Econ G6215, Fall 2011
Columbia University
Prof. Stefania Albanesi

Assignment 3, due October 4, 2011 in Nicolas' mailbox.

You are encouraged to work in groups on the assignment, however, you must turn in an individual solution.

Question 1

Suppose that a planner chooses to maximize, by choice of $c_0, c_1, c_2, ...$, the objective function:

$$u(c_0) + \delta\left[\beta u(c_1) + \beta^2 u(c_2) + ...\right], u(c_t) = \ln(c_t), \tag{1}$$

subject to:

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, \tag{2}$$

for $t \geq 0$ with k_0 given, where $0 < \delta < \beta < 1$. If $\delta = 1$, this is planning problem for the canonical growth model.

a) Let $k_{t+1} = g(k_t)$ denote the policy function that solves this problem, for $t = 0, 1, 2, ...$ From the perspective of period 0, the part of the problem that starts at $t = 1$ is the same as the problem starting at time 0 with $\delta = 1$. From our analysis of the canonical growth model, we know that the optimized value of $u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + ...$ has the form $v(k_1)$. We can also solve for the function v, which, given the assumption on u, will have a simple log-linear form.

Derive $v(k_1)$ and using the properties of v, show that the optimal choice of k_1 has the form:

$$k_1 = g k_0^\alpha,$$

where g is a scalar. Obtain an explicit formula relating g to the parameters of the model, β, α, δ. How does the capital accumulation rate from period $t = 1$ onward compare with the capital accumulation rate at $t = 0$?

b) Is there a unique k^* with the property that $k_t \to k^*$ as $t \to \infty$ for all k_0? Display a formula relating k^* to the parameters of the model.

c) Suppose $\beta = 1/1.03$, $\alpha = 0.36$, $\delta = 0.8$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, ..., k_5$ that solve the problem at date $t = 0$.

d) Suppose that at date 1 the planner decides to re-optimize with respect to $k_2, k_3, ...$ The initial condition for this problem is k_1. From the perspective of date $t = 1$, the planner's preferences over sequences of consumption are given by:

$$u(c_1) + \delta\left[\beta u(c_2) + \beta^2 u(c_3) + ...\right]$$

and the resource constraint is still (2) for $t \geq 1$, with k_1 given. What values will the planner choose for k_2, k_3, k_4, k_5? Are they the same as those chosen at time 0? Why not? If the planner decides to re-optimize in this way every period, to what value will k_t actually tend?

e) The planner in this problem is time inconsistent. If allowed to re-optimize in each period, her decisions would be based on the following reasoning: " I am very impatient today (the discount factor is $\delta\beta$), but I will be a lot more patient from tomorrow onward (the discount factor in all future periods is β). I will consume a lot today, but I will consume less from tomorrow onward." Time inconsistent preferences have been used to explain addiction (Gary Becker) and low saving (David Laibson, and Tony Smith and Per Krusell, various papers).

Here, you will explore an alternative solution concept for models with time inconsistent preferences. This solution concept treats the planner in different periods as a different decision makers not bound by decisions made in previous periods. A planner is considered as a separate "self" at each date t. Each self takes as given the optimal choices that future selves will make, and decisions of past selves only have an effect on the current decision via the outstanding level of capital k_t. The decisions of future selves can be represented as a policy function:

$$k_{t+j+1} = dk_{t+j}^{\alpha}, for j > 0. \tag{3}$$

Given that future selves set capital according to (3), we characterize the optimal capital accumulation decision for the date t self. This amounts to solving for a Nash equilibrium in the game between the current self and all future selves.

i) Let $v(k_t; d)$ denote the present discounted value of utility starting at time t, if preferences are given by the analogue of (1) at time t, and in all periods $t > t$ capital is set according to (3). Display an explicit formula for $v(k_t; d)$.

ii) Let $g(k_t; d)$ denote the policy rule for a planner with preferences (1), who expects her future selves will follow policy (3). Show that $g(k; d) = D(d)\, k^{\alpha}$ and derive an explicit formula for $D(d)$.

iii) The equilibrium value of d is a d^* that solves: $d^* = D(d^*)$. The equilibrium capital accumulation rate is the fixed point of the map $D(d)$. Display a formula relating d^* to the parameters of the model. How does the value of utility in equilibrium in this concept compare with the optimal value of utility in a)-b)?

Question 2

Consider the canonical growth model with $u(c) = \ln(c)$, $\beta = 1/1.03$, $\delta = 0.10$, $\alpha = 0.36$.

a) Compute the value of capital corresponding to the stationary point, k^*, of this economy. (You do not need to reproduce the derivation. See the handout on characterizing the policy function.)

b) Compute the first order Taylor series expansion of the policy rule at $k = k^*$. Using the resulting approximate policy function, compute k_1, k_2, k_3, k_4, k_5 for two different initial values of k: i) $k_0 = 0.9k^*$, ii) $k_0 = 0.5k^*$.

c) Compute the second order Taylor series expansion of the policy rule around k^* in case i) and in case ii). Is the approximate policy rule derived from the second order Taylor "better" than the one derived in point b)? How would you be able to tell?

Question 3

Suppose the representative household has the following preferences:

$$ {}_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}. $$

The resource constraint is:

$$ c_t + i_t \leq k_t^{\alpha}, k_{t+1} = (1-\delta) k_t + i_t q_t, $$

where

$$ q_t = \gamma^t, \gamma > 1. $$

a) Derive the formula for the ratio i_t/k_t, along the balanced growth path for this economy. Express this as a function of σ, β, α, δ.

b) For a closed economy, i_t/k_t is equal to the saving rate. A case has been made that γ increased in the 1980's, and that at the same time, the savings rate fell in the US. Are there parameter values that can give rise to this joint outcome in this model?