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Columbia University
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1 Characterizing the Policy Function for the Canonical Growth Model

The purpose of this handout is to illustrate a strategy for approximating the policy function for the canonical growth model based on the recursive solution of the planning problem.

The functional equation representation of the planning problem is given by:

$$v(k) = \left\{ \max_{k' \in \Gamma(k)} u(F(k) - k') + \beta v(k') \right\} \qquad \text{FE}$$

where $\Gamma(k) = \{k' \in X : \ F(k) - k' \geq 0\}$ and $X = \left[0, \max\left\{k_0, \bar{k}\right\}\right]$, $\bar{k}$ satisfies $\bar{k} = F\left(\bar{k}\right)$, and k_0 is the initial level of capital in the canonical economy. Here, $u : \mathcal{R}_+ \to \mathcal{R}$ and $F : \mathcal{R}_+ \to \mathcal{R}$ are strictly increasing, twice continuously differentiable, strictly concave and satisfy $\lim_{c \to 0} u'(c) = +\infty$ and $\lim_{k \to 0} F'(k) = +\infty$, $\lim_{k \to +\infty} F'(k) = 1 - \delta$, where $\delta \in (0,1)$.

Under these assumptions on u and F, assumptions 4.1-4.9 in Stokey -Lucas hold, so that v is strictly increasing, strictly concave and differentiable (Theorems 4.7, 4.8, 4.10). In addition, the policy function $g(k)$, which solves:

$$v(k) = u(F(k) - g(k)) + \beta v(g(k),), \qquad \text{(G)}$$

is continuous and single valued (Theorem 4.8). Also, by the definition of $\Gamma(k)$, $g(0) = 0$.

The solution to FE is characterized by the following couple of equations:

$$u'(F(k) - k') = \beta v'(k'), \qquad \text{(E)}$$

$$v'(k) = u'(F(k) - g(k)) F'(k). \qquad \text{(ENV)}$$

1.1 Characterization without differentiability

In this section, we will show that the policy function $g(k)$ is monotone increasing and that it has a stationary point $k^* = g(k^*) \neq 0$. We will also show that, and that starting from any k in the interior of X, the sequence generated by the policy function $g(k)$ converges monotonically to the stationary point k^*.

1.1.1 Monotonicity

We can establish monotonicity using a direct argument. Let:

$$MC(k'; k) = u'(F(k) - k'),$$

$$MB(k') = \beta v'(k'),$$

be the marginal cost and the marginal benefit of increasing k'. Then, $g(k) = k'$, where k' solves $MC(k'; k) = MB(k')$ from (E). The function MC is increasing in k' since u is strictly concave and asymptotes at $k' = \bar{k}$,

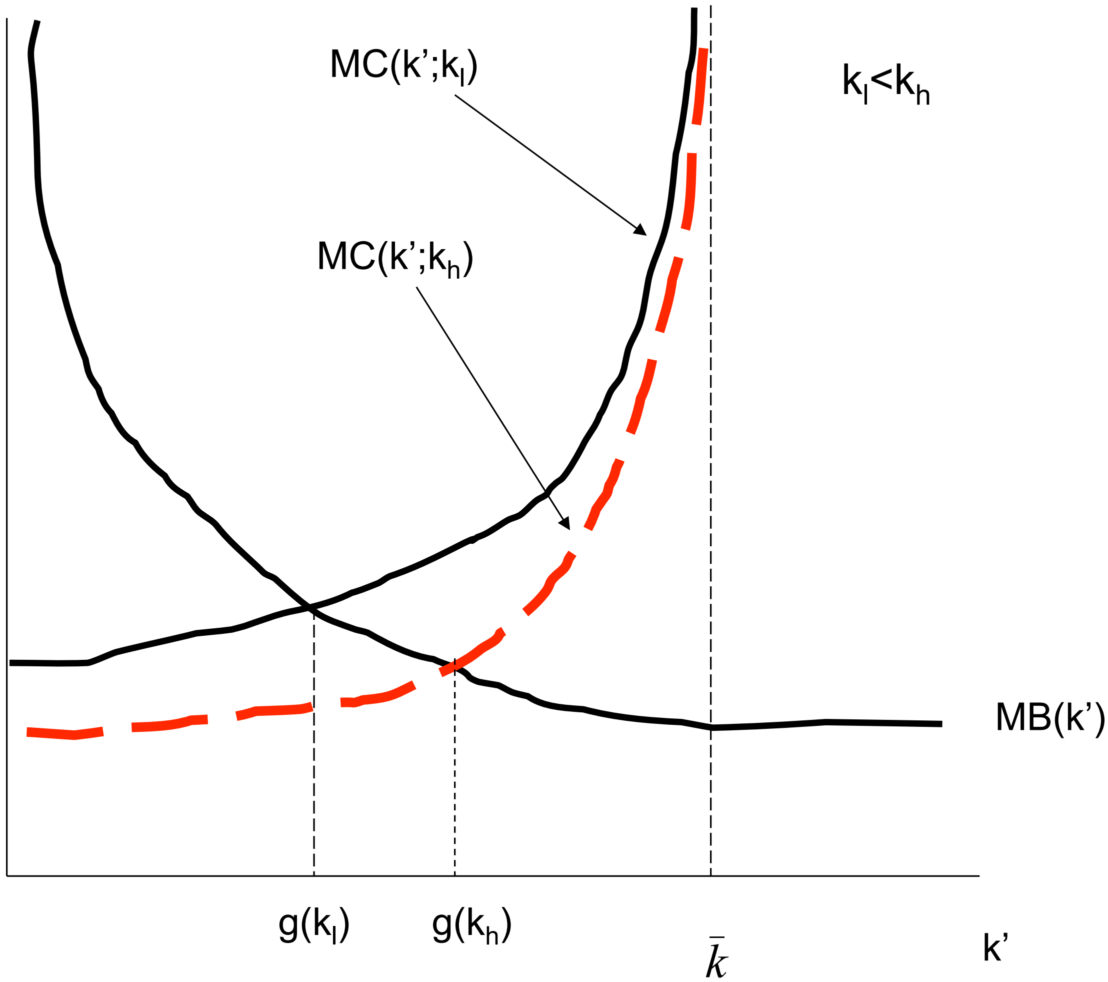

Figure 1: Characterizing the slope of $g(k)$

by the Inada conditions on u. By the same properties of u, MB is decreasing in k' and asymptotes at $k' = 0$. In addition, the strict concavity of u implies that MC is decreasing in k. Then, for $k_l < k_h$, $g(k_l) < g(k_h)$, so that g is monotone increasing. See figure 1.

Note that feasibility implies that $g(k) < F(k)$. It follows that the asymptotic slope of $g(k)$ is at most equal to $1-\delta$. To proceed further in our characterization of the policy function, we need to make assumptions on the differentiability of g. See section 1.2.

1.1.2 Stationary Points and Monotone Convergence

A stationary point is a value of k such that $k = g(k)$. Feasibility implies that $k = 0$ is stationary point for the policy function $g(k)$. Here, we are interested in identifying necessary and sufficient conditions for stationary point $k^* = g(k^*) > 0$.

Assume k^* is a stationary point and satisfies $k^* = g(k^*) > 0$. By (E) and (ENV) evaluated at k^*:

$$u'(F(k^*) - k^*) = \beta u'(F(k^*) - k^*) F'(k^*).$$

Then,

$$F'(k^*) = 1/\beta, \tag{1}$$

is a necessary condition for a stationary point.

We now show that (1) is a sufficient condition for a stationary point. Note that, since v is strictly concave:

$$[v'(k) - v'(g(k))] * [k - g(k)] \leq 0, \tag{2}$$

with strict equality *if and only* if $k = g(k)$. Using (ENV) to substitute for $v'(k)$ and (E) to substitute for $v'(g(k))$, we are left with:

$$\left[u'(F(k) - g(k)) F'(k) - \frac{1}{\beta} u'(F(k) - g(k))\right] * [k - g(k)] \leq 0, \tag{3}$$

which is equivalent to: $\left[F'(k) - \frac{1}{\beta}\right] * [k - g(k)] \leq 0$, since $u'(F(k) - g(k)) > 0$. If (1) holds, (2) holds with equality. Then, by the strict concavity of v, $k = g(k)$.

Note that (1) identifies the *unique* stationary point for policy function $g(k)$, since it is both necessary and sufficient.

The strict concavity of v implies that convergence to the stationary point from any k in the interior of X is monotone. To see this, note that (3) implies:

$$\left[F'(k) - \frac{1}{\beta}\right] * [k - g(k)] < 0,$$

for $k \neq k^*$. Since F is strictly concave, for $k > k^*$, $F'(k) < 1/\beta$, and $k > g(k)$. Similarly, $k < k^*$, implies $F'(k) > 1/\beta$, and $k < g(k)$.

1.2 Characterization with differentiability

To make further progress in characterizing the policy function, we assume that the derivatives up to order $n \geq 1$ of the function $g(k)$ exist on the interior of X and we denote them with $\{g'(k), g''(k), ...g^n(k)\}$. We can then use Taylor's Theorem to *locally* approximate the policy function in a neighborhood of any point $\hat{k} \in X$:

$$\begin{aligned} g(k) &= g\left(\hat{k}\right) + g'\left(\hat{k}\right)\left(k - \hat{k}\right) + \frac{1}{2} g''\left(\hat{k}\right)\left(k - \hat{k}\right)^2 + ... + \frac{1}{(n-1)!} g^{(n-1)}\left(\hat{k}\right)\left(k - \hat{k}\right)^{(n-1)} \\ &\quad + \frac{1}{n!} g^n\left(\tilde{k}\right)\left(k - \hat{k}\right)^n, \end{aligned}$$

where $\tilde{k} \in \left[\min\left\{k, \hat{k}\right\}, \max\left\{k, \hat{k}\right\}\right]$. Assuming the term $\frac{1}{n!} g^n\left(\tilde{k}\right)\left(k - \hat{k}\right)^n$ is negligible, we can simply write:

$$\hat{g}(k) = g\left(\hat{k}\right) + g'\left(\hat{k}\right)\left(k - \hat{k}\right) + \frac{1}{2} g''\left(\hat{k}\right)\left(k - \hat{k}\right)^2 + ... + \frac{1}{(n-1)!} g^{(n-1)}\left(\hat{k}\right)\left(k - \hat{k}\right)^{(n-1)}, \tag{4}$$

where $\hat{g}(k)$ is our *approximate* policy function.

To adopt this method, we need to know the value of $g\left(\hat{k}\right)$. If $\hat{k} = k^*$, then $g(k^*) = k^*$, by the definition of a stationary point. So we will focus on locally approximating the policy function in a neighborhood of a stationary point. We also need to assume that the functions u and F are $(n+1)$-order differentiable.

Note that (E) and (ENV) jointly imply:

$$R(k) = u'(F(k) - k') - \beta u'(F(g(k)) - g(g(k)))\, F'(g(k)) = 0, \tag{R}$$

for any k in the interior of X, since $g(k)$ is a policy function for the problem (FE). The function $R(k)$ is known as Euler error or residual. Note that (R) is a functional equation where the unknown is the function $g(\cdot)$.

(R) implies:

$$\begin{aligned} R'(k) &= 0, \\ R''(k) &= 0, \\ &.... \\ R^{(n-1)}(k) &= 0, \end{aligned} \tag{5}$$

for k in X. This is simply from the fact that $R(k)$ is a constant function on the interior of X. Note that (dR) evaluated at a particular value of k in X defines a system of $(n-1)$ equations in the unknowns $\left\{g'(k), g''(k), ..., g^{(n-1)}(k)\right\}$, where each $g^i(k)$ is simply a number. Each $R^i(k)$ involves terms in $g^i(k)$, for $i = 1, 2, ...n-1$.

Let's apply this approximation method to our the canonical problem. We assume that $g(k)$ is continuously differentiable on a neighborhood of k^*, and use the equation:

$$R'(k^*) = 0,$$

to characterize $g'(k^*)$. Note that:

$$\begin{aligned} R'(k) = \;& u''(F(k) - g(k))\,[F'(k) - g'(k)] \\ & -\beta\left\{u''(F(g(k)) - g(g(k)))\,[F'(g(k)) - g'(g(k))]\, g'(g(k))\right\} F'(g(k)) \\ & -\beta\left\{u'(F(g(k)) - g(g(k)))\right\} F''(g(k))\, g'(g(k)). \end{aligned}$$

Then:

$$R'(k^*) = u''(F(k^*) - k^*)\left\{[F'(k^*) - g'(k^*)]\,(1 - \beta F'(k^*)\, g'(k^*)) - \beta\frac{u'(F(k^*) - k^*)}{u''(F(k^*) - k^*)} F''(k^*)\, g'(k^*)\right\},$$

by $k^* = g(k^*)$. Then, the equation $R'(k^*) = 0$, using $F'(k^*) = 1/\beta$, can be rewritten as:

$$\left[\frac{1}{\beta} - g'(k^*)\right](1 - g'(k^*)) - \frac{u'(F(k^*) - k^*)}{u''(F(k^*) - k^*)}\frac{F''(k^*)}{F'(k^*)} g'(k^*) = 0,$$

which is a second order equation in $g\prime(k^*)$.

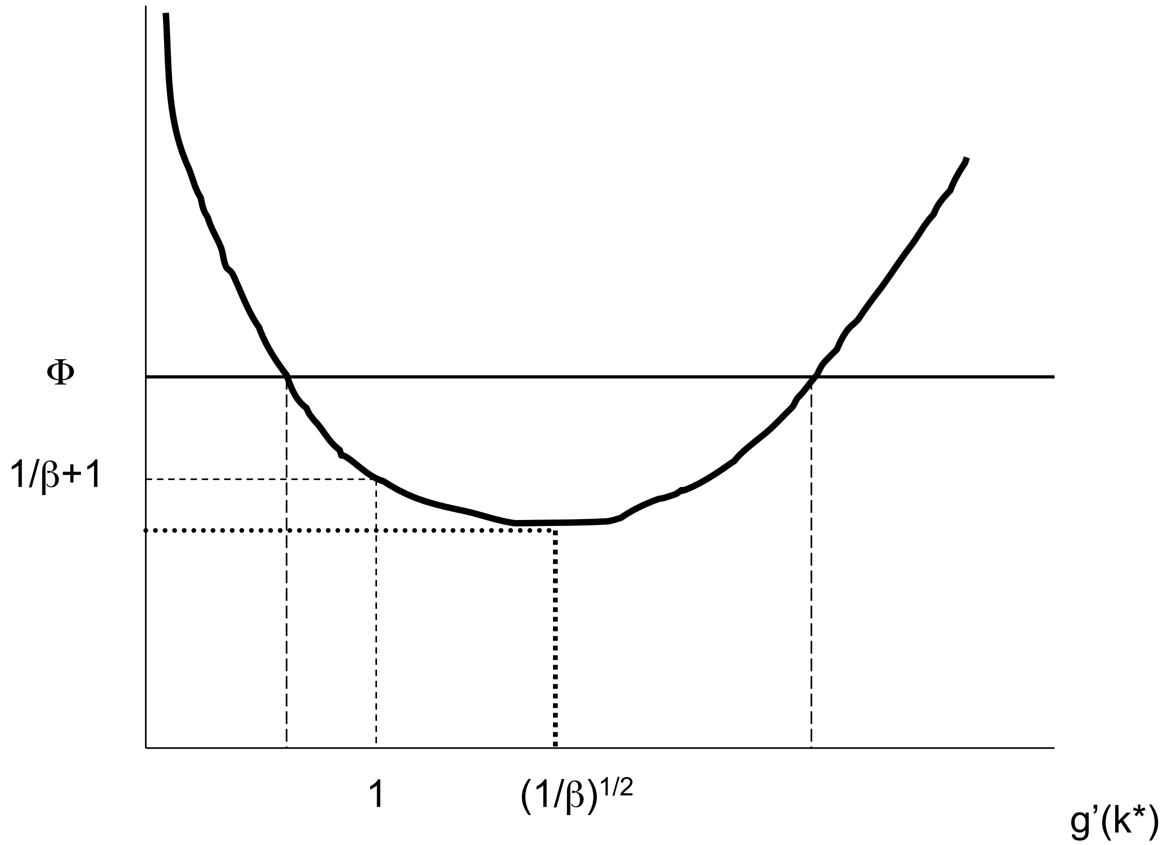

FIGURE 2: Solving for $g'(k)$

It is useful to rewrite this equation in the following form:

$$\Phi = g'(k^*) + \frac{1}{\beta}\frac{1}{g'(k^*)}, \tag{g'*}$$

where $\Phi = \beta^{-1} + 1 + \frac{u'(F(k^*)-k^*)}{u''(F(k^*)-k^*)}\frac{F''(k^*)}{F'(k^*)}$. The equation is represented in figure 2.

Equation (g'*) has two non-negative roots since the function in (g'*) asymptotes for $g'(k^*) \to 0$. In addition, since the term $\frac{u'(F(k^*)-k^*)}{u''(F(k^*)-k^*)}\frac{F''(k^*)}{F'(k^*)}$ is strictly positive, the smallest root, which we call the stable root, is smaller than 1. We can use the stable root for equation (g'*) to construct a first order approximation of the policy function $g(k)$ in a neighborhood of k^* according to (4), see figure 3.

Denote with g'^* the stable root of (g'*). The value of g'^* is decreasing in Φ. for given β, higher values of Φ correspond to higher values of $\frac{u'(F(k^*)-k^*)}{u''(F(k^*)-k^*)}\frac{F''(k^*)}{F'(k^*)} = \frac{d\ln F'(k^*)}{d\ln u'(F(k^*)-k^*)}$, where the numerator is the curvature of the production function F, while the denominator is the curvature of the utility function. Note from figure 3 that a lower value of g'^* corresponds to *faster* convergence to the stationary point. So the rate of convergence to the stationary point is increasing in the curvature of the production function and decreasing in the curvature of the utility function. The economic interpretation of this property is as follows:

- More curvature in the production function corresponds to a greater degree of decreasing returns. The the optimal rate of investment, given by $g(k)/k$, will be smaller around the steady state.

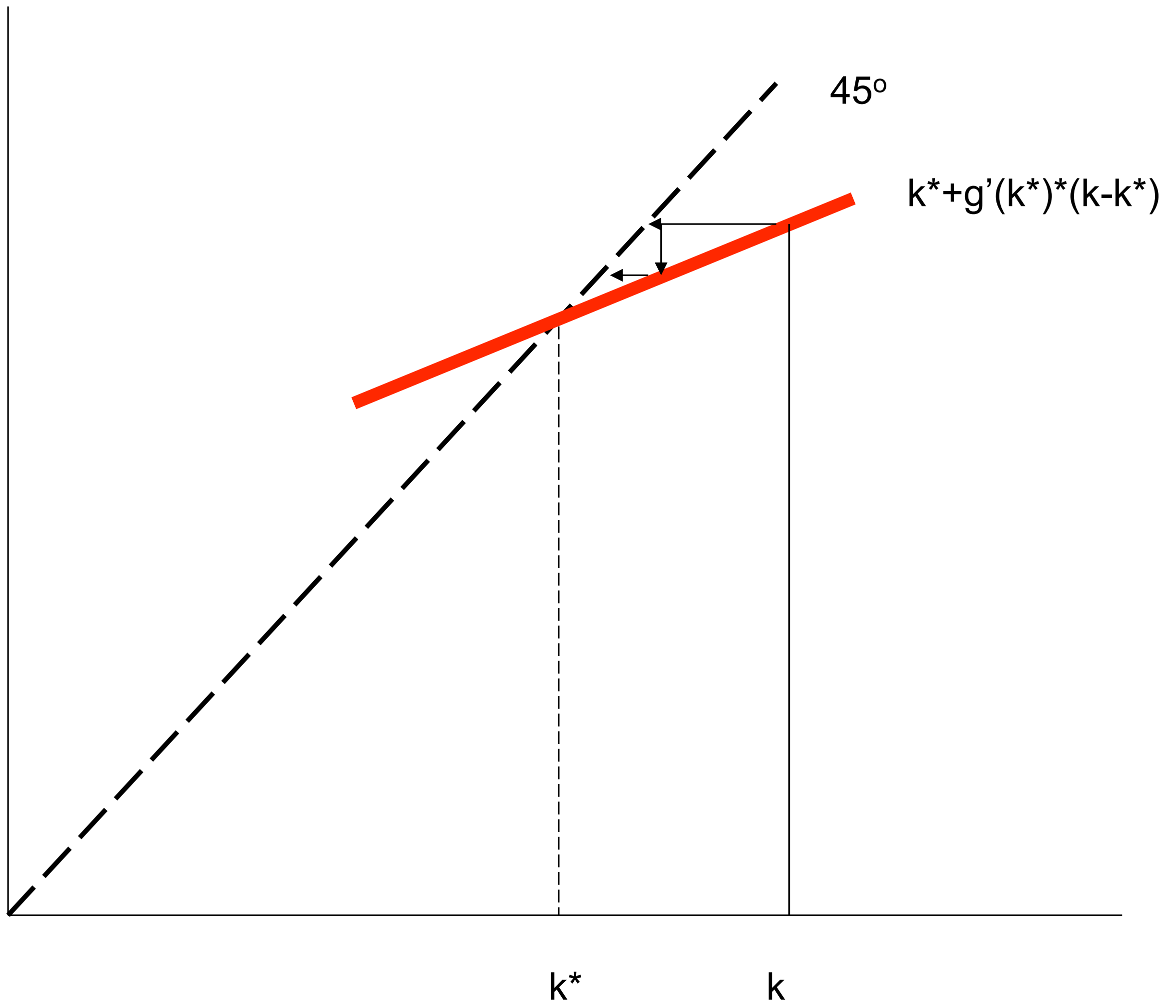

Figure 3: Stable root solution

- More curvature in the utility function corresponds to a smaller degree of intertemporal substitution. Households' preference for a smooth consumption path is stronger the higher the curvature of the utility function. As the economy converges to the stationary point, the marginal return to capital is non-constant, which, from (E), implies that the consumption path is not smooth. Slower convergence to the stationary point implies that a smoother consumption path along the transition to the stationary point.

References

[1] Stokey, Nancy, Robert E. Lucas, with Edward C. Prescott, 1989, "Recursive Methods in Economic Dynamics", Harvard University Press.