) $0 \leq \bar{u} < 1 - 1/x$. In this case, if we solve for $\bar{u}$ we obtain $\bar{u} = (x-1)/2$. This is a valid solution since for $x < 2$ we have $\bar{u} = (x-1)/2 < 1 - 1/x$.

(b) $1 - 1/x \leq \bar{u} \leq 1/x$. In this case, if we solve for $\bar{u}$ we obtain a quadratic equation such that $2\bar{u} = 1 - (1 - x\bar{u})/(x(1-\bar{u}))$. Now we have, $x + (\sqrt{2x - x^2})/(2x) > 1/x$ on $x \in (1, 2)$ so this solution is infeasible for all $x < 2$. In addition, we have that $1 - 1/x > x - 1 - \sqrt{1 - 2(1 - 1/x)}/2$ for all $x \in (1, 2)$. So, both solutions are infeasible.

(c) $1/x < \bar{u} \leq 1$. In this case, if we solve for $\bar{u}$ we obtain $\bar{u} = \sqrt{1/2x}$. This is a contradiction since $\sqrt{1/(2x)} - 1/x < 0$ for $x < 2$.

So we may conclude that when $1 \leq x \leq 2$ we may have exactly one Nash equilibrium: $(x-1)/2$.

Hence, the unique Nash equilibrium for the two-stage game is given by

$$\bar{u} = \min\left\{\frac{x-1}{2}, \frac{1}{2}\right\}$$

■

We see that the equilibrium threshold is weakly increasing in x, the total demand. This makes intuitive sense, since if x is small an agent would expect to have a good chance in getting the resource in both time periods, and would therefore be more inclined to participate in period 1.

[7] It is clear that following the same analysis, we can deduce that for any symmetric distribution the Nash equilibrium will be the mean/median of the distribution.

Proposition 4 *Fix* $U \sim \text{Unif}[0,1]$. *The unique Nash equilibrium for the two stage game induced by priority mechanism 2 is given by*

$$\min\left\{\frac{1}{2(x-1)}, \frac{x-1}{2}\right\}$$

Proof. Applying Equation (2) performing the identical analysis as for Proposition 3 yields the result. ■

In this case, and in contrast to Mechanism 1, the equilibrium threshold tends to zero as x becomes large.

4.4 Mechanism Comparisons

4.4.1 Market-Clearing Equilibria

In order to intelligently compare the two mechanisms we need to somehow refine our notion of the Nash equilibrium of the game. As we saw, absent any refinement, we can be led to situations with a multiplicity of equilibria. In that case it is not clear how we would go about making meaningful comparisons between the two models.

Therefore, let us introduce the following definition:

Definition 1 *An equilibrium that exhausts all available supply in period 1 is called a market clearing (MC) equilibrium.*

In other words, a MC equilibrium strategy $\bar{u}$ is a solution to the equilibrium equation such that $x\bar{F}(\bar{u}) \geq 1$. It is quite natural to wish to examine equilibria of this sort, since they have the appealing efficiency property of using up all the available supply of the resource. Furthermore, such equilibria have attractive analytical properties.

Before we begin our analysis we should note that MC equilibria need not in general exist. Consider the following example.

Example 1. Consider the setting of either mechanism with $U \sim |N(0,1)|, x = 2.01$ Since $x \geq 2$ a sufficient condition for the unique equilibrium of the game to not be MC is if $u = E[U]/(xF(u))$ and $F(u) > 1 - 1/x$.

In our case, we note that $F(u) = P[|N(0,1)| \leq u] = P[-u \leq N(0,1) \leq u] = 2\Phi(u) - 1$. Furthermore,

$$E[U] = 2\int_0^\infty \frac{1}{\sqrt{2\pi}} u e^{-\frac{u^2}{2}} du = 2\frac{1}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$$

Thus we wish to find u such that $u \cdot (2\Phi(u) - 1) = \sqrt{2/\pi}/2.01 \approx .3970$ and $2\Phi(u) - 1 > 1 - 1/x = .505 \Rightarrow \Phi(u) > .7525 \Rightarrow u > .68$. But since $\Phi(.69) = .7549$ and $.69 \cdot (2\Phi(.69) - 1) = .3518 < .3970$ and the function $u \cdot (2\Phi(u) - 1)$ is monotone increasing, this will indeed be the case. ■

Now, let us focus our attention on the cases where MC equilibria do exist. A number of attractive properties emerge.

Corollary 1 *If priority mechanisms 1 and 2 induce MC Nash equilibria, then these equilibria are unique.*

Proof. For mechanism 1, please refer to the proof of Proposition (1). Once we restrict ourselves to thresholds $\bar{u}$ such that $x\bar{F}(\bar{u}) \geq 1$ it is straightforward to see that uniqueness will hold. For mechanism 2, the reasoning is identical upon applying Equation (2). ■

Now we note the following intuitive fact regarding the MC equilibria of the two mechanisms.

Corollary 2 *The MC equilibrium of priority mechanism 1 is weakly greater than that of mechanism 2.*

Proof. Again refer to the proof of Proposition (1) and let $\bar{u}_1, \bar{u}_2$ denote the MC equilibria of models 1 and 2 respectively. Now let us distinguish between two cases:

(i) $x < 2$. In this case, both market clearing equilibria will equal $\bar{u}_1 = \bar{u}_2 = (x-1)E[U]$.

(ii) $x \geq 2$. In this case we will have $\bar{u}_2 = E[U]/(x-1)$. On the other hand, if $F(\bar{u}_1) \leq 1/x$, $\bar{u}_1$ will equal $(x-1)E[U]$ and if $1/x < F(\bar{u}_1) \leq 1 - 1/x$ $\bar{u}_1$ will have to satisfy $\bar{u}_1 = E[U](1/(F(\bar{u}_1)) - 1) \geq E[U](x/(x-1) - 1) = E[U]/(x-1) = \bar{u}_2$. In either case we have $\bar{u}_1 \geq \bar{u}_2$.

■

4.4.2 Comparing Priority Mechanisms 1 and 2

Now, we can compare the two mechanisms and their unique MC equilibria in terms of their welfare and fairness properties. Let u_1 and u_2 be MC equilibria for Mechanisms 1 and 2, respectively. We refer to the ex-ante expected utility of mechanism 1 (2), as a function of the equilibrium strategies, as $U^1(u_1)$ ($U^2(u_2)$). Furthermore, the ex-ante expected probability that an agent is granted the resource in mechanism 1 (2) is denoted by $P^1(u_1)$ ($P^2(u_2)$). Given that we are restricting attention to MC equilibria we may, after some simplification, write:

$$
\begin{aligned}
U^1(u_1) &= F(u_1)E[U]\min\left\{1, \frac{1}{xF(u_1)}\right\} + E[U]\frac{(1-xF(u_1))^+}{x} + \int_{u_1}^{u_{max}} u\frac{1}{x\bar{F}(u_1)}dF(u) \\
&= \frac{1}{x}E[U] + \frac{\int_{u_1}^{u_{max}} udF(u)}{x\bar{F}(u_1)} \\
U^2(u_2) &= \frac{x-1}{x}E[U]\min\left\{1, \frac{1}{x-1}\right\} + \frac{1}{x}E[U](2-x)^+ + \int_{u_2}^{u_{max}} u\frac{1}{x\bar{F}(u_2)}dF(u) \\
&= \frac{1}{x}E[U] + \frac{\int_{u_2}^{u_{max}} udF(u)}{x\bar{F}(u_2)} \\
P^1(u_1) &= F(u_1)\min\left\{1, \frac{1}{xF(u_1)}\right\} + \frac{1}{x} + (1 - \frac{1}{x\bar{F}(u_1)})\frac{(1-xF(u_1))^+}{x} \\
P^2(u_2) &= \frac{x-1}{x}\min\left\{1, \frac{1}{x-1}\right\} + \frac{1}{x}
\end{aligned}
$$

From now on we suppress the dependence of these quantities on u_1 and u_2. We are ready to state the following proposition.

Proposition 5 *The MC equilibria of mechanisms 1 and 2 satisfy the following:*

(i) $U^1 \geq U^2$

(ii) $P^1 \leq P^2$

Proof. We begin by proving (i). Note that we can write, for general $\tilde{u}$:

$$\frac{\int_{\tilde{u}}^{u_{max}} udF(u)}{x\bar{F}(\tilde{u})} = \frac{1}{x}E[U|U > \tilde{u}]$$

This is a function which is increasing in $\tilde{u}$. We may rewrite U^1 and U^2 in the following way.

$$
\begin{aligned}
U^1 &= \frac{1}{x}E[U] + \frac{\int_{u_1}^{u_{max}} udF(u)}{x\bar{F}(u_1)} = \frac{1}{x}(E[U] + E[U|U > u_1]) \\
U^2 &= \frac{1}{x}E[U] + \frac{\int_{u_2}^{u_{max}} udF(u)}{x\bar{F}(u_2)} = \frac{1}{x}(E[U] + E[U|U > u_2])
\end{aligned}
$$

Now, from Corollaries 1 and 2 we know that for $x \geq 2$, $u_1 > u_2$, unless $F(u_1) = 1 - 1/x$ when we have $u_1 = u_2 = E[U]/(x-1)$, whereas for $x < 2$ we have $u_1 = u_2 = (x-1)E[U]$. In any case we have $u_1 \geq u_2 \Rightarrow E[U|U > u_1] \geq E[U|U > u_2]$ and therefore $U^1 \geq U^2$.

We turn to proving (ii) and distinguish between two cases:

(a) $x \geq 2$. In this case we can easily see that $P^2 = 2/x$. To get the value of P^1 we can consider different cases on whether $xF(u_1) \geq 1$ or $xF(u_1) < 1$. In the first case $P^1 = 2/x$, whereas in the second $P^1 = 2/x - (1 - xF(u_1))/(x^2F(u_1)) < 2/x$. Thus, we will have $P^1 \leq P^2$.

(b) $x < 2$. In this case $P^2 = 1$ and $P^1 < 1$.

■

4.4.3 Comparing Priority Mechanisms 1 and 2 to Canonical Allocations

This section compares the two priority based approaches to the canonical efficient (EF) and random (RD) allocations in terms of their efficiency (as measured by ex-ante expected utility) and equity (as measured by the ex-ante probability of being granted the resource). We assume that the two priority mechanisms give rise to MC equilibria. The efficient allocation satisfies the demand of the agents with the highest utility realizations, whereas the random one selects agents arbitrarily. Both clear the market in each time period.

What we deduce is that mechanisms I and II are ex-ante more efficient than RD. They also afford a higher ex-ante probability of receiving the resource than both EF and RD.

Ex-Ante Utility Comparison. Clearly the efficient allocation will dominate every other. But let us look at the welfare ratio between mechanism 1 and EF. Let u_E denote the value of u such that $x\bar{F}(u_E) = 1$. The ex-ante utility of an agent in the EF allocation is $2E[U|U > u_E]/x$. So, denoting the MC equilibrium of mechanism I by u_1 the ratio of its ex-ante utility over that of EF is:

$$\frac{\frac{1}{x}(E[U] + E[U|U > u_1])}{2\frac{1}{x}E[U|U > u_E]} = \frac{(E[U] + E[U|U > u_1])}{2E[U|U > u_E]}$$

Similarly, for mechanism 2 we get

$$\frac{(E[U] + E[U|U > u_2])}{2E[U|U > u_E]}$$

On the other hand, bearing in mind that the efficiency of the random allocation is equal to $2E[U]/x$, we obtain the following ratio for mechanisms $i = 1, 2$ and the random allocation:

$$\frac{E[U] + E[U|U > u_i]}{2E[U]}$$

Ex-Ante Probability Comparison. In this case, we do not need to distinguish between EF and RD since they both give a probability of $1-(1-1/x)^2$. But we do need to distinguish between different values of x. Let R_1 (R_2) denote the ratios for mechanisms 1 (2).

Let us first focus on the case when $x \geq 2$. Here we have:

$$R^2 = \frac{\frac{2}{x}}{1 - (\frac{x-1}{x})^2} = \frac{2x}{2x-1} > 1$$

$$R^1 = \begin{cases} \frac{\frac{2}{x}}{1-(\frac{x-1}{x})^2} = \frac{2x}{2x-1} > 1, & \text{if } xF(u_1) \geq 1 \\ \frac{\frac{2}{x} - \frac{1-xF(u_1)}{x^2 \bar{F}(u_1)}}{1-(\frac{x-1}{x})^2} = \frac{x + x\bar{F}(u_1) - 1}{2x-1} \geq 1, & \text{if } xF(u_1) < 1 \end{cases}$$

We see that both mechanisms 1 and 2 outperform the generic allocations.

Now consider $x < 2$. This (together with the fact that the equilibria are MC) immediately implies that $xF(u) < 1$. Mechanism 2 clearly dominates either generic allocation since $P^2 = 1$. Writing down the ratios again we obtain:

$$R^2 = \frac{1}{1 - (\frac{x-1}{x})^2} = \frac{x^2}{2x-1} > 1$$

$$R^1 = \frac{x + x\bar{F}(u_1) - 1}{2x - 1} \geq 1$$

Once again, Mechanisms 1 and 2 dominate the generic allocations.

4.5 Monotonicity Comments

Are Equilibrium Thresholds Monotonic in Demand for Mechanism 1? As we saw in Proposition 3, the equilibrium strategy for the Uniform [0,1] case grows linearly in x, the total demand. Furthermore, for $x \geq 2$ we can easily see that this would be true for any distribution. The same assertion would be true if we restricted ourselves to (unique) MC equilibria for either value of x (see proof of Corollary 2. But, the question arises, does this monotonicity property hold in general? In particular, given a utility distribution, are equilibrium thresholds weakly increasing in x? Intuitively, such a relationship would seem to be true, since a higher x implies that there is more to lose by participating in the first round, since participation in the second as a class 2 agent will be less profitable due to high demand. Our numerical experiments seemed to support this fact. However, this turns out not to be true.

Refer to the proof of Proposition 1 and the utility distribution which we constructed. Consider a demand $x_1 < 2$ and the associated utility distribution $\bar{F}(u) = (x_1 - 1)/(x_1 u)$ on a suitably defined support $[b_1, b_2]$ such that we have $E[U] = 1$. In the proof of the proposition we showed that this distribution will have a continuum of equilibria, namely the whole interval $[F^{-1}(1 - 1/x_1), F^{-1}(1/x_1)]$.

Now, let us fix the distribution and consider a demand x_2 such that $x_1 < x_2 \leq 2$. We will show that for a suitably chosen x_2, the equilibrium threshold will be strictly less than $F^{-1}(1/x_1)$, and therefore will be strictly less than some of the equilibria for x_1.

Let us consider u such that $u \geq F^{-1}(1/x_2)$. In this interval we know that the equilibrium equation can be written as:

$$u - E[U]\frac{1}{x_2 F(u)} = 0$$

which implies:

$$\begin{aligned} u - \frac{1}{x_2(1 - \frac{x_1-1}{x_1}\frac{1}{u})} &= 0 \Leftrightarrow u \cdot x_2(1 - \frac{x_1-1}{x_1}\frac{1}{u}) = 1 \Leftrightarrow \\ u \cdot x_2 - \frac{x_1-1}{x_1}x_2 &= 1 \Leftrightarrow u = \frac{1}{x_2} + \frac{x_1-1}{x_1} = \frac{x_1 + (x_1-1)x_2}{x_1 x_2} \end{aligned}$$

Now if we can show that this choice of u is such that:

$$F^{-1}(\frac{1}{x_2}) \leq u < F^{-1}(\frac{1}{x_1})$$

then we will be done. So, let us calculate $F(u)$. We have:

$$\begin{aligned} \bar{F}(u) &= \frac{x_1-1}{x_1}\frac{1}{u} = \frac{(x_1-1)x_2}{x_1+(x_1-1)x_2} \Rightarrow \\ F(u) &= 1 - \frac{(x_1-1)x_2}{x_1+(x_1-1)x_2} = \frac{x_1}{x_1+(x_1-1)x_2} \end{aligned}$$

Now, if we can find x_1, x_2 such that $x_1 < 2, x_2 > x_1, x_2 \leq 2$ and

$$\frac{1}{x_2} \leq \frac{x_1}{x_1+(x_1-1)x_2} < \frac{1}{x_1}$$

then we are done. But this can be easily found. Consider, for example, $x_1 = 1.1, x_2 = 1.3$. This choice of x_1, x_2 yields:

$$\frac{1}{x_2} = .769 < \frac{x_1}{x_1+(x_1-1)x_2} = .894 < \frac{1}{x_1} = .909,$$

and the result is established. ■

The Monotonicity of the Probability of Receiving the Resource. Once again, we would be reasonable in expecting that in both Mechanisms I and II the probability of a class 1 agent receiving the resource is higher in period 2 than it is in period 1. This turns out, however, to not always be the case. Let us take the utility distribution to be the following: $F(u) = (1-e^{-x})/(1-e^{-k})$ for $u \in [0, k]$ and $F(u) = 1$ for $u \geq k$. Now, consider the equilibrium condition for the last threshold $\bar{u}$, and assume that demand is exhausted in both time periods.[8] The equilibrium equation then reduces to:

$$\frac{\bar{u}}{\bar{F}(\bar{u})} = \frac{E[U]}{F(\bar{u})} \Rightarrow \bar{u} = E[U]\frac{\bar{F}(\bar{u})}{F(\bar{u})} = (1 - e^{-k} - ke^{-k})\frac{e^{-\bar{u}} - e^{-k}}{1 - e^{\bar{u}}}$$

Let k now be very large so that $ke^{-k} \approx 0$ and the above equation is close to $\bar{u} = E[U]e^{-\bar{u}}/(1 - e^{-\bar{u}})$. This equation has a unique solution $\bar{u} \approx .807 < 1$. Hence, we will have $F(\bar{u}) > \bar{F}(\bar{u})$. Now, take x such that total demand is exhausted in both time periods. So, we pick x large enough to satisfy the following:

$$\begin{aligned} x\bar{F}(.807) &\geq 1 \\ xF(.807) &\geq 1 \end{aligned}$$

Denoting the probabilities of receiving the resource as a class 1 agent in periods 1 and 2 by $\alpha(1,1), \alpha(1,2)$ respectively, we now obtain,

$$\begin{aligned} \alpha(1,1) &= \frac{1}{x \cdot \bar{F}(\bar{u})} \\ \alpha(1,2) &= \frac{1}{x \cdot F(\bar{u})} \end{aligned}$$

Now, $F(\bar{u}) > \bar{F}(\bar{u})$ immediately implies $\alpha(1,2) < \alpha(1,1)$. Picking x so that $x\bar{F}(.807) = 1$ establishes the result for Mechanism 2. ∎

Again, this is a somewhat unexpected result, since we would expect, if we fix a class i, our probability of getting the resource $\alpha(i,k)$ to be weakly increasing in k.

5 N-Period Models

5.1 General Framework

When there are N time periods, we need to make certain assumptions to fully define the problem. In particular, we assume that the players are able to observe in which class they belong, but are unable to tell the composition of the other classes. Thus, they can only infer the expected value of the composition of other classes, given a certain threshold policy. Let us also define some notation.

[8] Total demand x will be adjusted to make sure this condition holds.

- u_i^k: threshold strategy of class i agent at time k
- U_i^k: expected utility of a class i agent at time k
- S_i^k: supply of resource for class i agent at time k
- D_i^k: demand of resource for class i agent at time k
- P_i^k: probability of being a class i agent at time k
- $\alpha(i,k)$: probability of a class i agent receiving the resource at time k, assuming she participates

All the above quantities are determined by our threshold policy. At this point, we restrict our attention to the analysis of Priority Mechanism 1. Similarly to the 2-period case, a treatment of Mechanism 2 would be virtually identical. So, fixing Mechanism 1, we note the following:

$$
\begin{aligned}
D_i^k &= xP_i^k \bar{F}(u_i^k) \\
S_i^k &= \left(1 - x\sum_{j<i} P_j^k \bar{F}(u_j^k)\right)^+ = \left(1 - \sum_{j<i} D_j^k\right)^+ \\
P_i^k &= P_i^{k-1} F(u_i^{k-1}) + P_{i-1}^{k-1} \bar{F}(u_{i-1}^{k-1}) \\
\alpha(i,k) &= \min\left\{1, \frac{S_i^k}{D_i^k}\right\} \\
U_i^k &= \int_{u_i^k}^{u_{max}} u\alpha(i,k)dF(u) + U_{i+1}^{k+1}\bar{F}(u_i^k) + U_i^{k+1}F(u_i^k)
\end{aligned}
$$

Figure 2 depicts the way that an agent can either stay in the same class or get bumped down a class according to her decision. For example, at time k an agent who is class i (where $1 \leq i \leq k$) can either demand the resource and be bumped down to class $i+1$ in time $k+1$ or she can decline and remain at class i. Now, following the same logic as before we may conjecture that the following vector of thresholds u_i^k, will be a Nash equilibrium.

$$
u_i^k = \min\left\{\frac{U_i^{k+1} - U_{i+1}^{k+1}}{\alpha(i,k)}, u_{max}\right\} \tag{3}
$$

If the denominator of the above fraction is zero (ie. if the supply for an agent of type i at time k is zero) then this means that an agent will get the good with probability 0. Hence, in this case

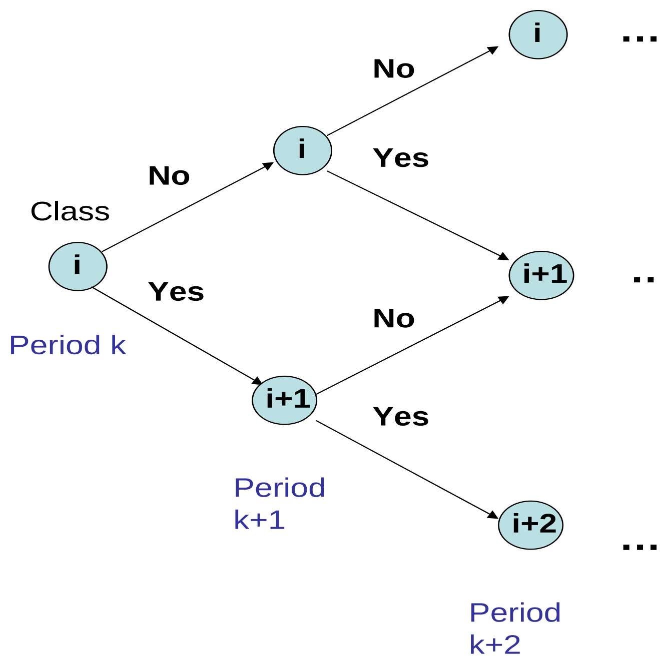

Figure 2: An illustration of priority classes in a multiple period problem. "Yes" and "No" denote an agent's decision to request the resource or not.

it is not worthwhile to participate in the mechanism and we assign a threshold of u_{max}, thereby ensuring non-participation.

We begin our analysis with the following lemma.

Lemma 1 *Let u_i^k be a solution to Equation 3. We have $\alpha(i,k) \geq \alpha(i+1,k)$ for all i and k. Furthermore, given a time period k, we may have exactly one i such that $0 < \alpha(i,k) < 1$.*

Proof. Clearly, if we have $S_{i+1}^k = 0$ the above inequality holds trivially. So let's assume that $S_{i+1}^k > 0$. This implies that:

$$S_{i+1}^k > 0 \Leftrightarrow 1 - x \sum_{j<i+1} P_j^k \bar{F}(u_j^k) > 0 \Leftrightarrow$$
$$1 - x \sum_{j<i} P_j^k \bar{F}(u_j^k) - x P_i^k \bar{F}(u_i^k) > 0 \Leftrightarrow 1 - x \sum_{j<i} P_j^k \bar{F}(u_j^k) - D_i^k > 0 \Leftrightarrow$$
$$\frac{S_i^k}{D_i^k} > 1 \Rightarrow \min\left\{1, \frac{S_i^N}{D_i^N}\right\} = 1$$

Hence we will have:

$$\alpha(i+1,k) = \min\{1, \frac{S_{i+1}^k}{D_{i+1}^k}\} \leq 1 = \min\left\{1, \frac{S_i^k}{D_i^k}\right\} = \alpha(i,k)$$

and the result is proved. Furthermore, notice that the above argument establishes that we can have at most one class receiving the resource with positive probability that is strictly less than 1.

■

Proposition 6 *Let u_i^k be a solution to Equation 3. We have the following holding:*

(i) $U_i^k \geq U_i^{k+1}$

(ii) $U_i^k \geq U_{i+1}^k$

for all i and k.

Proof. First we establish (i). Recall that

$$\begin{aligned} u_i^k &= \min\left\{\frac{U_i^{k+1} - U_{i+1}^{k+1}}{\alpha(i,k)}, u_{max}\right\} \\ U_i^k &= \int_{u_i^k}^{u_{max}} u\alpha(i,k) dF(u) + U_{i+1}^{k+1} \bar{F}(u_i^k) + U_i^{k+1} F(u_i^k) \end{aligned}$$

If $u_i^k = u_{max}$ then $U_i^k = U_i^{k+1}$. If $u_i^k < u_{max}$, then we have $u_i^k \alpha(i,k) + U_{i+1}^{k+1} = U_i^{k+1}$, which implies that $u\alpha(i,k) + U_{i+1}^{k+1} \geq U_i^{k+1}$, for all $u \geq u_i^k$. Thus, $\int_{u_i^k}^{u_{max}} u\alpha(i,k)dF(u) + U_{i+1}^{k+1}\bar{F}(u_i^k) \geq \bar{F}(u_i^k)U_i^{k+1}$, which establishes $U_i^k \geq U_i^{k+1}$. ■

Now we prove (ii) by backward induction on the time period k.

Base Case. First we establish base case:

$$U_i^N \geq U_{i+1}^N$$

Since $U_i^N = E[U]\alpha(i,N)$ and $U_{i+1}^N = E[U]\alpha(i+1,N)$, we simply need to prove:

$$\min\left\{1, \frac{S_i^N}{D_i^N}\right\} \geq \min\left\{1, \frac{S_{i+1}^N}{D_{i+1}^N}\right\}$$

But this is true by applying Lemma 1 to $k = N$. Hence, the base case is proved.

Inductive Step. Let us assume that:

$$U_i^{k+1} \geq U_{i+1}^{k+1}$$

We remind ourselves of the following relations:

$$\begin{aligned}
u_i^k &= \min\left\{\frac{U_i^{k+1} - U_{i+1}^{k+1}}{\alpha(i,k)}, u_{max}\right\} \\
u_{i+1}^k &= \min\left\{\frac{U_{i+1}^{k+1} - U_{i+2}^{k+1}}{\alpha(i+1,k)}, u_{max}\right\} \\
U_i^k &= \int_{u_i^k}^{u_{max}} u\alpha(i,k)dF(u) + U_{i+1}^{k+1}\bar{F}(u_i^k) + U_i^{k+1}F(u_i^k) \\
U_{i+1}^k &= \int_{u_{i+1}^k}^{u_{max}} u\alpha(i+1,k)dF(u) + U_{i+2}^{k+1}\bar{F}(u_{i+1}^k) + U_{i+1}^{k+1}F(u_{i+1}^k)
\end{aligned}$$

and consider two cases:

Case 1 - $S_{i+1}^k = 0$. In this case we have $u_{i+1}^k = u_{max}$ and consequently $U_{i+1}^k = U_{i+1}^{k+1}$. On the other hand U_i^k is the sum of a non-negative number and a convex combination of two numbers weakly greater than U_{i+1}^{k+1}. Hence we will necessarily have:

$$U_i^k \geq U_{i+1}^k$$

Case 2 - $S_{i+1}^k > 0$. Recall that by Lemma 1 we have:

$$\alpha(i,k) \geq \alpha(i+1,k)$$

Hence, it suffices to prove the result for the sub-case $\alpha(i+1,k) = \alpha(i,k) = 1$. We may rewrite

$$\begin{aligned} U_i^k &= \int_{u_i^k}^{u_{max}} udF(u) + U_{i+1}^{k+1}\bar{F}(u_i^k) + U_i^{k+1}F(u_i^k) \\ U_{i+1}^k &= \int_{u_{i+1}^k}^{u_{max}} udF(u) + U_{i+2}^{k+1}\bar{F}(u_{i+1}^k) + U_{i+1}^{k+1}F(u_{i+1}^k) \end{aligned}$$

Let us, once again, distinguish between two cases:

Case 2a - $u_i^k \le u_{i+1}^k$. In this case we have:

$$\int_{u_i^k}^{u_{max}} udF(u) \ge \int_{u_{i+1}^k}^{u_{max}} udF(u)$$

Furthermore, by the inductive hypothesis we have:

$$U_{i+1}^{k+1}\bar{F}(u_i^k) + U_i^{k+1}F(u_i^k) \ge U_{i+2}^{k+1}\bar{F}(u_{i+1}^k) + U_{i+1}^{k+1}F(u_{i+1}^k)$$

These two inequalities establish the result.

Case 2b - $u_i^k > u_{i+1}^k$. In this case, it is useful to write U_i^k and U_{i+1}^k in the following ways.

$$\begin{aligned} U_i^k &= \int_{u_i^k}^{u_{max}} \left(u + U_{i+1}^{k+1}\right)dF(u) + \int_0^{u_{i+1}^k} U_i^{k+1}dF(u) + \int_{u_{i+1}^k}^{u_i^k} U_i^{k+1}dF(u) \\ U_{i+1}^k &= \int_{u_i^k}^{u_{max}} \left(u + U_{i+2}^{k+1}\right)dF(u) + \int_0^{u_{i+1}^k} U_{i+1}^{k+1}dF(u) + \int_{u_{i+1}^k}^{u_i^k} \left(u + U_{i+2}^{k+1}\right)dF(u) \end{aligned}$$

Now we note the following three inequalities:

$$\begin{aligned} &\int_{u_i^k}^{u_{max}} \left(u + U_{i+1}^{k+1}\right)dF(u) \ge \int_{u_i^k}^{u_{max}} \left(u + U_{i+2}^{k+1}\right)dF(u) \\ &\int_0^{u_{i+1}^k} U_i^{k+1}dF(u) \ge \int_0^{u_{i+1}^k} U_{i+1}^{k+1}dF(u) \\ &\int_{u_{i+1}^k}^{u_i^k} U_i^{k+1}dF(u) \ge \int_{u_{i+1}^k}^{u_i^k} \left(u + U_{i+1}^{k+1}\right)dF(u) \ge \int_{u_{i+1}^k}^{u_i^k} \left(u + U_{i+2}^{k+1}\right)dF(u) \end{aligned}$$

It is clear that the first two inequalities follow directly from the induction hypothesis. The third follows from the definition of u_i^k as the threshold value of utility above which it is profitable to participate and compete for the resource for an agent of class i at time k. In particular, recall that for $\alpha(i,k) = 1$ we will have:

$$\begin{aligned} u_i^k &= \min\left\{\frac{U_i^{k+1} - U_{i+1}^{k+1}}{\alpha(i,k)}, u_{max}\right\} \Rightarrow \\ u_i^k &= U_i^{k+1} - U_{i+1}^{k+1} \end{aligned}$$

This implies:

$$U_i^{k+1} \geq u + U_{i+1}^{k+1} \text{ for all } u \leq u_i^k$$

Since in this case we assume $u_{i+1}^k < u_i^k$, this establishes that

$$\int_{u_{i+1}^k}^{u_i^k} U_i^{k+1} dF(u) \geq \int_{u_{i+1}^k}^{u_i^k} (u + U_{i+1}^{k+1}) dF(u)$$

Then we simply use the induction hypothesis to establish the inequality. Putting the three inequalities together concludes the proof of the inductive step. ■

We now turn to proving that the above solution will in fact constitute a Nash equilibrium in the multi-period setting.

Proposition 7 *The solution of Equation 3 is a subgame-perfect Nash equilibrium threshold policy.*

Proof. Let $\{u_i^k\}$ denote the solution to Equation 3. This solution (and any other set of thresholds for that matter) uniquely determines the U_i^k. We wish to show that any deviation from this vector will yield a (weakly) lower expected profit. Recall that:

$$U_i^k = \int_{u_i^k}^{u_{max}} (u\alpha(i,k) + U_{i+1}^{k+1}) dF(u) + \int_0^{u_i^k} U_i^{k+1} dF(u)$$

Assume that all the agents are behaving according to $\{u_i^k\}$ and suppose some agent uses a different vector of thresholds $\tilde{u} = \{\tilde{u}_i^k\}$ where $\tilde{u} \neq u$, which gives rise to a different set of expected utilities, $\tilde{U}_i^k$. We will show that, at every point in time, this threshold strategy can do no better than the equilibrium one. That is, we will prove that:

$$\tilde{U}_i^k \leq U_i^k, \text{ for all } i, k$$

To wit, let us consider $\tilde{u}_{i_l}^{k_l} \neq u_{i_l}^{k_l}$, where k_l is the highest time period index among all elements of $\tilde{u}$ that are not equal to u. We note that U_i^k for $k > k_l$ are not affected by this change in thresholds, i.e. $U_i^k = \tilde{U}_i^k$ for all $k > k_l$ and i. We will have the following holding:

$$\begin{aligned} \tilde{U}_{i_l}^{k_l} &= \int_{\tilde{u}_{i_l}^{k_l}}^{u_{max}} (u\alpha(i_l,k_l) + U_{i_l+1}^{k_l+1}) dF(u) + \int_0^{\tilde{u}_{i_l}^{k_l}} U_{i_l}^{k_l+1} dF(u) \\ &\leq \int_{u_{i_l}^{k_l}}^{u_{max}} (u\alpha(i_l,k_l) + U_{i_l+1}^{k_l+1}) dF(u) + \int_0^{u_{i_l}^{k_l}} U_{i_l}^{k_l+1} dF(u) = U_{i_l}^{k_l} \end{aligned}$$

The inequality follows from the definition of $u_{i_l}^{k_l}$ and the fact that $\tilde{u}_{i_l}^{k_l} \neq u_{i_l}^{k_l}$. We can similarly prove $\tilde{U}_j^{k_l} \leq U_j^{k_l}$ for any other j such that $\tilde{u}_j^{k_l} \neq u_j^{k_l}$. For all other j we have $\tilde{U}_j^{k_l} = U_j^{k_l}$. Thus we have $\tilde{U}_i^{k_l} \leq U_i^{k_l}$ for all $i = 1, ..., k_l$. This will in turn imply for $i = 1, ..., k_l - 1$,

$$\begin{aligned}
\tilde{U}_i^{k_l-1} &= \int_{\tilde{u}_i^{k_l-1}}^{u_{max}} (u\alpha(i, k_l - 1) + \tilde{U}_{i+1}^{k_l}) dF(u) + \int_0^{\tilde{u}_i^{k_l-1}} \tilde{U}_i^{k_l} dF(u) \\
&\leq \int_{\tilde{u}_i^{k_l-1}}^{u_{max}} (u\alpha(i, k_l - 1) + U_{i+1}^{k_l}) dF(u) + \int_0^{\tilde{u}_i^{k_l-1}} U_i^{k_l} dF(u) \\
&\leq \int_{u_i^{k_l-1}}^{u_{max}} (u\alpha(i, k_l - 1) + U_{i+1}^{k_l}) dF(u) + \int_0^{u_i^{k_l-1}} U_i^{k_l} dF(u) = U_i^{k_l-1}
\end{aligned}$$

The second inequality follows from the previously shown $\tilde{U}_{i_l}^{k_l} \leq U_{i_l}^{k_l}$, while the last equality follows from the definition of $u_{i_l-1}^{k_l-1}$. Recursing backwards the result is proven. ■

The following lemma suggests that the maximum in the equilibrium equation is attained by u_{max} only if the probability of an agent receiving the resource is zero. This shows that agents with a positive probability of receiving the resource will always participate in the process, provided their observed utility is high enough.

Lemma 2 *Let u_i^k be a solution to Equation 3. Then for every (i, k) such that $\alpha(i, k) > 0$, we will have $u_i^k = (U_i^{k+1} - U_{i+1}^{k+1})/\alpha(i, k) < u_{max}$. In other words, the minimum in Equation 3 will not be attained by u_{max}.*

Proof. We consider two cases, according to the value of $\alpha(i, k)$.

Case 1 - $0 < \alpha(i, k) < 1$. Recall that we have defined $\alpha(i, k) = \min \{1, S_i^k / D_i^k\}$. Hence we will have $0 < \alpha(i, k) = S_i^k / D_i^k < 1$. This implies that $S_i^k > 0$ and $D_i^k > 0$. But recall that $D_i^k = x P_i^k \bar{F}(u_i^k)$. Thus, $D_i^k > 0 \Rightarrow u_i^k < u_{max}$, since $\bar{F}(u) = 0$ for all $u \geq u_{max}$. This in turn implies that we must have $u_i^k = (U_i^{k+1} - U_{i+1}^{k+1})/\alpha(i, k) < u_{max}$. Note that we do not consider the degenerate case when $S_i^k = D_i^k = 0$. Since $S_i^k = 0$ we can simply take $\alpha(i, k) = 0$, no matter what demand we may have.

Case 2 - $\alpha(i, k) = 1$. Let us look at the quantity $U_i^k - U_{i+1}^k$. By Proposition 6, we have $U_{i+1}^k \geq U_{i+1}^{k+1}$.

Thus we obtain that,

$$\begin{aligned}
U_i^k - U_{i+1}^k &\leq U_i^k - U_{i+1}^{k+1} = \alpha(i, k) \int_{u_i^k}^{u_{max}} u dF(u) + \bar{F}(u_i^k) U_{i+1}^{k+1} + F(u_i^k) U_i^{k+1} - U_{i+1}^{k+1} \\
&< u_{max} \bar{F}(u_i^k) + F(u_i^k)(U_i^{k+1} - U_{i+1}^{k+1})
\end{aligned}$$

Now we can establish the desired inequality via a simple backward induction argument. Clearly we will have $U_i^N - U_{i+1}^N < u_{max}$. Now, assume the inductive hypothesis $U_i^{k+1} - U_{i+1}^{k+1} < u_{max}$. Then the previous inequality will imply:

$$U_i^k - U_{i+1}^k < u_{max}(\bar{F}(u_i^k) + F(u_i^k)) = u_{max}$$

This completes the inductive step and the proof is complete since $u_i^k = U_i^k - U_i^{k+1} < u_{max}$. ■

Finally, we note the following lemma.

Lemma 3 *The equilibrium thresholds satisfy the following relation:*

$$\begin{aligned}\alpha(i,k-1)u_i^{k-1} &= \alpha(i,k)u_i^k + \alpha(i,k)\int_{u_i^k}^{u_{max}}(u-u_i^k)dF(u) \\ &- \alpha(i+1,k)\int_{u_{i+1}^k}^{u_{max}}(u-u_{i+1}^k)dF(u)\end{aligned}$$

Proof. We begin with the following observation:

$$\begin{aligned}U_i^k &= \int_{u_i^k}^{u_{max}}(u\alpha(i,k) + U_{i+1}^{k+1})dF(u) + F(u_i^k)U_i^{k+1} \Rightarrow \\ U_i^k &= \alpha(i,k)\int_{u_i^k}^{u_{max}} udF(u) + \bar{F}(u_i^k)(U_{i+1}^{k+1} - U_i^{k+1}) + U_i^{k+1} \Rightarrow \\ U_{i+1}^k &= \alpha(i+1,k)\int_{u_{i+1}^k}^{u_{max}} udF(u) + \bar{F}(u_{i+1}^k)(U_{i+2}^{k+1} - U_{i+1}^{k+1}) + U_{i+1}^{k+1}\end{aligned}$$

Subtracting the last two equations leads to:

$$\begin{aligned}U_i^k - U_{i+1}^k = U_i^{k+1} - U_{i+1}^{k+1} &+ \int_{u_i^k}^{u_{max}} \alpha(i,k)u + U_{i+1}^{k+1} - U_i^{k+1}dF(u) \\ &- \int_{u_{i+1}^k}^{u_{max}} \alpha(i+1,k)u + U_{i+2}^{k+1} - U_{i+1}^{k+1}dF(u)\end{aligned}$$

Recall that the equilibrium thresholds are such that:

$$u_i^k = \min\left\{\frac{U_i^{k+1} - U_{i+1}^{k+1}}{\alpha(i,k)}, u_{max}\right\}$$

Thus the last equation becomes:

$$\alpha(i,k-1)u_i^{k-1} = \alpha(i,k)u_i^k + \alpha(i,k)\int_{u_i^k}^{u_{max}}(u-u_i^k)dF(u) - \alpha(i+1,k)\int_{u_{i+1}^k}^{u_{max}}(u-u_{i+1}^k)dF(u)$$

Note that if $u_i^k = u_{max}$ we have that $\alpha(i,k) = 0$ so the equation still holds. ■

5.2 Limiting Equilibrium as x grows

In this section we note an interesting property of our model. Namely, that given any utility distribution $U \sim F(\cdot)$ and any number of time periods N, there exists a high enough value of aggregate demand x, call it $x(F,N)$, such that for all $x \geq x(F,N)$ we can easily identify an equilibrium strategy that is insensitive to aggregate demand.

Our intuition derives from the simple fact that $\alpha(i,k) < 1 \Rightarrow \alpha(i,k+1) = 0$. So, if we assume that $\alpha(1,k) < 1$ for all k then we are left with a simple set of $N-1$ equations which can be solved via backwards recursion. We formally state our result with the following proposition.

Proposition 8 *Given a utility distribution $F(\cdot)$ and a time horizon N, there exists a threshold demand level $x(F,N)$ such that for all $x \geq x(F,N)$ there exists a common equilibrium strategy.*

Proof. The proof is by construction. First, let us assume that $\alpha(1,k) \leq 1$ for all time periods k and that $\alpha(j,k) = 0$ for all $j \neq 1$ and k. So we need only consider class 1 agents at any time period. We thus suppress the thresholds' dependence on class and let u_k denote the threshold for time period k. The equilibrium conditions can be rewritten as:

$$\frac{u_k}{xF(u_1)F(u_2)...\bar{F}(u_k)} = U_1^{k+1}$$

Recall that,

$$\begin{aligned}
U_1^N &= \alpha(1,N)E[U] = \frac{E[U]}{xF(u_1)F(u_2)...F(u_{N-1})} \\
U_1^k &= \int_{u_k}^{u_{max}} \alpha(1,k)udF(u) + U_1^{k+1}F(u_k) = \frac{\int_{u_k}^{u_{max}} udF(u)}{xF(u_1)F(u_2)...\bar{F}(u_k)} + U_1^{k+1}F(u_k)
\end{aligned}$$

With these equations we can easily recurse backwards and compute the thresholds. This leads to a set of thresholds u_k such that $u_k \in (0, u_{max})$ for all $k \in \{1, ..., N-1\}$ (obviously, $u_N = 0$). Now, having computed the u_k, let us take $x(F,N)$ to be the minimum value of x such that the following conditions hold:

$$\begin{aligned}
x \cdot F(u_1)F(u_2)...F(u_{N-1}) &\geq 1 \\
x \cdot F(u_1)F(u_2)...\bar{F}(u_{N-1}) &\geq 1 \\
x \cdot F(u_1)F(u_2)...\bar{F}(u_{N-2}) &\geq 1 \\
&... \\
x \cdot \bar{F}(u_1) &\geq 1
\end{aligned}$$

This will ensure feasibility of the threshold vector we have computed via the recursive procedure outlined above. Since for $x \geq x(F,N)$ feasibility is not affected, the same vector will still be an equilibrium. ∎

Proposition 9 *The equilibrium vector computed above is unique and strictly monotone decreasing in the time period k.*

Proof. First let us show the monotonicity property. We have for $1 \leq k \leq N-1$,

$$\frac{u_k}{xF(u_1)F(u_2)...\bar{F}(u_k)} = U_1^{k+1}$$

and,

$$\begin{aligned} U_1^N &= \frac{E[U]}{xF(u_1)F(u_2)...F(u_{N-1})} \\ U_1^k &= \frac{1}{xF(u_1)F(u_2)...\bar{F}(u_k)} \int_{u_i}^{u_{max}} udF(u) + U_1^{k+1}F(u_k) \end{aligned}$$

Since $\alpha(1,k) > 0$ for all k, this implies that $u_k < u_{max}$ for all k. From the proof of Proposition 6 we can immediately note that

$$U_1^k > U_1^{k+1}$$

Now let us compare u_k and u_{k+1} for $k \in \{1, N-2\}$. We have the following holding:

$$\begin{aligned} \frac{u_k}{xF(u_1)F(u_2)...\bar{F}(u_k)} &= U_1^{k+1} \\ \frac{u_{k+1}}{xF(u_1)F(u_2)...\bar{F}(u_{k+1})} &= U_1^{k+2} \end{aligned}$$

denoting $C = x \cdot F(u_1)F(u_2)...F(u_{k-1})$, we get

$$\begin{aligned} \frac{u_k}{\bar{F}(u_k)} &= C \cdot U_1^{k+1} \\ \frac{u_{k+1}}{\bar{F}(u_{k+1})} &= C \cdot F(u_k)U_1^{k+2} \end{aligned}$$

Thus we can write:

$$\frac{u_k}{\bar{F}(u_k)} > \frac{u_{k+1}}{\bar{F}(u_{k+1})}$$

And since the function $f(x) = \frac{x}{\bar{F}(x)}$ is strictly monotone increasing we will have that:

$$\frac{u_k}{\bar{F}(u_k)} > \frac{u_{k+1}}{\bar{F}(u_{k+1})} \Rightarrow u_k > u_{k+1}$$

We finally note that $u_N = 0$ so that $u_{N-1} > u_N$.

Now we show that the solution computed with the above procedure will be unique. This follows from a simple observation. Let us begin solving for u_{N-1} and then recurse backwards to compute the other thresholds. We will have:

$$u_{N-1} - \frac{\bar{F}(u_{N-1})}{F(u_{N-1})} E[U] = 0$$

which clearly admits a unique solution since $\frac{\bar{F}(u_{N-1})}{F(u_{N-1}}$ is monotone decreasing. Now let us go a step backwards and compute u_{N-2}. We will have:

$$\begin{aligned}
\frac{u_{N-2}}{xF(u_1)F(u_2)...\bar{F}(u_{N-2})} &= U_1^{N-1} \Rightarrow \\
\frac{u_{N-2}}{xF(u_1)F(u_2)...\bar{F}(u_{N-2})} &= \frac{\int_{u_{N-1}}^{u_{max}} udF(u)}{xF(u_1)F(u_2)...F(u_{N-2})\bar{F}(u_{N-1})} \\
&+ \frac{E[U]F(u_{N-1})}{xF(u_1)F(u_2)...F(u_{N-2})F(u_{N-1})} \Rightarrow \\
\frac{u_{N-2}}{\bar{F}(u_{N-2})} F(u_{N-2}) &= g(u_{N-1})
\end{aligned}$$

Thus, since $u_{N-2}F(u_{N-2})/\bar{F}(u_{N-2})$ is monotone increasing and since $g(u_{N-1})$ is a function of u_{N-1} and therefore a uniquely determined constant, u_{N-2} will also be uniquely determined.

In general, as we recurse backwards we will be solving equations of the form:

$$\frac{u_k}{\bar{F}(u_k)} F(u_k) = g(u_{k+1}, u_{k+2}, ..., u_{N-1})$$

Thus, at every step of the way the thresholds will be uniquely determined. ■

5.3 Comparison to Canonical Allocations

In this section, we generalize the results of earlier sections and show that the efficiency of market clearing equilibria is not too much lower than that of the efficient allocation, and in fact outperform the random one. Let u_E denote the value of u such that $\bar{F}(u) = \frac{1}{x}$.

Proposition 10 *Market clearing equilibria of priority mechanisms are ex-ante more efficient than the random allocation. The ratio between the ex-ante efficiency of market clearing equilibria to the efficient allocation is bounded from below by the quantity* $E[U]/E[U|U > u_E]$.

Proof. It is clear that the ex-ante utility of the efficient allocation is equal to:

$$N\bar{F}(u_E)E[U|U > u_E] = \frac{N}{x}E[U|U > u_E]$$

By contrast, the random allocation's ex-ante utility is given by:

$$\frac{N}{x}E[U]$$

Now let us fix a market clearing equilibrium strategy. Let P_i^k denote the probability of being a class i agent at time k, and let R_i^k denote the expected reward such an agent derives at time k. The market clearing equilibrium utility will be equal to the following quantity:$\sum_{k=1}^N \sum_{i=1}^k P_i^k R_i^k$.

Consider the term $\sum_{i=1}^k P_i^k R_i^k$ for arbitrary k. Since we assume the equilibrium is market clearing, there exists i^* such that $\alpha(i,k) = 1$ if $i < i^*$, $\alpha(i^*, k) < 1$, and $\alpha(i,k) = 0$ if $i > i^*$.

Thus, we have that:

$$\begin{aligned}
P_i^k R_i^k &= P_i^k \int_{u_i^k}^{u_{max}} udF(u) = xP_i^k \bar{F}(u_i^k)\frac{E[U|U > u_i^k]}{x}, \text{ if } i < i^* \\
P_i^k R_i^k &= P_i^k \frac{1 - x\sum_{i<i^*} P_i^k \bar{F}(u_i^k)}{xP_i^k \bar{F}(u_i^k)} \int_{u_i^k}^{u_{max}} udF(u) = (1 - x\sum_{i<i^*} P_i^k \bar{F}(u_i^k))\frac{E[U|U > u_i^k]}{x}, \text{ if } i = i^* \\
P_i^k R_i^k &= 0, \text{ if } i^* < i \leq k
\end{aligned}$$

Since $E[U|U > u_i^k] \geq E[U|U > 0] = E[U]$, we will have:

$$\sum_{i=1}^k P_i^k R_i^k \geq \frac{E[U]}{x} \quad \forall k$$

which in turn implies:

$$\sum_{k=1}^N \sum_{i=1}^k P_i^k R_i^k \geq N\frac{E[U]}{x}$$

So immediately we note that priority mechanisms outperform the random allocation. Furthermore, the ratio between the ex-ante efficiency of market clearing equilibria to the efficient allocation is bounded from below by:

$$\frac{N\frac{E[U]}{x}}{N\frac{E[U|U>\tilde{u}]}{x}} = \frac{E[U]}{E[U|U > u_E]}$$

■

6 Extensions

6.1 Inefficiency of non-MC equilibria

Unfortunately, we cannot guarantee that the canonical priority mechanisms will always be efficient. In the following example we illustrate how a non MC equilibrium strategy can do worse that even a random allocation. This suggests that, in certain cases, our mechanism introduces a non-trivial tradeoff between efficiency and equity.

Example 2. This argument applies to both Mechanisms. Fix $x < 2$ and consider the distribution in the proof of Proposition 1. We have a multiplicity of equilibria to choose from, so let's pick the highest one, namely $\bar{u} = F^{-1}(1/x)$. Since $\bar{F}(u) = x - 1/(xu)$ we will have:

$$1 - \bar{F}(\bar{u}) = \frac{1}{x} \Rightarrow \bar{F}(\bar{u}) = \frac{x-1}{x} \Rightarrow \frac{x-1}{x}\frac{1}{\bar{u}} = \frac{x-1}{x} \Rightarrow \bar{u} = 1$$

Recall that $E[U] = 1$. Thus, the utility of the random allocation will simply be $2/x$.

Now, for our choice of $\bar{u}$ we will have the following utility:

$$\int_1^{b_2} u\frac{x-1}{x}\frac{1}{u^2}du + \frac{1}{x}E[U] = \frac{x-1}{x}[\ln(b_2) - \ln(1)] + \frac{1}{x} = \frac{x-1}{x}\ln(b_2) + \frac{1}{x}$$

For concreteness, let us fix $b_1 = 1/2 \Rightarrow b_2 = 1/2e^{\frac{x}{(x-1)^2}}$. Now if we pick $x = 1.9$ we will have $b_2 = 1/2e^{\frac{1.9}{1.8}} < e \Rightarrow \ln(b_2) < 1$. This in turn implies that

$$\frac{x-1}{x}\ln(b_2) + \frac{1}{x} < 1 < \frac{2}{x}$$

■

6.2 A Priority Generalization

The discussion of the previous section exemplifies the inefficiency that may arise in the priority scheme we propose. Unfortunately, there is no way to guarantee that a non-MC clearing equilibrium will be able to do at least as well as a completely random allocation. To rectify this problem, we need to be more flexible in our priority scheme.

Recall that in Mechanism 1 we assume that if an agent chooses to not demand the resource then he retains his priority class status in the next time period *with probability 1*. Conversely, if she chooses to demand the resource then she gets bumped down a priority class in the next time period *with probability 1*. This 0-1 dimension of our priority scheme needs to be relaxed if we want to get rid of non-MC equilibria and the efficiency issues they might pose.

We introduce the following generalization to our model. Consider a class i agent at time k. Assume that, given that she chooses to participate, her probability of "getting lucky" and keeping her priority status is $p^1_{i,k}(u)$. If she chooses to delay participation, then her probability of keeping her priority status is $p^2_{i,k}(u)$ (for simplicity the above are abbreviated by p^1, p^2). Then, our notion of fairness would imply that $p^2 > p^1$ -indeed if this is not the case, there is never any incentive not to participate. Note that this is a direct generalization of Mechanisms 1 and 2. In Mechanism 1 we assumed $p^1 = 0, p^2 = 1$, while in Mechanism 2 our scheme was $p^1 = 1 - \alpha(i,k), p^2 = 1$.

Given this new environment we can rewrite the equilibrium equations as:

$$u^k_i \alpha(i,k) + p^1_{i,k}(u)U^{k+1}_i + (1 - p^1_{i,k}(u))U^{k+1}_{i+1} = p^2_{i,k}(u)U^{k+1}_i + (1 - p^2_{i,k}(u))U^{k+1}_{i+1}$$

which may be rearranged into:

$$u^k_i = \min\left\{(p^2_{i,k}(u) - p^1_{i,k}(u))\frac{(U^{k+1}_i - U^{k+1}_{i+1})}{\alpha(i,k)}, u_{max}\right\} \tag{4}$$

Furthermore, the dynamics of the game can be generalized to:

$$\begin{aligned}
D^k_i &= xP^k_i \bar{F}(u^k_i) \\
S^k_i &= \left(1 - \sum_{j<i} D^k_j\right)^+ \\
\alpha(i,k) &= \min\left\{1, \frac{S^k_i}{D^k_i}\right\} \\
P^k_i &= P^{k-1}_i[F(u^{k-1}_i)p^2_{i,k-1}(u) + \bar{F}(u^{k-1}_i)p^1_{i,k-1})] \\
&+ P^{k-1}_{i-1}[F(u^{k-1}_{i-1})(1 - p^2_{i-1,k-1}(u)) + \bar{F}(u^{k-1}_{i-1})(1 - p^1_{i-1,k-1}(u)) \\
U^k_i &= \int_{u^k_i}^{u_{max}} u\alpha(i,k)dF(u) + [\bar{F}(u^k_i)p^1_{i,k} + F(u^k_i)p^2_{i,k}]U^{k+1}_i \\
&+ [\bar{F}(u^k_i)(1 - p^1_{i,k}) + F(u^k_i)(1 - p^2_{i,k})]U^{k+1}_{i+1}
\end{aligned}$$

Now we are ready to state some easy results:

Proposition 11 *Let u^k_i be a solution to Equation 4. We have the following inequalities holding:*

(i) $U^k_i \geq U^k_{i+1}$

(ii) $U^k_i \geq U^{k+1}_i$

Proof. Identical to Proposition 6. ■

Given this new structure, we can always find probabilistic priority schemes which yield MC equilibria.

Corollary 3 *The following scheme yields MC equilibria:*

- $p_{i,k}^2(u) = 1 - \alpha(i,k)$, $p_{i,k}^1(u) \leq p_{i,k}^2(u)$

Proof. We note the attractive property that when $\alpha(i,k) = 0$ an agent always keeps her priority in the next time period. Now, assume that we have an equilibrium that is not MC. This would imply that all $\alpha(i,k) = 1$. But, applying this to Equation 4 would in turn ensure that $u_i^k = 0$ for all i. But then the market would most certainly clear since $x > 1$. ■

This brief argument establishes that an enhanced priority framework can do away with the kinds of inefficiencies that plague simpler models, while still maintaining their appealing fairness properties. For this reason, it deserves a vigorous investigation, which is left for future research.

7 Conclusion

In this paper, we develop a novel framework for the repeated allocation of a scarce resource. We focus on two intuitive priority mechanisms that aim to instill a measure of fairness to the dynamic allocation procedure. They do so by dynamically placing agents into priority classes in a way that takes into account their prior allocation history.

Focusing on market-clearing equilibria, we are able to show that, in addition to having appealing fairness properties, priority mechanisms are relatively efficient. However, the bounds that we establish are extremely crude and do not provide much insight as to the actual efficiency of a priority mechanism vis-a-vis the generic allocations. In addition, a major concern is that sometimes market-clearing equilibria may not exist. In those instances our mechanisms can be considerably inefficient, faring worse than even a completely random allocation. A way of overcoming this problem is by relaxing our priority framework adding flexibility to the way that priority classes are determined. We have only scratched the surface of this topic in Section 6, and a more complete examination is warranted.

While our treatment of the 2-period case is comprehensive, multiple-period models present significant challenges. In some special cases, as when aggregate demand is high enough, we are able to make detailed and meaningful statements about the structure of equilibrium strategies. Unfortunately, we cannot extend the scope of our insights much further. The complexity of the dynamic game that we examine makes the establishment of even the most basic structural results, let alone the actual computation of closed-form equilibria, extremely difficult. It is not entirely clear how this situation can be remedied, but we note that such concerns are common in these kinds of models. Even Casella's [3] notable work in storable votes suffers from this kind of intractability. One potential avenue of future research that may be amenable to more structural insights would be to examine models having any or all of the following properties: a limited number of priority classes; a discrete number of agents; an infinite time horizon.

A deeper criticism of this work involves its most basic assumptions regarding agents' ability to collect information. Recall that in the outline of our model in Section 2, we stated that as the allocation procedure moves forward, agents are not able to observe the composition of different priority classes and adapt their strategies accordingly. Instead, the best they can do is make ex-ante expected value calculations about the status quo in the market. So, in a sense, while our problem is a dynamic one, we are imposing a non-adaptive static structure to it. We view this assumption as a serious objection to our model, but note that it is not completely unreasonable in the context of very large markets. Furthermore, an adaptive model would present even greater analytical difficulties than the -already substantial- ones that we encounter in our work.

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