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1 Necessary and Sufficient Conditions for Household Optimization in a Cash-Credit Good Economy

This section discusses the technical aspects of the household problem in the cash-credit good model in a sequence of markets equilibrium. Here, the price system $\{P_t, W_t, R_t\}_{t\geq 0}$, where P_t is the price of consumption goods in terms of money at time t, W_t the nominal wage and R_t the gross nominal interest rate at time t, monetary policy $\{M_t\}_{t\geq 0}$ and the sequence of dividend payments by firms $\{D_t\}_{t\geq 0}$ are taken as given.

The basic results in this section are available in the literature, for example, in Woodford (1994).

1.1 Setup of the Household Problem

Households have the following preferences over streams of consumption and labor:

$$\sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}, n_t),$$

where the one-period utility function $u(\bullet)$ is strictly increasing, differentiable and strictly concave. The households face the following constraints for each t:

$$M_t^d + B_t \leq A_t, \tag{1}$$

$$A_{t+1} = W_t n_t + D_t + T_t + R_t B_t + (M_t^d - P_{1t}c_{1t}) - P_{2t}c_{2t}, \tag{2}$$

$$P_{1t}c_{1t} \leq M_t^d, \tag{3}$$

$$n_t,\ c_{1t},\ c_{2t},\ 1 - n_t \geq 0, \tag{4}$$

$$A_0 \text{ given.} \tag{5}$$

Here, c_{1t} denotes consumption of the cash good, c_{2t} denotes consumption of the credit good, n_t denotes labor effort, and $1 - n_t$ is leisure, since the time endowment is normalized at unity. The household faces the following borrowing constraint at each t :

$$A_{t+1} \geq -\frac{1}{q_{t+1}} \sum_{j=1}^{\infty} q_{t+j+1} I_{t+j}, \tag{6}$$

where A_{t+1} is nominal asset holdings at the beginning of period $t+1$ and

$$I_t \equiv W_t + T_t + D_t. \tag{7}$$

Here, I_t denotes the household's period t non-interest income. We include the market value of the time endowment in income. The discount factor, q_t, is defined as follows:

$$q_t = \prod_{j=0}^{t-1} \frac{1}{R_j},\ q_0 \equiv 1.$$

It is useful, for later purposes, to simplify the expression of the household's asset evolution equation. Thus,

$$\begin{aligned} A_{t+1} &= W_t n_t + D_t + T_t + R_t B_t + (M_t^d - P_{1t}c_{1t}) - P_{2t}c_{2t} \qquad (8) \\ &= W_t n_t + D_t + T_t + R_t B_t + R_t M_t^d + (1 - R_t)M_t^d - P_{1t}c_{1t} - P_{2t}c_{2t} \\ &\leq W_t n_t + D_t + T_t + R_t A_t + (1 - R_t)M_t^d - P_{1t}c_{1t} - P_{2t}c_{2t} \\ &\equiv I_t + R_t A_t - S_t, \end{aligned}$$

where the weak inequality reflects (1) and

$$S_t \equiv (R_t - 1)M_t^d + P_{1t}c_{1t} + P_{2t}c_{2t} + W_t(1 - n_t). \tag{9}$$

Note that in our measure of spending, S_t, we include not just purchases of goods, but also purchases of leisure and spending on interest foregone by holding money.

We place restrictions on the prices faced by the household that we know must hold in equilibrium:

$$P_{1t},\ P_{2t},\ W_t > 0,\ R_t \geq 1,\ lim_{t\to\infty} \sum_{j=0}^{t} q_{j+1} I_j \text{ finite.} \tag{10}$$

If these did not hold, then the household's consumption opportunity set would be unbounded above, something that is incompatible with equilibrium in an economy with bounded resources and non-satiation in utility. In addition, we know that A_0 satisfies (6) at $t = 0$. We shall assume, in addition, that (6) is satisfied as a *strict* inequality at $t = 0$. This guarantees that the household's constraint set has a non-empty interior. This is a restriction on the set of equilibria, which may not be binding.

1.2 Characterizing the Household's Intertemporal Consumption Opportunity Set

We refer to the characterization of the household's budget constraint presented in the previous subsection as the 'borrowing constrained' characterization. In this subsection we establish that there are two other ways to characterize the household's consumption opportunity set, in addition to what we have above. The first simply replaces (6) by the following restriction:

$$\lim_{T\to\infty} q_T A_T \geq 0. \tag{11}$$

We refer to this characterization as the 'don't die in debt' characterization. The second replaces (1) (2) and (6) with a single intertemporal budget equation. We refer to this as the 'intertemporal budget constraint' characterization. We establish that the three ways of characterizing the household's consumption opportunities are equivalent.

The first result is established in the following proposition:

Proposition 1 *Suppose (1) and (2) are satisfied. Then, (6) holds if, and only if (11) holds.*

Proof We first establish that (11) implies (6). Recursively solving for assets using (8) and (1) from t to T yields:

$$q_T A_T \leq \sum_{j=0}^{T-t-1} q_{t+j+1} I_{t+j} + q_t A_t - \sum_{j=0}^{T-t-1} q_{t+j+1} S_{t+j}. \tag{12}$$

Taking into account $q_{t+j+1} S_{t+j} \geq 0$ and rewriting this expression, we obtain

$$q_t A_t \geq q_T A_T - \sum_{j=0}^{T-t-1} q_{t+j+1} I_{t+j}.$$

Fixing t, taking the limit, $T \to \infty$, and using (11) yields (6).

We now show that (6) implies (11). Note first that the limit in (10) being finite implies

$$\lim_{t\to\infty} \sum_{j=1}^{\infty} q_{t+j+1} I_{t+j} = 0.$$

Using this and (6), (11) follows trivially.■

The second result is established in the following proposition:

Proposition 2 *A sequence, $\{c_{1t},\ c_{2t},\ n_t,\ M_t^d\}$ satisfies (3), (4), (1), (2) and (6) if, and only if it satisfies (3), (4) and:*

$$\sum_{j=0}^{\infty} q_{j+1} S_j \leq \sum_{j=0}^{\infty} q_{j+1} I_j + A_0. \tag{13}$$

where S and I are defined in (9) and (7), respectively.

Proof Suppose $\{c_{1t}, c_{2t}, n_t, M_t^d\}$ satisfies (3), (4), (1), (2) and (6). Conditions (1) and (2) imply (12). Substituting out for $q_T A_T$ in (12) using (6), setting $t = 0$, and rearranging the result, we obtain

$$\sum_{j=0}^{T-1} q_{j+1} S_j \leq \sum_{j=0}^{\infty} q_{j+1} I_j + A_0.$$

The object on the left of the equality is a non-decreasing sequence in T. The inequality guarantees that it is bounded above. Hence, the sequence converges and (13) holds.

Now suppose $\{c_{1t}, c_{2t}, n_t, M_t^d\}$ satisfies (3), (4) and (13). We establish that sequences, $\{B_t, A_t\}$, can be found with the property that $\{c_{1t}, c_{2t}, n_t, M_t^d, B_t, A_t\}$ satisfy (1), (2) and (6). Candidate sequences, $\{B_t, A_t\}$, can be found as follows. First, solve for $\{B_t\}$ using the following expression:

$$\begin{aligned} R_t B_t &= \frac{1}{q_{t+1}} \sum_{j=1}^{\infty} q_{t+j+1} \left(S_{t+j} - I_{t+j} \right) \\ &\quad -M_t^d + P_{1t} c_{1t} + P_{2t} c_{2t} + W_t(1 - n_t) - D_t - T_t - W_t. \end{aligned}$$

The fact that (13) holds guarantees that the infinite sums here are well defined. Moving the terms on the second line to the left of the equality and using (2) implies:

$$A_{t+1} = \frac{1}{q_{t+1}} \sum_{j=1}^{\infty} q_{t+j+1} \left(S_{t+j} - I_{t+j} \right), \tag{14}$$

which guarantees (6) because of the non-negativity of $q_{t+j+1} S_{t+j}$. It remains to establish that (1) holds.

To do this, we first establish that (2) and (13) imply:

$$\sum_{j=0}^{\infty} q_{t+j+1} S_{t+j} \leq \sum_{j=0}^{\infty} q_{t+j+1} I_{t+j} + q_t A_t, \ t = 1, 2, \ldots . \tag{15}$$

Suppose this expression holds at t. We show that it holds at $t+1$. First, shift the terms corresponding $j = 0$, and combine them with $q_t A_t$ to obtain:

$$\begin{aligned} \sum_{j=1}^{\infty} q_{t+j+1} S_{t+j} &\leq \sum_{j=1}^{\infty} q_{t+j+1} I_{t+j} + q_t A_t + q_{t+1} I_t - q_{t+1} S_t \\ &= \sum_{j=1}^{\infty} q_{t+j+1} I_{t+j} + q_{t+1} \left(R_t A_t + I_t - S_t \right) \\ &= \sum_{j=1}^{\infty} q_{t+j+1} I_{t+j} + q_{t+1} A_{t+1}. \end{aligned}$$

The first equality makes use of (2). This gives us the result we seek.

From (14), we have:

$$\begin{aligned}
\frac{A_{t+1}}{R_t} &= \frac{1}{q_t}\sum_{j=1}^{\infty} q_{t+j+1}(S_{t+j} - I_{t+j}) \\
&= \frac{1}{q_t}\sum_{j=0}^{\infty} q_{t+j+1}(S_{t+j} - I_{t+j}) + \frac{q_{t+1}}{q_t}(I_t - S_t) \\
&= \frac{1}{q_t}\sum_{j=0}^{\infty} q_{t+j+1}(S_{t+j} - I_{t+j}) + \left[\frac{1}{R_t}A_{t+1} - \left(M_t^d + B_t\right)\right],
\end{aligned}$$

where the last equality makes use of (2), (9) and (7). Rewriting the last expression, we obtain:

$$M_t^d + B_t = \frac{1}{q_t}\sum_{j=0}^{\infty} q_{t+j+1}(S_{t+j} - I_{t+j}) \leq A_t,$$

using (15).■

1.3 Characterizing Household Optimization

In the previous subsection we developed three different ways to characterize the household's budget constraint: the borrowing constraint characterization, the don't die in debt characterization and the intertemporal budget constraint characterization. In this section we establish necessary and sufficient conditions for household optimality. We do so for the second two characterizations. In both cases, the Euler equations are part of the necessary and sufficient conditions. In both cases, one extra constraint is needed too. In the don't die in debt case, the additional condition is that (11) be satisfied as an equality. This is the usual transversality condition. In the intertemporal budget constraint case, the additional condition is that (13) be satisfied with an equality.

1.3.1 Don't Die in Debt

The result is established in the following proposition:

Proposition 3 *A sequence, $\{c_{1t},\ c_{2t},\ n_t,\ M_t^d, B_t\}$, solves the don't die in debt version of the household problem if, and only if the following conditions are satisfied. The 'Euler Equations' are:*

$$\frac{u_{1t}}{P_{1t}} = R_t \frac{u_{2t}}{P_{2t}}, \tag{16}$$

$$\frac{-u_{3t}}{u_{2t}} = \frac{W_t}{P_{2t}}, \tag{17}$$

$$\frac{u_{1t}}{P_{1t}} = \beta R_t \frac{u_{1t+1}}{P_{1t+1}}, \tag{18}$$

$$(R_t - 1)(P_{1t}c_{1t} - M_t^d) = 0. \tag{19}$$

The transversality condition is (11) with equality:

$$\lim_{t\to\infty} q_t A_t = 0. \tag{20}$$

Proof We begin by showing that if a sequence $\{c_{1t},\ c_{2t},\ n_t,\ M_t^d, B_t\}$ satisfies (16)-(20), then that sequence solves the household's problem. To do this, establish that:

$$D = \lim_{T\to\infty}\left[\sum_{t=0}^{T}\beta^t u(c_{1t}, c_{2t}, n_t) - \sum_{t=0}^{T}\beta^t u(c'_{1t}, c'_{2t}, n'_t)\right] \geq 0.$$

Here, $\{c'_{1t}, c'_{2t}, n'_t, M_t^{d\prime}, B'_t\}_{t=0}^{\infty}$ is any other plan consistent with the don't die in debt version of the household's constraints. Note first that the Euler equations imply:

$$\beta^t u_{1,t} = q_t P_{1t}\frac{u_{1,0}}{P_{1,0}},$$

$$\beta^t u_{2,t} = q_{t+1} P_{2t}\frac{u_{1,0}}{P_{1,0}},$$

$$\beta^t u_{3,t} = -q_{t+1} W_t\frac{u_{1,0}}{P_{1,0}},$$

where $u_{i,t}$ is the derivative of u with respect to its i^{th} argument. By concavity and the fact that the candidate optimal plan satisfies (16) and (17) we can write:

$$D \geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\sum_{t=0}^{T}\left[q_t P_{1t}(c_{1t} - c'_{1t}) + q_{t+1}P_{2t}(c_{2t} - c'_{2t}) - q_{t+1}W_t(n_t - n'_t)\right]$$

$$= \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\sum_{t=0}^{T} q_t\left[\frac{S_t}{\acute{R}_t} + \frac{(1-R_t)}{R_t}(M_t^d - P_{1t}c_{1t}) - \frac{S'_t}{\acute{R}_t} - \frac{(1-R_t)}{R_t}(M_t^{d\prime} - P_{1t}c'_{1t})\right]$$

$$\geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\sum_{t=0}^{T}\left[q_{t+1}S_t - q_{t+1}S'_t\right],$$

where the equality is obtained by using the definition of S_t and the second inequality is obtained by using $R_t \geq 1$, (19) and $(1-R_t)(M_t^{d\prime} - P_{1t}c'_{1t}) \leq 0$ (see (3)). Iterating on (1) and (2) for the two plans this implies

$$D \geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\left[\sum_{t=0}^{T} q_{t+1}S_t + q_{T+1}A'_{T+1} - \sum_{t=0}^{T} q_{t+1}I_t - A_0\right] \tag{21}$$

$$\geq \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}}\left[\sum_{t=0}^{T} q_{t+1}S_t - \sum_{t=0}^{T} q_{t+1}I_t - A_0\right]$$

$$= \lim_{T\to\infty}\frac{u_{1,0}}{P_{1,0}} q_{T+1}A_{T+1} \geq 0,$$

by (20) and the fact, $\frac{u_{1,0}}{P_{1,0}} > 0$.

Now we establish that if $\{c_{1t}, c_{2t}, n_t, M_t^d, B_t\}$ is optimal, then (16)-(20) is true. That (16)-(19) are necessary is obvious. It remains to show that (20) is necessary. Suppose (20) is not true. We show this contradicts the hypothesis of optimality.

We need only consider the case where $\lim_{T\to\infty} q_T A_T$ is strictly positive. The strictly negative case is ruled out by the don't die in debt constraint, (11). So, suppose

$$\lim_{T\to\infty} q_T A_T = \Delta > 0.$$

We construct a deviation from the optimal sequence which is consistent with the budget constraint and results in an increase in utility. Fix some particular date, τ. We replace $c_{1\tau}$ by $c_{1\tau} + \varepsilon/P_{1\tau}$, where $0 < \varepsilon \leq \Delta/q_\tau$. Consumption at all other dates and $c_{2\tau}$ are left unchanged, as well as employment at all dates. We finance this increase in consumption by replacing M_τ^d with $M_\tau^d + \varepsilon$ and B_τ with $B_\tau - \varepsilon$. Money holdings at all other dates are left unchanged. Debt and wealth after t, B_t, A_t, $t > \tau$ are different in the perturbed allocations. We denote the variables in the perturbed plan with a prime. From (2)

$$\begin{aligned} A'_{\tau+1} - A_{\tau+1} &= -R_\tau\varepsilon = -\frac{q_\tau}{q_{\tau+1}}\varepsilon \\ A'_{\tau+j} - A_{\tau+j} &= -R_{\tau+j-1}\cdots R_\tau\varepsilon = -\frac{q_\tau}{q_{\tau+j}}\varepsilon. \end{aligned}$$

Multiplying this last expression by $q_{\tau+j}$ and setting $T = \tau + j$:

$$q_T\left(A'_T - A_T\right) = -q_\tau\varepsilon.$$

Taking the limit, as $T \to \infty$

$$\lim_{T\to\infty} q_T A'_T = \Delta - q_\tau\varepsilon \geq 0.$$

We conclude that the perturbed plan satisfies all the restrictions of the don't die in debt budget constraint. But, utility is clearly higher in this plan. Contradiction.■

1.3.2 Intertemporal Budget Constraint

We now establish a proposition analogous to the one above for the intertemporal budget constraint version of the household's budget constraint.

Proposition 4 *A sequence, $\{c_{1t}, c_{2t}, n_t, M_t^d, B_t\}$, solves the intertemporal budget constraint version of the household problem if, and only if (i) the Euler equations, (16)-(19), are satisfied and (ii) the intertemporal budget constraint, (13), holds with a strict equality.*

Proof Sufficiency is established using a slightly modified version of the argument above. The reasoning up to (21) is unchanged. The result follows from the fact that the object in square brackets in the second term in (21) converges to zero when (13) holds with equality. Consider necessity. Suppose (13) holds with a strict inequality. Then, it is easy to find a perturbation to the household's spending plan which raises utility and is consistent with (13). Contradiction.■

References

[1] Woodford, Michael, 1994, 'Monetary Policy and Price Level Determinacy in a Cash-in-Advance Economy,' *Economic Theory*, vol. 4, pp. 345-380.