G6512, Fall 2011
Columbia University
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Assignment 2, due September 26 in class.

You are encouraged to work in groups on the assignment, however, you must turn in an individual solution.

Question 1

This question is designed to illustrate Blackwell's Theorem (Th. 3.3, pp. 54, Lucas and Stokey). The theorem presents conditions that are *sufficient* for a mapping T to be a contraction. This question illustrate that those conditions are not *necessary.*

Consider the mapping:

$$\begin{aligned} T(v) &= \max_{0\leq\lambda\leq A+1-\delta} \frac{(A+1-\delta-\lambda)^{1-\sigma}}{1-\sigma} + \beta\lambda^{1-\sigma} v \\ \text{where } \sigma &> 1 \text{ and } \beta(A+1-\delta)^{1-\sigma} < 1 \end{aligned}$$

(a) Show that $T(v) = +\infty$ for $v > 0$ and $T(0) = \frac{(A+1-\delta)^{1-\sigma}}{1-\sigma}$.

(b) Show that the derivative of $T(v)$ at $v = v_0 < 0$ is given by:

$$\frac{\partial T(v)}{\partial v}|_{v=v_0<0} = \beta\lambda(v_0)^{1-\sigma} > 0,$$

where

$$\lambda(v_0) = \arg\max_{0\leq\lambda\leq A+1-\delta} \frac{(A+1-\delta-\lambda)^{1-\sigma}}{1-\sigma} + \beta\lambda^{1-\sigma} v_0.$$

(c) Explain why T does not satisfy the conditions in Theorem 3.3 pp. 54. (Hint: Is discounting satisfied for T on the relevant space for which T is defined?)

(d) What happens to $\lambda(v)$ as $v \to -\infty$?

(e) What does the graph of $T(v)$ versus v look like for $v \leq 0$? Does it cross the 45ˆ in the negative orthant? Draw the graph by hand, highlighting its qualitative features.

(f) Explain, using the graph you just developed, why $T^j(v_0) \to v^*$ for $j \to \infty$, for every $v_0 \leq 0$, where v^* is unique.

Question 2

Stokey-Lucas ex. 5.17a-b pp. 126-7

Question 3

Suppose the representative household has the following preferences:

$$\sum_{j=0}^{+\infty} \beta^j \frac{\left[c_{t+j}\left(1-l_{t+j}\right)^{\eta}\right]^{1-\gamma}}{1-\gamma},\ 0<\beta<1.$$

The technology for producing consumption goods is:

$$c_t \leq A\left(k_{c,t}\right)^{\alpha} l_t^{1-\alpha},\ 0<\alpha<1.$$

The technology for producing investment goods is:

$$I_t \leq b k_{I,t},\ b>0.$$

The technology for increasing capital is:

$$\mathrm{k}_{t+1}=(1-\delta) k_t+I_t,\ 0<\delta<1.$$

The allocation of capital must satisfy:

$$k_t=k_{It}+k_{ct},$$

where k_0 is given. Also, parameter values satisfy:

$$b>\delta,\ \beta(1-\delta+\beta)^{\sigma(1-\gamma)}<1.$$

a) Define a sequence of markets equilibrium for this economy and show that the allocation in that equilibrium coincides with the efficient allocation i.e. the one that solves the planning problem. Provide an explicit formula for the price of investment goods P_{It} in equilibrium. P_{it} is the price of I_t in units of period t consumption goods.

b) Provide a formula for the equilibrium growth rate of capital: $\lambda_t=k_{t+1}/k_t$.

c) Let y_t denote aggregate output measured in consumption units for this economy. Provide a formula in terms of model parameters for:

$$\lambda_y=y_{t+1}/y_t,\ \lambda_c=c_{t+1}/c_t,\ \lambda_{P_I}=P_{It+1}/P_{It}.$$

d) Can you find parameter values for which $\lambda_y=1.015$, $\lambda_{P_I}=0.97$ and $wl/y=2/3$ (i.e. the share of labor in total income)?

Question 4

(Stokey-Lucas ex. 6.7a-g pp. 157-8)

Question 5

Problem 2.9, Lucas and Stokey pp. 28