Prof. Stefania Albanesi
Columbia University
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Suggested Answers

Question 1- Shocks in a Limited Participation Model

(a) The representative household's problem is

$$\max_{\{C_t, L_t, M_{t+1}, N_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$
$$\text{s.t. } P_t C_t \leq M_t - N_t + W_t L_t \quad (1)$$
$$0 \leq N_t \leq M_t, \quad (2)$$
$$M_{t+1} = M_t - N_t + W_t L_t - P_t C_t + R_t N_t + \pi_t^f. \quad (3)$$

We do not need to impose a no-Ponzi game condition by (2).

Assume that P_t, W_t, R_t and π^f are strictly positive and bounded. The Lagrangian associated to the problem above is:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t [U(C_t, L_t) - \lambda_t (P_t C_t - M_t + N_t - W_t L_t) \\ &\quad -\mu_t \left(M_{t+1} - M_t + N_t - W_t L_t + P_t C_t - R_t N_t - \pi_t^f - \pi_t^{fi}\right)], \end{aligned}$$

where λ_t and μ_t are the Lagrange multipliers associate to the cash-in-advance constraint and the money evolution equation, respectively. This Lagrangian assumes an interior solution with respect to the constraint (2).

The following conditions are jointly necessary and sufficient:

$$\begin{aligned} C_t &: \frac{U_{C,t}}{P_t} = \lambda_t + \mu_t \\ L_t &: \frac{-U_{L,t}}{W_t} = \lambda_t + \mu_t \\ M_{t+1} &: \mu_t = \beta E_t \left(\mu_{t+1} + \lambda_{t+1}\right) \\ N_t &: E_{t-1} (\lambda_t + \mu_t) = E_{t-1} \mu_t R_t, \\ TVC &: \lim_{T \to \infty} \beta^T (\lambda_T + \mu_T) M_t = 0. \end{aligned}$$

(b) The representative firm's problem is:

$$\max_{\{K_t,H_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{U_{C,t+1}}{P_{t+1}} \left(P_t\theta_t H_t - W_t H_t R_t\right)$$

The FONC is:

$$E_t\left[\beta^t \frac{U_{C,t+1}}{P_{t+1}} \left(P_t\theta_t - W_t R_t\right)\right] = 0 \tag{4}$$

(c) A sequence-of-markets equilibrium is a sequence of quantities and prices such that:

- given prices, the households choose $\{C_t, L_t, M_{t+1}, N_t\}_{t=0}^{\infty}$ solving the problems $\forall t$:

$$\max_{\{C_j,L_j,M_{j+1},N_j\}_{j=t}^{\infty}} E_t \sum_{j=t}^{\infty} \beta^{j-t} U(C_j, L_j)$$
$$\text{s.t. } P_j C_j \leq M_j - N_j + W_j L_j$$
$$M_{j+1} = M_j - N_j + W_j L_j - P_j C_j + R_j N_j + \pi_j^f + \pi_j^{fi}$$

- given prices, the firms choose $\{H_t\}_{t=0}^{\infty}$ solving the problems $\forall t$:

$$\max_{\{H_j\}_{j=t}^{\infty}} E_t \sum_{j=t}^{\infty} \beta^{j-t} \frac{U_{C,j+1}}{P_{t+1}} \left(P_j\theta_j H_j - W_j H_j R_j\right) \ \forall t$$

- markets clear $\forall t$:

$$C_t = Y_t = \theta_t H_t \tag{5}$$
$$L_t = H_t \tag{6}$$
$$M_{t+1} - M_t = X_t \tag{7}$$
$$W_t H_t = N_t + X_t \tag{8}$$
$$K_t = 1 \tag{9}$$

(d) Since $(M_t + X_t)/M_t = 1 + x_t$, a positive shock to x_t corresponds to a positive shock to X_t. Combining the cash-in-advance constraint (1), the loans market clearing condition (8), and the labor market clearing condition (6):

$$P_t C_t = M_t + X_t$$

Dividing (8), after having substituted $H_t = L_t$, by the previous equation:

$$\Gamma_t \equiv \frac{W_t L_t}{P_t C_t} = \frac{N_t + X_t}{M_t + X_t}.$$

Since $N_t < M_t$, a positive shock to X_t increases Γ_t, i.e. $\frac{d\Gamma_t}{dX_t} > 0$.

Combining the FONC for C_t, and the one for N_t derived in **(a)**:

$$\frac{W_t}{P_t} = \frac{-U_{L,t}}{U_{C,t}}$$

Substituting the previous equation and the goods market clearing condition, $C_t = \theta_t L_t$, into the expression for Γ_t:

$$\Gamma_t = \frac{1}{\theta_t} \frac{-U_{L,t}}{U_{C,t}}$$

Totally differentiating:

$$d\Gamma_t = \frac{-U_{LL,t}U_{C,t} + U_{L,t}U_{CL,t}}{\theta_t \left[U_{C,t}\right]^2} dL_t$$

Rearranging:

$$\frac{dL_t}{d\Gamma_t} = \frac{\theta_t \left[U_{C,t}\right]^2}{-U_{LL,t}U_{C,t} + U_{L,t}U_{CL,t}}$$

given the usual assumptions on U, i.e. $U_{L,t} < 0$, $U_{LL,t} < 0$, and the additional assumption $U_{CL,t} = 0$, $\frac{dL_t}{d\Gamma_t} > 0$. Hence, $\frac{dL_t}{dx_t} > 0$.
Consider equation (4). Since P_t, H_t, θ_t, W_t, and R_t are all known as of time t, it implies:

$$\left(P_t\theta_t - W_t R_t\right) E_t \left[\beta^t \frac{U_{C,t+1}}{P_{t+1}}\right] = 0$$

$$R_t = \frac{P_t\theta_t}{W_t}$$

Since $H_t = L_t$, and $C_t = \theta_t L_t$,

$$R_t = \frac{P_t C_t}{W_t L_t} = \frac{1}{\Gamma_t}$$

Hence, $\frac{dR_t}{dx_t} = \frac{dR_t}{d\Gamma_t}\frac{d\Gamma_t}{dX_t}\frac{dX_t}{dx_t} < 0$.

(e) Given the assumed functional form for U:

$$\begin{aligned} U_{C,t} &= \frac{1}{C_t} \\ U_{L,t} &= \frac{-\alpha}{1-L_t} \\ \frac{W_t}{P_t} &= \frac{-U_{L,t}}{U_{C,t}} = \frac{\alpha C_t}{1-L_t} \end{aligned}$$

Substituting into the expression for Γ_t, and solving for L_t:

$$\begin{aligned} \Gamma_t &= \frac{W_t}{P_t}\frac{L_t}{C_t} = \frac{\alpha L_t}{1-L_t} \\ L_t &= \frac{\Gamma_t}{\Gamma_t + \alpha} \end{aligned}$$

Γ_t cannot change in response to a technology shock ε_t, because it does not affect X_t, and N_t is determined before the realization of ε_t. Hence, $\frac{dL_t}{d\theta_t} = 0$. Since $R_t = \frac{\alpha}{\Gamma_t}$, $\frac{dR_t}{d\theta_t} = 0$.

Question 2

(a) Differentiate the final goods producer's profits with respect to Y_{it}, and set the result to zero, to get:

$$\frac{Y_t}{Y_{it}} = \left(\frac{P_{it}}{P_t}\right)^{\eta}. \tag{1}$$

This is the final goods' producer's demand for Y_{it}.. Rewriting this,

$$P_{it} = Y_t^{\frac{1}{\eta}} Y_{it}^{\frac{-1}{\eta}} P_t.$$

Raise both sides of the last expression to the power $(1-\eta)$, and integrate over i from 0 to 1:

$$\int_0^1 P_{it}^{1-\eta} di = Y_t^{\frac{1-\eta}{\eta}} P_t^{1-\eta} \int_0^1 Y_{it}^{\frac{\eta-1}{\eta}} di.$$

Raise both sides of this to the power $\eta/(1-\eta)$, and the result is:

$$P_t = \left[\int_0^1 P_{it}^{1-\eta}\right]^{\frac{1}{1-\eta}}. \tag{2}$$

(b) We can think of the i^{th} intermediate food firm as maximizing profits subject to its cost function. In particular, its profits are given by:

$$E_s \left[\beta \frac{u_{c,t+1}}{P_{t+1}} \left(P_{it} Y_{it} - P_t z_t Y_{it}\right)\right], \ s = t, t-1$$

where $P_t z_t Y_{it}$ is the cost of efficiently producing the quantitiy of final goods Y_{it}, and $P_t z_t$ is the marginal dollar cost. The variable z_t is the marginal cost, denominated in consumption good units, and it is the multiplier in the following cost minimization problem:

$$\min_{K_{it}, L_{it}} r_t K_{it} + w_t L_{it} + z_t \left(Y_{it} - \theta_t K_{it}^{\alpha} H_{it}^{1-\alpha}\right).$$

The FONC's associated to the previous problem, assuming interiority, are:

$$\begin{aligned} r_t &= z_t \alpha \theta_t \left(\frac{H_{it}}{K_{it}}\right)^{1-\alpha} = z_t MP_{K,t} \\ w_t &= z_t (1-\alpha) \theta_t \left(\frac{H_{it}}{K_{it}}\right)^{-\alpha} = z_t MP_{H,t} \\ Y_{it} &= \theta_t K_{it}^{\alpha} H_{it}^{1-\alpha}, \end{aligned} \tag{3}$$

which we view as three equations in the three unkonwns, z_t, K_{it} and H_{it}. Here, $MP_{K,t}$ and $MP_{H,t}$ denote the marginal product of capital and labor. It is readily confirmed that z_t has the following solution to this problem[1]:

$$z_t = w_t^{1-\alpha} r_t^{\alpha} \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{1}{\theta_t}.$$

The values of K_{it} and H_{it} which solve these equations may be found as follows. The ratio of these variables is obtained as follows:

$$\frac{r_t}{w_t} = \frac{\alpha}{1-\alpha} \frac{(H_{it}/K_{it})^{1-\alpha}}{(H_{it}/K_{it})^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{H_{it}}{K_{it}}.$$

Given the ratio, H_{it} may be found by solving

$$Y_{it} = \theta_t \left(\frac{K_{it}}{H_{it}}\right)^{\alpha} H_{it}.$$

The variable K_{it} then follows trivially.

The intermediate good producer chooses price, P_{it}, conditional on quantity being determined by the demand curve (1). Thus the problem is

$$\max_{P_{it}} E_s \left[\beta \frac{u_{c,t+1}}{P_{t+1}} Y_t P_t^{\eta} \left(P_{it}^{1-\eta} - P_t z_t P_{it}^{-\eta}\right)\right],$$

for $s = t$ and $t-1$. Given the firm's choice of Y_{it}, its factor input decisions are determined as just discussed.

To find P_{it}, solve the firm problem stated above. It is strictly concave, so setting the derivative to zero is a sufficient condition for a solution:

$$E_s \left[\beta \frac{u_{c,t+1}}{P_{t+1}} Y_t P_t^{\eta} \left(P_{it}^{-\eta} + \frac{\eta}{1-\eta} P_t z_t P_{it}^{-\eta-1}\right)\right] = 0,$$

after dividing by the constant $1-\eta$. Consider case (i) first, so that E_s is E_t. Then, this experssion becomes:

$$E_t \left[\beta \frac{u_{c,t+1}}{P_{t+1}} Y_t P_t^{\eta} \left(P_{it}^{-\eta} + \frac{\eta}{1-\eta} P_t z_t P_{it}^{-\eta-1}\right)\right] =$$
$$\left[Y_t P_t^{\eta} P_{it}^{-\eta-1} \left(P_{it} + \frac{\eta}{1-\eta} P_t z_t\right)\right] E_t \left(\beta \frac{u_{c,t+1}}{P_{t+1}}\right) = 0,$$

which implies, dividing by $Y_t P_t^{\eta} P_{it}^{-\eta-1} E_t \left(\beta \frac{u_{c,t+1}}{P_{t+1}}\right)$:

$$P_{it} = \frac{\eta}{\eta-1} P_t z_t \tag{4}$$

[1] That z_t is indeed $d\text{Cost}_t/dY_{it}$ is readily confirmed by adding $r_t K_{it} + w_t H_{it}$ and using the above equations for r_t and w_t to get that at the cost-minimizing choice of K_{it}, H_{it}, $r_t K_{it} + w_t H_{it} = z_t Y_{it}$.

Now consider case (ii):

$$E_{t-1}\left[\beta\frac{u_{c,t+1}}{P_{t+1}}Y_tP_t^{\eta}\left(P_{it}^{-\eta}+\frac{\eta}{1-\eta}P_tz_tP_{it}^{-\eta-1}\right)\right]=$$

$$\beta E_{t-1}\left[m_tY_tP_t^{\eta}\left(P_{it}^{-\eta}+\frac{\eta}{1-\eta}P_tz_tP_{it}^{-\eta-1}\right)\right]=0,\ m_t=E_t\left(\frac{u_{c,t+1}}{P_{t+1}}\right),$$

by the law of iterated mathematical expectations. Think of E_{t-1} as summing each of the objects inside the square brackets over the period t states of nature. Now, m_t, Y_t, and z_t can be expected to vary non-trivially over these states of nature. However, since P_{it} is selected before the period t state of nature is realized, it is a constant in this sum. Similarly, since the equilibrium aggregate price level is an integral of the intermediate good prices, it is a constant too. As a result, this last expression can be written as:

$$\beta P_t^{\eta}P_{it}^{-\eta-1}E_{t-1}\left[m_tY_t\left(P_{it}+\frac{\eta}{1-\eta}P_tz_t\right)\right]=0,$$

which implies:

$$P_{it}=\frac{\eta}{\eta-1}\frac{P_tE_{t-1}\left(m_tY_tz_t\right)}{E_{t-1}\left(m_tY_t\right)}. \tag{5}$$

(c) The Lagrangean associated to the household problem is:

$$\max_{\{c_t,B_t,M_{t+1},K_{t+1},L_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t\{u\left(c_t,1-L_t\right)+\lambda_t\left[M_t+M_t^s\left(G_t-1\right)-B_t-P_tc_t\right]+$$
$$\mu_t[M_t+M_t^s\left(G_t-1\right)+P_t\left(w_tL_t+r_tK_t\right)-P_tc_t-P_tT_t-$$
$$P_t\left(K_{t+1}-\left(1-\delta\right)K_t\right)+\pi_t+B_t\left(R_t-1\right)-M_{t+1}]\}.$$

Differentiating this, and then susbstituting out the multipliers we get:

$$\frac{u_{c,t}}{P_t}=\beta R_tE_t\frac{u_{c,t+1}}{P_{t+1}}. \tag{6}$$

At the time the households make their consumption purchase, they have the option to reduce these purchases by 1 dollar, at a cost of $1/P_t$ goods, or, $u_{c,t}/P_t$ in utility terms. The gain is R_t additional dollars in the next period, which can be used to purchase R_t/P_{t+1} goods next period. Now, P_{t+1} depends on next period's state of nature. But, the utility function makes it easy to value this nevertheless. The date t contribution to utility is just $\beta R_tE_t\left(u_{c,t+1}/P_{t+1}\right)$. A necessary condition for (interior) maximization is that this type of intertemporal reallocation does not raise utility, i.e., that the first order condition stated above be satisfied as an equality.

The first order condition for K_{t+1} is:

$$\frac{u_{c,t}}{R_t}=\beta E_t\left[\frac{u_{c,t+1}}{R_{t+1}}\left(r_{t+1}+1-\delta\right)\right]. \tag{7}$$

The purchase of one extra unit of K_{t+1} costs P_t dollars that must be delivered at the end of the period. But, P_t dollars at the beginning of next period is equivalent, via the loan market, to P_t/R_t dollars today. This in turn is equivalent to $1/R_t$ units of consumption goods today. Thus, one extra unit of K_{t+1} costs $1/R_t$ units of period t consumption goods. In utility terms, the cost is $u_{c,t}/R_t$. The extra unit of K_{t+1} generates an additional inflow of $P_{t+1}(r_{t+1}+1-\delta)$ dollars at the end of next period. This money comes in at the end of next period, too late to be useful for satisfying next period's cash constraint on consumption. This money that is available at the end of next period is equivalent to $P_{t+1}(r_{t+1}+1-\delta)/R_{t+1}$ dollars at the beginning of next period, for the same reason outlined a moment ago. These extra dollars at the beginning of next period will buy $(r_{t+1}+1-\delta)/R_{t+1}$ extra consumption goods at the beginning of next period, with a period t utility value of $\beta E_t[u_{c.t+1}(r_{t+1}+1-\delta)/R_{t+1}]$. This completes the explanation of the household intertemporal Euler equation for investment.

The Euler equation for labor is:

$$u_{l,t} = w_t u_{c,t}\frac{1}{R_t} \tag{8}$$

where $u_{l,t}$ is the partial derivative of the utility function with respect to *leisure*. If a household works an extra unit of time, it looses $u_{l,t}$ in utility terms, and it gains $P_t w_t$ dollars. However, these dollars cannot be spent on consumption goods until next period, because they cannot be used to satisfy period t cash-in-advance constraint on consumption. But an extra dollar next period is identical to $1/R_t$ dollars in the goods market. So, the $P_t w_t$ dollars to be received at the end of the period correspond to w_t/R_t units of consumption goods, or, $u_{c,t}w_t/R_t$ units of utility. A necessary condition for an interior equilibrium is that the marginal utility cost equal the marginal benefit, i.e., the above equation be satisfied as an equality.

(d) Let $s_t = (\varphi_t, \psi_t)$, and let $s^t = (s_0, s_1, ..., s_t)$. Then, $c(s^t)$ denotes consumption in history s^t. Similarly, $K_i(s^t)$ denotes the capital decision of the i^{th} intermediat good firm (i.e. $K_{i,t+1}$) in history s^t, and $M(s^t)$ is the money decision (i.e. M_{t+1}), and so on for $w(s^t)$, $B(s^t)$, $Y(s^t)$, $Y_i(s^t)$. Case (i) is obvious, and will not be considered here. Consider case (ii). Then, the price in history s^t is the same for all s^t sharing a common s^{t-1}. Thus, the history s^t prices are $P(s^{t-1})$ and $P_i(s^{t-1})$.

The household problem in history s^t is to take $P(s^j)$, $w(s^j)$, $M(s^{t-1})$, $K(s^{t-1})$ as given, $j \geq t$, and choose $c(s^j)$, $B(s^j)$, $L(s^j)$, $K(s^j)$, $M(s^j)$, for all s^j, $j \geq t$, that are possible given s^t, to maximize thier utility function, subject to the cash-in-advance constraint and wealth evolution constraint being satisfied for each possible s^j, $j \geq t$. The household solves this problem in each s^t, $t \geq 0$. $K(s^{-1}) = K_0$ and $M(s^{-1}) = M_0$ are given numbers.

In history s^t, the final good firm takes $P(s^{t-1})$ and $P_i(s^{t-1})$ as given and chooses $Y(s^{t-1})$ and $Y_i(s^{t-1})$ to maximize profits.

In history s^{t-1}, the i^{th} intermediate good firm chooses $P_i(s^{t-1})$ to maximize profits, defined as the history s^{t-1} value of the sum of cash flows across all s^t consistent with s^{t-1}. In doing so, the firm takes as given $w(s^t)$, $Y(s^t)$, $P(s^{t-1})$, and the requirement that its demand curve be satisfied in each s^t.

A symmetric sequence of markets equilibrium is a set of sequnces $\{c(s^t), B(s^t), L(s^t), K(s^t), M(s^t), w(s^t), P(s^{t-1}), P_i(s^{t-1}), K_i(s^t), H_i(s^t), Y_i(s^t), Y(s^t)\}$ with the properties:

(i) $P_i\left(s^{t-1}\right),\ K_i\left(s^t\right),\ H_i\left(s^t\right)$, and $Y_i\left(s^t\right)$ are the same for all $i,\ s^{t-1},\ s^t$ (this is the 'simmetry' part);

(ii) given the prices, the quantities solve the household problem for each s^t;

(iii) given the prices, the quantities solve the final good firm problem for each s^t;

(iv) given $w\left(s^t\right)$, $Y\left(s^t\right)$, $P\left(s^{t-1}\right)$, the objects $P_i\left(s^{t-1}\right)$, $K_i\left(s^t\right)$, $H_i\left(s^t\right)$, $Y_i\left(s^t\right)$ solve the i^{th} intermediate good firm problem;

(v) factor markets clear, $K_i\left(s^t\right) = K\left(s^t\right)$ and $L\left(s^t\right) = H_i\left(s^t\right)$;

(vi) the resource constraint is satisfied for all s^t :

$$c\left(s^t\right) + K\left(s^t\right) - (1-\delta) K\left(s^{t-1}\right) \le \theta\left(s^t\right) K\left(s^{t-1}\right)^\alpha H\left(s^t\right)^{1-\alpha};$$

(vii) the money market clears:

$$M\left(s^t\right) = M^s\left(s^t\right);$$

(viii) the loan market clears:

$$B\left(s^t\right) = 0.$$

We now collect the various equations that characterize the equilibrium. Note that $P_{it} = P_t$ because of symmetry and (2). Consider case (i). Combining the first two equations of (3) with (4), we get:

$$\begin{aligned} r_t &= \frac{\eta - 1}{\eta} MP_{K,t} \\ w_t &= \frac{\eta - 1}{\eta} MP_{H,t}. \end{aligned} \tag{9}$$

We also have:

$$P_t c_t = M_t G_t = M_{t+1},\ t \ge 0. \tag{10}$$

Using $P_t = M_{t+1}/c_t$ and $M_{t+1}/M_t = G_t$ in (6):

$$u_{c,t} c_t = R_t \beta E_t \left[\frac{u_{c,t+1} c_{t+1}}{G_{t+1}}\right],\ t \ge 0. \tag{11}$$

Substituting out for r_t from (9) in (7):

$$\frac{u_{c,t}}{R_t} = \beta E_t \left[\frac{u_{c,t+1}}{R_{t+1}} \left(\frac{\eta - 1}{\eta} MP_{K,t} + 1 - \delta\right)\right],\ t \ge 0. \tag{12}$$

Substituting out for w_t from (9) in (8):

$$u_{l,t} = \frac{\eta - 1}{\eta} MP_{H,t} u_{c,t} \frac{1}{R_t},\ t \ge 0. \tag{13}$$

Finally,

$$c_t + K_{t+1} - (1-\delta) K_t \le \theta_t K_t^\alpha H_t^{1-\alpha},\ t \ge 0. \tag{14}$$

Suppose G_t, θ_t are given stochastic processes. Then equations (11) – (14) represent four equations in four unknown stochastic processes, $c_t,\ H_t,\ K_t,\ R_t$. Given some boundary conditions on K_t, it is possible to solve these equations. Then, given M_0, the stochastic process on G_t, implies a stochastic process for P_t via (10).

(e1) The result follows directly by setting $R_t = R$ in the household's Euler equation for capital. Athough the household's intertemporal Euler equation coincides with that in the one-sector growth model, the equilibrium allocations associated with the $R_t = R$ policy do differ from those in the growth model. Monetary distortions on capital are not eliminated *entirely* by this policy: it is just that the capital-labor ratio is not distorted in (12). The stock of capital itself is distorted by the fact that the labor decision is distorted unless $R = 1$. Another distinction between this and the one-sector growth model is the presence of the monopoly distortion, which enters because $(\eta - 1)/\eta \neq 1$.

(e2) Under the policy $R_t = R$, (11) – (14) is a closed system Note that only G_1 enters this system, not G_0. Because G_0 is left unrestricted, so is P_0. Thus the price level is not pinned down.

(e3) In the log case (11) reduces to:

$$1 = R\beta E_t \left[\frac{1}{G_{t+1}} \right], \ t \geq 0.$$

This equation is clearly satisfied for the money supply process given in the question. What is going on here is that in the log case the nominal interest rate is roughly equal to the expected money growth rate. There are many random money growth rates consistent with a given expected money growth rate.