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Solving a Limited Information Model

Consider a dynamic stochastic general equilibrium model in which there is an exogenous state x, which follows an l state Markov chain, with transition matrix Π (an $l*l$ matrix) and support X (an $l*1$ vector)[1]. A private sector equilibrium is given by a set of variables $\{q_t, z_t\}_{t\geq 0}$ where, in each period, $z \in R^m$ is determined after the exogenous state x is realized and $q \in R^n$ is determined before x is realized i.e. under limited information. This implies that the private sector equilibrium in each period depends on $\{x_{t-1}, x_t\}$, the realization of the exogenous state in the previous period and in the current period. For economies with this limited information structure the *economic* state is given by $s_t = \{x_{t-1}, x_t\}$. We denote with $S \in R^{l^2}$, the support of the state s.

Suppose that the equilibrium conditions can be represented by the following system of equations:

$$\Gamma_1(q_t, z_t; s_t) = 0, \quad (1)$$

$$E_{t-1}\Gamma_2(q_t, z_t; s_t) = 0, \quad (2)$$

for $s_t \in S$, where Γ_1 is a function from R^{n+m+2} to R^m and Γ_2 is a function from R^{n+m+2} to R^n. The operator E_{t-1} denotes expectations conditional on the realization of the exogenous state x_{t-1}.

A stationary equilibrium for this class of economies is a set of functions $Q(x_{-1})$ and $Z(s)$, where $s = \{x_{-1}, x\}$ such that the private sector equilibrium restrictions are satisfied for $q = Q(x_{-1})$ and $z = Z(s)$ at all possible values of x_{-1} and s.

1 Computing the Stationary Equilibrium

The restriction defined by Γ_1 typically allows to solve for z as a function of $\{q, x\}$. If this is the case, we can rewrite (2) as:

[1] The fact that we consider one exogenous state only is without loss of generality. The structure we develop here is compatible with having more than one random variable determining the exogenous state (say a productivity shock *and* a government spending shock), as long as all random variables have a discrete support and a first order Markovian structure. If there are multiple random variables determining the exogenous state, the support of the state is given by all the possible combinations of the realizations of the individual random variables and the distribution of the state can be derived from the distribution of the individual random variables if their correlation structure is known. Moreover, this procedure can be used with random variables on continuous supports, where discretization is used as an approximation technique.

$$E_{-1}\hat{\Gamma}\left(q;s\right)=0, \tag{3}$$

where E_{-1} is the expectations operator conditional on the realization of x_{-1}. We can use this restriction to solve for $Q\left(x_{-1}\right)$.

Given that we are considering the case in which x is distributed as a discrete state Markov chain, (3) is, in fact, a linear (in the probabilities) system of equations. We are looking for $\{q_1, q_2, ...q_l\}$ such that (3) is satisfied at each possible value of x_{-1}. Analogously, (1) can be represented as follows:

$$\Gamma_1\left(q_i, z_{ij}; s_{ij}\right)=0, \text{ for } i,j=1,2,..l,$$

where $s_{ij}=\{x_i, x_j\}$ for $i,j=1,2,...l$.

Often, (2) and, consequently, (3) are originally derived from intertemporal Euler equations. In this case, it is possible to rewrite (3) as:

$$E_{-1}\xi(q;x_{-1},x)=E_{-1}\left(E\left(\eta\left(q;s,x^{'}\right)\right)\right),$$

where ξ and η are functions. In this case, we have a linear system with the structure:

$$\sum_{j=1}^{l}\pi_{ij}\xi\left(q_i;x_i,x_j\right)=\beta\sum_{j=1}^{l}\pi_{ij}\sum_{k=1}^{l}\pi_{jk}\eta\left(q_i;s_{ij},x_k\right), \text{ for } i=1,...l, \tag{4}$$

where $\pi_{ij}=\Pi(i,j)$ is element i,j (row,column) of matrix Π, while $\Pi\left(i,:\right)$ is row i of matrix Π.

1.1 Computing Impulse Responses

We can now analyze the effect of an innovation in the exogenous shock x on the economy. Assume that at an arbitrary time, say $t=0$, the state is i, while at $t=1$ the state is j, that is:

$$\begin{aligned} x_t &= x_i, \text{ for } t=0, \\ x_t &= x_j, \text{ for } t=1.^2 \end{aligned}$$

The impulse response function to a particular shock for a variable in the vector $\{q;z\}$ is simply the percentage difference at time $t=1,2,3,...$ between the value of this variable conditional on the shock having taken place and the value of that variable conditional on the shock not having occurred. Here, we will provide an algorithm for computing impulse responses for a variable of interest. The algorithm is written in Matlab notation[3] with all vectors in column form.

[3] In Matlab notation, if Π is a matrix, then $\Pi\left(i,j\right)$ is the element in row i, column j. The expression $\Pi\left(i,:\right)$ correspond to row i of the matrix, the expression $\Pi\left(:,j\right)$ corresponds to column j of the matrix. The symbol $'$ after a vector or matrix is the transpose operator. The symbol $*$ is a dot product.

The impulse in the exogenous shock at $t = 1$, measured in percentage terms, is given by:

$$\gamma x = \log(1 + x_j) - \Pi\,(i, :) * \log(1 + x)'.$$

Note that the term $\Pi\,(i, :) * \log(1 + x)'$ is the expected value of x at time 1, conditional on the state at time 0 being x_i. Hence, γx is simply the difference between the actual value of x_1 under the shock and the expected value conditional on information available at time 0.

The responses of the other variables are also measured in percentage terms and typically rescaled by the initial impulse in absolute value. The algorithm for computing the responses of variables that are determined after the exogenous shock is realized in each period is given by:

$$\begin{aligned}
\gamma z_1 &= \left[\log\left(z(i,j)\right) - \Pi\,(i,:) * \log(z(i,j)')\right] / |\gamma x|, \\
\gamma z_2 &= \left[\omega z\,(j) - \Pi\,(i,:) * \omega z\right] / |\gamma x|, \\
\gamma z_3 &= \left[\Pi\,(j,:) * \omega z - \Pi\,(i,:) * (\Pi * \omega z)\right] / |\gamma x|, \\
\gamma z_t &= \left[\Pi\,(j,:) * \left(\Pi^{t-2} * \omega z\right) - \Pi\,(i,:) * \left(\Pi^{t-1} * \omega z\right)\right] / |\gamma x| \text{ for } t = 4, ..T,
\end{aligned}$$

where ωz is an $n * 1$ vector that corresponds to a one period ahead expectation conditional on the current state being x_i:

$$\omega z\,(i) = \Pi\,(i,:) * \log\left(z\,(i,:)'\right) \text{ for } i = 1, 2, ...l.$$

To implement the algorithm, recall the formula for the power of a matrix:

$$\Pi^t = P * \Lambda^t * inv\,(P)\,,$$

where P is the matrix containing the eigenvectors of Π (columns) and Λ is the matrix having the eigenvalues of Π on its diagonal.

For variables that are predetermined with respect to the shock, the algorithm is as follows:

$$\begin{aligned}
\gamma q_1 &= \left[\log\left(q(i)\right) - \log(q(i))\right] / |\gamma x| = 0, \\
\gamma q_2 &= \left[\log(q(j)) - \Pi\,(i,:) * \log(q(:))\right] / |\gamma x|, \\
\gamma q_3 &= \left[\Pi\,(j,:) * q - \Pi\,(i,:) * \omega q\right] / |\gamma x|, \\
\gamma q_4 &= \left[\Pi\,(j,:) * \omega q - \Pi\,(i,:) * (\Pi * wq)\right] / |\gamma x|, \\
\gamma q_t &= \left[\Pi\,(j,:) * \left(\Pi^{t-3} * \omega z\right) - \Pi\,(i,:) * \left(\Pi^{t-2} * \omega z\right)\right] / |\gamma x| \text{ for } t = 4, ..T,
\end{aligned}$$

where

$$\omega q\,(i) = \Pi\,(i,:) * \log\left(q\,(:)\right) \text{ for } i = 1, 2, ...l.$$

2 Example: A Limited Participation Model

To illustrate the procedure, we will apply it to the basic limited participation model discussed in the handout on monetary models.

We will start from a stationary equilibrium. Let $s = \{x_{-1}, x\}$. The stationary equilibrium allocations are defined from the following set of equations:

$$R(s) = \lambda \frac{1+x}{d(x_{-1}) + x}, \tag{5}$$

$$c(s) = \left[1 + \gamma \frac{1+x}{d(x_{-1}) + x}\right]^{-1}, \tag{6}$$

$$p(s) = (1+x)\left[1 + \gamma \frac{1+x}{d(x_{-1}) + x}\right], \tag{7}$$

$$E_{-1}\left(\frac{1}{1+x}\right) = \beta E_{-1}\left[\left(\lambda \frac{1+x}{d+x}\right) \frac{1}{1+x} E\left(\frac{1}{1+x'}\right)\right]. \tag{8}$$

We can map these equilibium conditions in the general notation of (1)-(2) as follows:

$$\begin{aligned} z \quad &: \quad = [R, c, p]', \\ q \quad &: \quad = d. \end{aligned}$$

Condition (1) corresponds to the system of equations (5)-(7), where Γ_1 is implicitly defined by this system. Condition (2) corresponds to (8), which also implicitely defines Γ_2.

We assume that x follows a two state Markov chain, with support $\{x_l, x_h\}$, where $x_l < x_h$. The transition probability matrix is given by:

$$\Pi = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix},$$

where $\pi_{ij} = prob\,(x = x_j | x_{-1} = x_i)$.

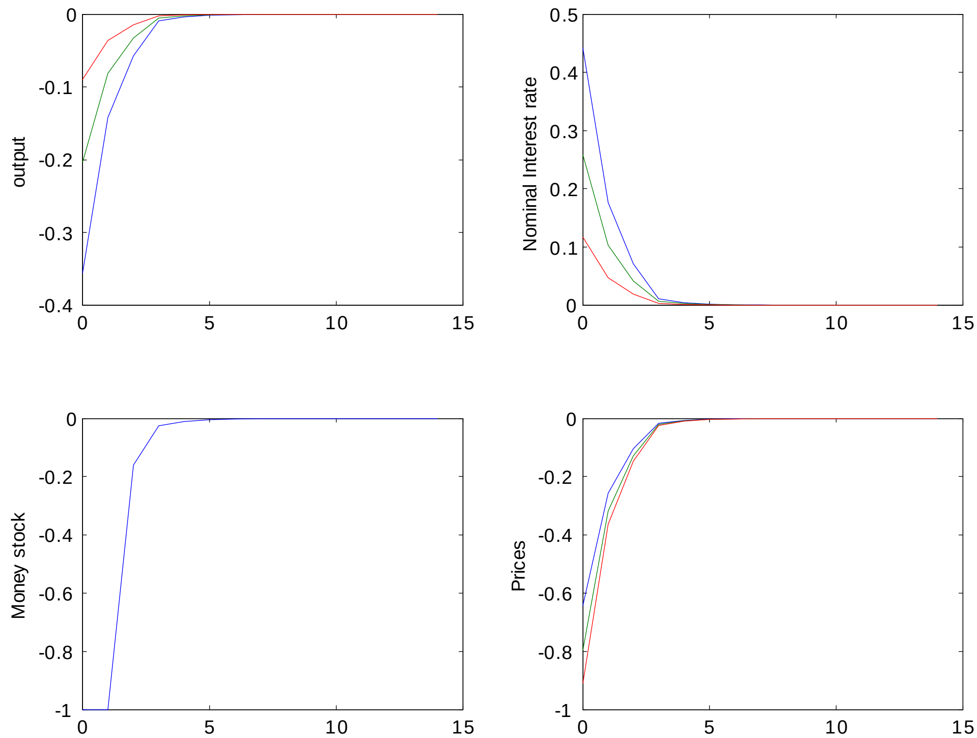
To fully solve for the equilibrium, we can start from (8). Since x_{-1} has two possible values, we can rewrite this equation as:

$$\sum_{j=l,h} \pi_{ij}\left(\frac{1}{1+x_j}\right) = \beta \sum_{j=l,h} \pi_{ij}\left[\left(\lambda \frac{1}{d(x_i) + x_j}\right) \sum_{k=l,h} \pi_{jk}\left(\frac{1}{1+x_{jk}}\right)\right] \text{ for } i = l, h.$$

This delivers $d(x_l)$, $d(x_h)$. We can use the solution for d in the first three equations, to obtain $[R, c, p]$. Note that the state s has four possible values: $s_1 = \{x_l, x_l\}$, $s_2 = \{x_l, x_h\}$, $s_3 = \{x_h, x_l\}$, $s_4 = \{x_h, x_h\}$.

The directory "Samples" contains Matlab files to run to solve the equilibrium and compute impulse response functions for this model, subjet to the assumption that the process for x is a three-state Markov chain with support $[z-\sigma,\, z,\, z+\sigma]$, with $z = 0.0095$, $\sigma = 0.005$. The transition matrix is given by:

$$\Pi = \begin{bmatrix} 1-\alpha_1-\alpha_2 & \alpha_1 & \alpha_2 \\ (1-\phi)/2 & \phi & (1-\phi)/2 \\ \alpha_2 & \alpha_1 & 1-\alpha_1-\alpha_2 \end{bmatrix},$$



where $\alpha_1 = 0.4$, $\alpha_2 = 0.1$ and $\phi = 0.5$.

The lead file is *lp.m*. This is a script file, which means that the file just executes a series of commands. The other two files are *alloc.m* and *interE.m*. They are function files. The first file solves the system of equations (5)-(7). The second file sets up the intertemporal Euler equation. One of the commands in lp.m solves the intertemporal equation for d.

The impulse responses for $\lambda = 0.7$ (blue), 0.8 (green), 0.9 (red) are displayed in the following chart.

Note that lower λ, which corresponds to higher mark-up, implies larger response of output and the nominal interest rate and a smaller response of prices.