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1 Three Notions of Private Sector Equilibrium in DGE Models

The purpose of this handout is to illustrate three equilibrium concepts for dynamic general equilibrium economies. These concepts will be described for the canonical real business cycle model and are easily extendable to most infinitely lived economies in macroeconomics. Stokey and Lucas with Prescott (1989), chapter 2.3 is the main reference.

For overlapping generations economies, see Prescott and Rios-Rull (2000).

1.1 A canonical economy

The economy is populated by households and firms. There is no risk, time is discrete and infinite.

Households own all factors of production and the capital stock, k_t. The initial level of capital stock is given by k_0. Factor and share ownership is equally distributed among households. Rental markets for labor and capital and the goods market are competitive.

Households' preference ordering over current and future consumption and labor at time $t \geq 0$ is given by:

$$\sum_{j \geq t} \beta^{j-t} u\left(c_j, n_j\right), \tag{1}$$

where $u(\cdot)$ is strictly concave, increasing in c and decreasing in n.

Firms own nothing. They rent labor and capital on factor markets each period, sell output on the goods market, and transfer any profits back to households. Their production technology is:

$$y \leq f(n, k), \tag{2}$$

where f is a homogeneous of degree 1 function, increasing in n and k and concave. Firms maximize profits by choice of factor inputs.

The size and number of firms are indeterminate in equilibrium, due to the constant-returns to scale assumption.

1.2 Arrow-Debreu Equilibrium

In an Arrow-Debreu equilibrium, all transactions take place in a single once-and-for-all market that meets at time 0. All quantities and prices are determined simultaneously at that time and no further trades are negotiated later. In periods $t = 0, 1, 2, ...T$, for $T \to \infty$, agents simply deliver/receive quantities of factors and goods contracted on the time 0 market.

Then, p_t is the price in units of output at time 0 of one unit of output delivered in period t for $t = 0, 1, ...t$. It is therefore a *forward* price. Analogously, w_t and r_t denote the rental price in terms of output at time

0 of one unit of labor and capital, respectively, at time t. The price system for this equilibrium is given by $\{p_j, w_j, r_j\}_{j=0}^{T}$.

Households maximize (1) for $t = 0$, by choice of $\{c_j, n_j, k_{j+1}\}_{j=0}^{T}$. They are subject to the constraints:

$$x_t = k_{t+1} - k_t(1-\delta), \tag{3}$$

$$\sum_{t=0}^{T} [p_t(c_t + x_t) - w_t n_t - r_t k_t] - \pi \leq 0, \tag{4}$$

$$c_t \geq 0,\ n_t \in [0,1],\ x_t \geq 0, \tag{5}$$

with k_0 given. π denotes profits of firms.

Firms chose factor demands to solve the following problem:

$$\pi = \max_{n_t^d, k_t^d} \sum_{t=0}^{T} \left[p_t y_t - w_t n_t^d - r_t k_t^d\right], \tag{6}$$

subject to (2). The first order conditions of this problem are:

$$\begin{aligned} r_t &= p_t f_k(n_t, k_t), \\ w_t &= p_t f_n(n_t, k_t) \end{aligned}$$

for $t = 0, 1, \ldots T$.

Definition 1 *An Arrow-Debreu equilibrium is a price system* $\{p_t, w_t, r_t\}_{t=0}^{T}$, *with* $p_t, w_t, r_t \geq 0$ *for all* t, *an allocation for the representative firm* $\{y_t, k_t^d, n_t^d\}_{t=0}^{T}$ *and an allocation for the representative household* $\{c_t, n_t, x_t, k_t\}_{t=0}^{T}$ *such that:*

1. $\{y_t, k_t^d, n_t^d\}_{t=0}^{T}$ *solves problem (6) for the stated price system;*
2. $\{c_t, n_t, x_t, k_t\}_{t=0}^{T}$ *maximizes (1) subject to (3), (4) and (5) for the stated price system;*
3. *all markets clear:* $k_t = k_t^d$, $n_t = n_t^d$, $c_t + x_t = y_t$, *for all* t.

By construction, the Arrow-Debreu equilibrium concept is applicable only if a full set of forward markets are available for all tradeable goods and factors. In general, however, markets are incomplete i.e. forward contracts cannot be written for all possible dates and states.

1.3 Sequence of Markets Equilibrium

In a sequence of markets equilibrium, in each period, spot markets for goods, rental capital and labor open and close. A price system for such an equilibrium, is given by $\{w_t, r_t\}$ where w_t is the price of one unit of labor to be delivered at time t in terms of output at time t. Analogously, r_t is the rental price of one unit of capital at time t in terms of output at time t. The price of output at time t in terms of output at time t, p_t, is by definition equal to 1.

Households maximize (1) in each t by choice of $\{c_t, x_t, n_t, k_t\}$. For each $t \geq 0$, they are subject to the following constraints:

$$c_j + x_j - w_j n_j - r_j k_j - \pi_j \leq 0, \tag{7}$$

as well as (3) and (5), for all $j \geq t$, for k_0 given.

The rationale behind the statement of the household problem is as follows. Households internalize the fact that they face a dynamic intertemporal problem, so that their choices today affect their choice set in subsequent periods. However, only spot markets for the current period are open, so households can only announce $\{c_t, x_t, n_t, k_t\}$ at t. Their optimal choice of $\{c_t, x_t, n_t, k_t\}$ at t depends on their expectations for the price system at $j > t$, which defines their constraint set for $j > t$ and, hence, their optimal choices for subsequent periods.

Firms face a completely static problem since they do not own any factors of production and don't accumulate capital. Their problem is as follows:

$$\pi_t = \max_{n_t^d, k_t^d} \left\{ y_t - r_t k_t^d - w_t n_t^d \right\}, \tag{8}$$

subject to (2). First order conditions for this problem are:

$$\begin{aligned} r_t &= f_k\left(n_t^d, k_t\right), \\ w_t &= f_n\left(n_t^d, k_t\right). \end{aligned}$$

Definition 2 *A sequence-of-markets equilibrium is a price system* $\{w_t, r_t\}$, *with* $w_t, r_t \geq 0$, *an allocation for the representative firm* $\left\{y_t, k_t^d, n_t^d\right\}$ *and an allocation for the representative household* $\{c_t, n_t, x_t, k_t\}$ *such that for all* t*:*

1. $\left\{y_t, k_t^d, n_t^d\right\}$ *solves problem (8) for the stated price system;*
2. $\{c_t, n_t, x_t, k_t\}$ *maximizes (1) subject to (3), (5) and (7) for the stated price system;*
3. *all markets clear:*

$$k_t = k_t^d,\ n_t = n_t^d,\ c_t + x_t = y_t,$$

for all t.

1.4 Recursive Equilibrium

A recursive equilibrium builds on the dynamic structure of a sequence of markets equilibrium and defines the household problem in a recursive form using dynamic programming. There is no explicit time dependence.

Equilibrium prices and allocations depend on the economy-wide state, which is given by the aggregate level of capital stock K for this economy. In general, K is constrained to lye in a set Γ. Only spot markets for rental capital and labor and output are open. Prices depend on the aggregate state: $w(K)$ denotes the price of one unit of labor on the rental market in terms of output when the economy-wide capital stock is given by K. $r(K)$ is defined analogously. The equilibrium allocation is only a function of the aggregate state K. However, given that households are competitive, their own choices depend on the economywide state and their own individual state, which for this model corresponds to their beginning of period capital stock, k. Thus, households solve the following problem:

$$v(k, K) = \left\{ \max_{c,n,x} u(c, n) + \beta v(k', K') \right\} \tag{9}$$

subject to

$$c, n, x, k' \geq 0, k' \in \Gamma, \tag{10}$$

$$x = k' - k(1 - \delta), \tag{11}$$

$$c + x - w(K)\, n - r(K)\, k - \pi(K) \leq 0, \tag{12}$$

$$K' = H(K), \tag{13}$$

where $H : \mathcal{R}_+ \rightarrow \mathcal{R}_+$ is the law of motion for the aggregate state. The functions $H(\cdot)$, $w(\cdot)$ and $r(\cdot)$ are taken as given by the households.

Note that households need to know the aggregate state K since the price system depends on K. They also need to know how the aggregate state evolves to be able to evaluate their utility in subsequent periods. All this information, which in a sequence of markets equilibrium is expressed in an infinite number of budget constraints, is summarized in the value function and in the law of motion for capital in a recursive equilibrium. The solution to the household problem is a value function $v(k, K) : \mathcal{R}^2_+ \rightarrow \mathcal{R}^1$ and a policy function for capital $k' = h(k, K)$, with $h : \mathcal{R}^2_+ \rightarrow \mathcal{R}^1_+$, and for the control variables $\{c, n, x\}(k, K) = p(k, K)$, with $p : \mathcal{R}^2_+ \rightarrow \mathcal{R}^3_+$.

Firms face the same problem as in a sequence of markets equilibrium. Expressed in reursive notation, this is given by:

$$\pi(K) = \max_{n^d, k^d} \left\{ y - w(K)\, n^d - r(K)\, k^d \right\}, \tag{14}$$

subject to (2).

Definition 3 *A recursive equilibrium is a price system* $\{w(K), r(K)\}$*, with* $w, r \geq 0$ *for all* $K \in \Gamma$*, an allocation for the representative firm* $\left\{y(K), k^d(K), n^d(K)\right\}$*, a value function* $v(k, K)$*, policy rules* $p(k, K)$ *and* $h(k, K)$*, and an allocation for the representative household* $\{c(K), n(K), x(K), k(K)\}$ *such that for all* t*:*

1. $\left\{y(K), k^d(K), n^d(K)\right\}$ *solves problem (14) for the stated price system at all* K*;*

2. the policy rules $p(k, K)$ *and* $h(k, K)$*, together with the value function* $v(k, K)$ *maximize (9) subject to (11), (10) and (12) for the stated price system at* $k, K \in \Gamma$*;*

3. household choices and aggregate quantities are consistent i.e. $\{c(K), n(K), x(K)\} = p(K, K)$ *and* $H(K) = h(K, K)$*;*

4. all markets clear:

$$k = k^d(K),\ n(K) = n^d(K),\ c(K) + x(K) = y(K),$$

for $K \in \Gamma$*.*

References

[1] Stokey, Nancy, Robert E. Lucas, with Edward C. Prescott, 1989, "Recursive Methods in Economic Dynamics", Harvard University Press.

[2] Prescott, Edward C., Jose'-Victor Rios-Rull, 2000, "On the equilibrium concept for overlapping generations organizations", Federal Reserve Bank of Minneapolis, Staff Report 282.