Prof. Stefania Albanesi
Columbia University
G 6612 Adv. Macro Analysis
Spring 2007

Practice Problems- Solution

Question 1- Equilibria in a Cash-Credit Good Model

(a) Recursively solving for assets using the asset evolution equation yields:

$$\begin{aligned} A_t^d &\geq \frac{q_{t+1}}{q_t} A_{t+1}^d + \frac{q_{t+1}}{q_t} (I_t - S_t) \geq \\ &\geq \frac{q_{t+2}}{q_t} A_{t+2}^d + \frac{q_{t+1}}{q_t} (I_t - S_t) + \frac{q_{t+2}}{q_t} (I_{t+1} - S_{t+1}) \geq \\ &\geq \ ... \geq \frac{q_{t+T}}{q_t} A_{t+T}^d + \sum_{j=0}^{T-1} \frac{q_{t+j+1}}{q_t} (I_{t+j} - S_{t+j}) \end{aligned}$$

Drive $T \to \infty$, use $\lim q_T A_T^d = 0$, and rearrange to get:

$$\sum_{j=0}^{\infty} \frac{q_{t+j+1}}{q_t} S_{t+j} \leq A_t + \sum_{j=0}^{\infty} \frac{q_{t+j+1}}{q_t} I_{t+j}.$$

(b) If $P_t = 0$ the household could purchase an unbounded amount of consumption goods at date t. This cannot be part of an equilibrium in an economy with bounded resources. Similarly for $P_t < 0$. If $W_t \leq 0$, the household would supply no labor, and there would be no output. Yet, there would be a demand for goods, owing to the fact that $A_0^d > 0$. Again, with demand exceeding supply there could not be an equilibrium. If $R_t < 0$, the demand for money would be ubounded. If $\lim_{T\to\infty} \sum_{j=0}^{T} q_{j+1} I_j = \infty$, the demand for cash and credit good would be unbounded.

(c) The Lagrangian associated to the household problem can be written as:

$$\sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} u(c_{1t}, c_{2t}, l_t) - \mu_t \left[\begin{array}{c} \left(B_t^d + M_t^d\right) - (1 + R_{t-1}) \left(M_{t-1}^d + B_{t-1}^d\right) \\ -\left(W_{t-1} L - P_{t-1} \tau_{t-1} + D_{t-1}\right) \\ + \left(R_{t-1} M_{t-1}^d + P_{t-1} (c_{1t-1} + c_{2t-1}) + W_{t-1} (L - l_{t-1})\right) \end{array} \right] \\ -\lambda_t \left(P_t c_{1t} - M_t^d\right) \end{array} \right\},$$

where $\beta^t \mu_t$ and $\beta^t \lambda_t$ denote the Lagrange multiplier associated to the asset evolution equation and to the cash-in-advance constraint, respectively.

The first order conditions are:

$$\begin{aligned} c_{1t} &: \quad \frac{u_{1t}}{P_t} = \mu_{t+1} + \lambda_t & (1) \\ c_{2t} &: \quad \frac{u_{2t}}{P_t} = \mu_{t+1} & (2) \\ l_t &: \quad -u_{3t} = \mu_{t+1} W_t & (3) \\ B_t^d &: \quad \mu_t = \beta \left(1 + R_t\right) \mu_{t+1} & (4) \\ M_t^d &: \quad \mu_t = \beta \left(\mu_{t+1} + \lambda_t\right) & (5) \end{aligned}$$

Equations (2) and (3) imply:

$$\frac{-u_{3t}}{u_{2t}} = \frac{W_t}{P_t} \tag{6}$$

Equations (4) and (5) imply:

$$\left(1 + R_t\right) \mu_{t+1} = \mu_{t+1} + \lambda_t$$

Substituting for μ_{t+1} from (2), for $\left(\mu_{t+1} + \lambda_t\right)$ from (1) and rearranging we obtain

$$\frac{u_{1t}}{u_{2t}} = 1 + R_t \tag{7}$$

Equations (1) and (5) imply

$$\mu_t = \beta \frac{u_{1t}}{P_t}.$$

Substituting into (4) and rearranging:

$$\frac{u_{1t}}{P_t} = \beta \left(1 + R_t\right) \frac{u_{1t+1}}{P_{t+1}}. \tag{8}$$

In addition we have the complementary slackness condition associated to the cash-in-advance constraint:

$$\lambda_t \left(P_t c_{1t} - M_t^d\right) = 0, P_t c_{1t} \leq M_t^d.$$

Equations (1), (2) and (7) imply

$$\lambda_t = R_t \mu_{t+1}.$$

With the usual concavity assumptions the asset evolution equation will hold with equality in equilibrium, implying $\mu_{t+1} \neq 0$. Thus, we can restate the complementarity slackness condition as:

$$R_t \left(M_t^d - P_t c_{1t}\right) = 0, P_t c_{1t} \leq M_t^d.$$

The transversality condition can be seen in two ways. First, by recognizing the equivalence between the transversality and flow budget constraint on the one hand, and the intertemporal budget constraint on the other, we see that the transversality condition simply corresponds to the situation in which the household exhausts its budget constraint. That is, suppose $\lim_{T\to\infty} q_T A_T > 0$. Then the household could deviate from its plan and consume more. This is inconsistent with optimization. Second, note that if $\lim_{T\to\infty} q_T A_T > 0$, then eventually A_t is growing at the rate of interest. That is, eventually the household reinvests both the principle and interest on A_t, and does not consume the proceeds. No optimizing household would engage in such a 'reverse Ponzi-game'. It would prefer at some stage to consume the proceeds of its investment.

Formally, the transversality condition can be written as:

$$\begin{aligned} \lim_{T\to\infty} \beta^T \left[\frac{u_{2T-1}}{P_{T-1}}, \frac{u_{2T-1}}{P_{T-1}}\right] \left[M_T^d, B_T^d\right]' &= 0 \\ \lim_{T\to\infty} \beta^T \frac{u_{2T-1}}{P_{T-1}} \left(B_T^d + M_T^d\right) &= 0 \end{aligned}$$

(2) and (4) imply:

$$\frac{u_{2t-1}}{P_{t-1}} = \beta\,(1+R_t)\,\frac{u_{2,t}}{P_t},$$

or, iterating backwards,

$$\frac{u_{2t-1}}{P_{t-1}} = \beta^{-t+1}\frac{q_t}{q_1}.$$

Thus, $\lim_{T\to\infty} q_T A_T^d = 0$.

(d) Recursively solving for debt using the government's budget constraint:

$$\begin{aligned} B_t &= \frac{q_{t+1}}{q_t} B_{t+1} + \frac{q_{t+1}}{q_t} P_t s_t = \\ &= \frac{q_{t+2}}{q_t} B_{t+2} + \frac{q_{t+1}}{q_t} P_t s_t + \frac{q_{t+2}}{q_t} P_{t+1} s_{t+1} = \\ &= \frac{q_{t+T}}{q_t} B_{t+T} + \sum_{j=0}^{T-1} \frac{q_{t+j+1}}{q_t} P_{t+j} s_{t+j} \end{aligned}$$

where s_t is the real government surplus, including seignorage revenues.

Taking limits for $T \to \infty$ and using $\lim_{T\to\infty} q_T B_T = 0$, we get

$$B_t = \sum_{j=0}^{\infty} \frac{q_{t+j+1}}{q_t} P_{t+j} s_{t+j}.$$

(e)

Definition 1 *An Equilibrium is a sequence,* $\{c_{1t}, c_{2t}, M_t, M_t^d, B_t, B_t^d, P_t, R_t, \tau_t, D_t, l_t\}$, *such that*

(i) the household problem is solved at each t
(ii) the firm first order condition is satisfied at each t
(iii) the monetary policy rule, $M_{t+1}/M_t = \mu$, is satisfied
(iv) the fiscal policy rule, $\lim_{T\to\infty} q_T B_T = 0$, is satisfied
(v) markets clear, i.e., $M_t^d = M_t$, $B_t^d = B_t$, $l_t = c_{1t} + c_{2t} + g$.

(f) **A-** Suppose we have an equilibrium.

Given the assumed functional form for the utlilty function, equations (6) – (8) simplify to:

$$\frac{c_{1t}^{-\sigma}}{P_t} = \beta\left(1+R_t\right)\frac{c_{1t+1}^{-\sigma}}{P_{t+1}} \tag{9}$$

$$c_{2t}^{\delta} = \frac{W_t}{P_t} \tag{10}$$

$$\frac{c_{1t}^{-\sigma}}{c_{2t}^{-\delta}} = 1+R_t \tag{11}$$

Combining (10) with the firm's first order condition $W_t/P_t = 1$, we obtain $c_{2t} = 1$. Thus, from (11), $c_{1t}^{-\sigma} = (1+R_t)$. Substituting into (6) and simplifying

$$\frac{1}{P_t} = \beta\frac{1}{P_{t+1}}c_{1t+1}^{-\sigma}.$$

Multiplying both sides of the previous expression by M_t, and using $M_{t+1}/M_t = \mu$

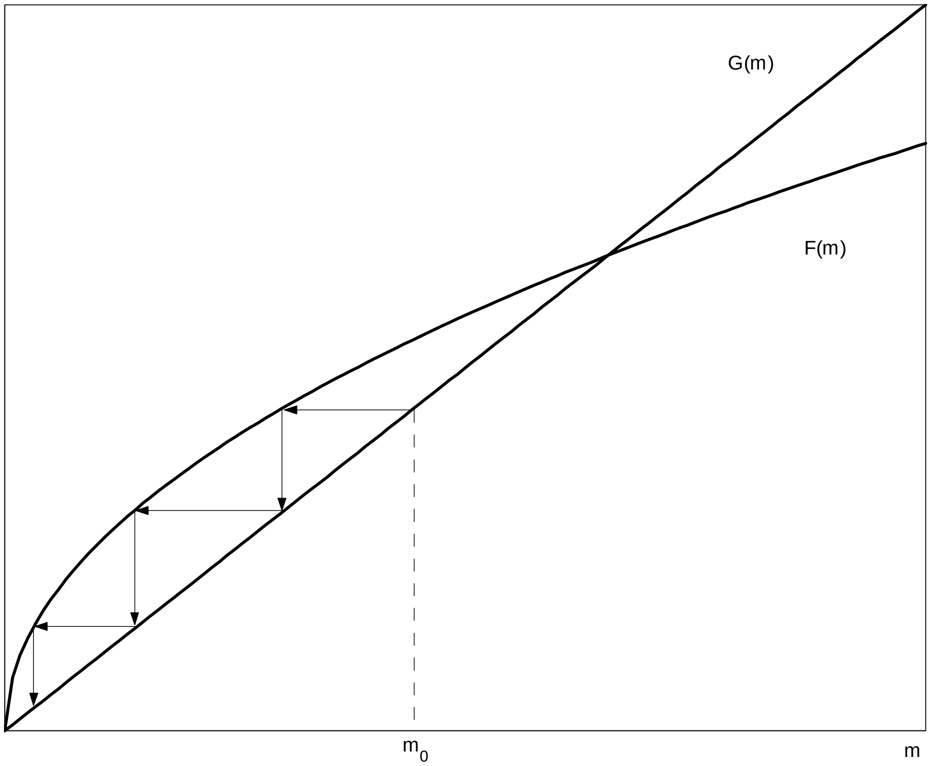
$$\begin{aligned} \frac{M_t}{P_t} &= \beta\frac{M_t}{M_{t+1}}\frac{M_{t+1}}{P_{t+1}}c_{1t+1}^{-\sigma} \\ m_t &= \frac{\beta}{\mu}\frac{m_{t+1}}{c_{1t+1}^{\sigma}} \end{aligned}$$

(ii) follows from the 'complementary slackness condition on the household, and the fact that, $u_1/u_2 = 1+R$ reduces to $c_1^{\sigma} = 1/(1+R)$ with our assumptions on utility. So, $c_{1t} < 1$ corresponds to $R > 0$.

The upper bound on condition (iii) corresponds to the requirement, $R \geq 0$, and the lower bound is just the non-negativity constraint on c_1.

Finally, the transversality condition is obtained by combining the household's transversality constraint with the solvency condition of the government. Now suppose we have a sequence of m_t's and c_{1t}'s that satisfy (i)-(iv). It is easy to verify that the necessary and sufficient conditions for household optimization are satisfied. For example, the transversality condition of the household follows by combining (iv) with the solvency condition on fiscal policy.

B- Trivial.



C- Write the difference equation in m_t as $G(m_t) = F(m_{t+1})$, where $G = m$, and $F = m^{1-\sigma}\beta/\mu$. Graph F and G with m on the horizontal axis and G, F on the vertical. Note that G is the 45 degree line. When $\sigma < 1$, then $F \to 0$ as $m \to 0$, and $F' \to \infty$ as $m \to 0$. So, F rises above the 45 degree line and then cuts it from above. It cuts only once since F is concave. A sequence, m_t, that satisfies this difference equation, with $m_t \to 0$ can be found by setting $0 < m_0 < (\beta/\mu)^{1/\sigma}$. Then, evaluate $G(m_0)$. Then, search horizontally for a value of m_1 such that $F(m_1) = G(m_0)$. Since F is concave, $m_1 < m_0$. Keep iterating in this way, and it is obvious that $m_t \to 0$. Obviously, (i)-(iv) are satisfied, so that this is an equilibrium. Another way to show this is to invert F and write the difference equation in the form, $m_{t+1} = [(\mu/\beta)\, m_t]^{1/(1-\sigma)}$. Note that this is a convex function in m_t, so that for m_0 less than the steady state, the iterates, m_t, converge to zero.

To determine inflation, note $m_{t+1}/m_t = \mu/\pi_{t+1}$, where $\pi_{t+1} =$

P_{t+1}/P_t. So,

$$\pi_{t+1} = \frac{m_t}{m_{t+1}} = (\mu/\beta)^{-1/(1-\sigma)} m_t^{\frac{-\sigma}{1-\sigma}} \to \infty,$$

as $m_t \to 0$.

There is a continuum of exploding hyperinflation equilibria, one for each possible $0 < m_0 < (\beta/\mu)^{1/\sigma}$.

D- If $R_t = 0$ for all t, then it must be that $c_{1t} = 1$, for all t. Then, (i) requires $m_{t+1} = (\mu/\beta) m_t$ for all t. Condition (ii) requires $m_0 \geq 1$. Condition (iii) is satisfied trivially. The rate of inflation is obtained from $m_{t+1}/m_t = \mu/\beta$, or, $\pi_{t+1} = \beta$ for all t. Also, $q_T = \beta^T (P_0/u_{c,0}) u_{c,T}/P_T = \beta^T P_0/P_T = 1$. This candidate equilibrium does not satisfy (iii) because $\beta^t m_t = \mu^t m_0 \nrightarrow 0$ as $t \to \infty$ when $\mu > 1$. If $\mu < 1$, then the transversality condition is satisfied. In this case, $R_t = 0$ and deflation at the rate of time preference is an equilibrium, even though $\mu > \beta$.

E- In this case, the household transversality condition is satisfied for all candidate equilibria, because of the nature of fiscal policy. As a result, the candidate deflation equilibrium is an actual equilibrium.

F- Seignorage revenue is

$$\begin{aligned}(M_{t+1} - M_t)/P_t &= \beta m_{t+1} - m_t = m_t (\beta\mu - 1) = \\ &= \left(\frac{\mu}{\beta}\right)^t (\beta\mu - 1) m_0 \to \infty,\end{aligned}$$

in the deflation equilibrium. Under a Ricardian fiscal policy, $M_T + B_T \to 0$ in the deflation equilibrium, while under the solvency condition, $B_T \to 0$ in this scenario. So, under our solvency condition the seignorage revenues are rebated in the form of tax subsidies, while under a Ricardian fiscal policy, they are used initially to drive B_T to zero, and then to drive B_T negative so that $M_T + B_T$ goes to zero.

Exercise 17.1

(a) Zero net inflation rate means that $\frac{p_{t-1}}{p_t} = 1$, so that $\alpha_t = 1$ as well.

(b) 100% inflation rate means that $\frac{p_{t-1}}{p_t} = 0.5$. This time:

$$\alpha_t = \frac{1.02 - 1}{1.02 - 0.5} = \frac{1}{26} = 0.038462.$$

Exercise 17.2

Note: questions **(b)**, **(c)** and **(d)** are all solved together in question **(b)**.

1. **Definition 2** *An equilibrium is a sequence of price* $\{p_t\}_{t=0}^{+\infty}$*, sequences of household consumption and money holding* $\{c_t, m_{t+1}\}_{t=0}^{+\infty}$*, sequences of government policy* $\{g_t, \tau_t, M_{t+1}\}_{t=0}^{+\infty}$ *such that the following conditions are satisfied:*

   (i) Optimality: given $\{p_t\}_{t=0}^{+\infty}$*,* $\{c_t, m_{t+1}\}_{t=0}^{+\infty}$ *solves the household problem.*

   (ii) Feasibility: the market for good and the market for money clear at all $t \geq 0$ *:*

$$\begin{aligned} c_t + g_t &= y \\ m_{t+1} &= M_{t+1}. \end{aligned}$$

   (iii) The government's budget constraint is satisfied for all $t \geq 0$ *:*

$$g_t = \tau_t + \frac{M_{t+1} - M_t}{p_t}.$$

(b) We form the Lagrangian of the household problem:

$$\sum_{t=0}^{+\infty} \beta^t \left\{ \ln(c_t) + \gamma \ln\left(\frac{m_{t+1}}{p_t}\right) - \lambda_t \left(c_t + \frac{m_{t+1}}{p_t} - y - \tau - \frac{m_t}{p_t}\right) \right\}.$$

The first order conditions are:

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ \frac{\gamma}{m_{t+1}} &= \frac{\lambda_t}{p_t} - \beta \frac{\lambda_{t+1}}{p_{t+1}}. \end{aligned}$$

Market clearing imposes that $c_t = y - g$ and $m_{t+1} = M_{t+1}$. This implies in turn $\lambda_t = \frac{1}{y-g}$. Replacing these expressions into the second first order condition gives:

$$\frac{\gamma}{M_{t+1}} = \frac{1}{y-g}\left(\frac{1}{p_t} - \beta\frac{1}{p_{t+1}}\right).$$

Note also that the market clearing and the household budget constraint imply the government budget constraint:

$$y - \tau - c_t = g - \tau = \frac{M_{t+1} - M_t}{p_t}.$$

Therefore, equilibrium price sequences and money holding sequences are characterized by the following system of difference equations:

$$\begin{aligned} \frac{\gamma}{M_{t+1}} &= \frac{1}{y-g}\left(\frac{1}{p_t} - \beta\frac{1}{p_{t+1}}\right), \ t \geq 0; \\ g - \tau &= \frac{M_{t+1} - M_t}{p_t}, \ t \geq 0. \end{aligned}$$

The first equation is the first order condition evaluated at the equilibrium allocation and the second is the government budget constraint.

Anticipating questions **(c)** and **(d)**, we rewrite this system of difference equations using auxiliary variables, the level of real balances $h_t = \frac{M_{t+1}}{p_t}$, and the return on currency $R_t = \frac{p_t}{p_{t+1}}$. We find, after some manipulations:

$$\begin{aligned} \gamma\left(y-g\right) &= h_t\left(1 - \beta R_t\right), \ t \geq 0; \\ g - \tau &= h_0 - \frac{M_0}{p_0}, \ t = 0; \\ g - \tau &= h_t - R_{t-1}h_{t-1}, \ t \geq 1; \\ R_t &= \frac{p_t}{p_{t+1}} \text{ and } h_t = \frac{M_{t+1}}{p_t}. \end{aligned}$$

The first equation allows to express $h_{t-1}R_{t-1}$ as a function of h_{t-1}. Replacing this expression in the third equation yields to:

$$\begin{aligned} R_t &= \frac{1}{\beta}\left(1 - \frac{\gamma\left(y-g\right)}{h_t}\right), \ t \geq 0; \\ g - \tau &= h_0 - \frac{M_0}{p_0}, \ t = 0; \\ h_t &= g - \tau - \frac{\gamma}{\beta}\left(y-g\right) + \frac{1}{\beta}h_{t-1}, \ t \geq 1; \\ R_t &= \frac{p_t}{p_{t+1}} \text{ and } h_t = \frac{M_{t+1}}{p_t}. \end{aligned}$$

The linear difference equation for h_t is easily solved. First we solve for its unique fixed point $h^* = g - \tau - \frac{\gamma}{\beta}\left(y-g\right) + \frac{1}{\beta}h^*$. Then we subtract the equation defining h^* to the difference equation to find $h_t - h^* = \frac{1}{\beta}(h_{t-1} - h^*)$. We iterate on it and obtain $h_t = h^* + \left(\frac{1}{\beta}\right)^t (h_0 - h^*)$.

Equilibria are constructed in the following way:

(1) Choose $h_0 > 0$.
(2) Solve for p_0 using $g - \tau = h_0 - \frac{M_0}{p_0}$.
(3) Solve for h_t using $h_t = h^* + \left(\frac{1}{\beta}\right)^t (h_0 - h^*)$.
(4) Solve for R_t using $R_t = \frac{1}{\beta}\left(1 - \frac{\gamma(y-g)}{h_t}\right)$.
(5) Solve for p_t and M_{t+1} using $R_t = \frac{p_t}{p_{t+1}}$ and $h_t = \frac{M_{t+1}}{p_t}$.
(6) Accept this equilibrium candidate only if p_0, h_t and R_t are positive.

Stationary equilibria are such that the rate of return on currency R_t is constant. From the above equations it implies that h_t is constant as well. But the difference equation for h_t has a unique fixed point $h^* = \frac{1}{1-\beta}(\gamma(y-g) - \beta(g-\tau))$. Thus there is a unique candidate stationary equilibrium:

$$
\begin{aligned}
h^* &= \frac{1}{1-\beta}(\gamma(y-g) - \beta(g-\tau)) \\
R^* &= \frac{1}{\beta}\frac{\gamma(y-g) - (y-\tau)}{\gamma(y-g) - \beta(y-\tau)} \\
\frac{M_0}{p_0^*} &= \frac{1}{1-\beta}(\gamma(y-g) - (y-\tau)).
\end{aligned}
$$

The positivity restrictions on h_t, R_t and p_0 imposes the following necessary and sufficient condition for existence of a stationary equilibrium:

$$\gamma(y-g) > (y-\tau).$$

(c) See question **(b)**.

(d) See question **(b)**.

(e) Assume that there exists a stationary equilibrium, so that $\gamma(y-g) > (y-\tau)$. Candidate equilibria are described by:

$$
\begin{aligned}
h_t &= h^* + \left(\frac{1}{\beta}\right)^t (h_0 - h^*) \\
R_t &= \frac{1}{\beta}\left(1 - \frac{\gamma(y-g)}{h_t}\right) \\
g - \tau &= h_0 - \frac{M_0}{p_0} \\
R_t &= \frac{p_t}{p_{t+1}} \text{ and } h_t = \frac{M_{t+1}}{p_t}.
\end{aligned}
$$

The restriction $h_t > 0$ imposes that $h_0 \geq h^*$. Otherwise, h_t would be negative for t large enough. Note that this implies that $h_t > h^*$. Since the equation defining R_t is increasing in h_t, we also have $R_t > R^* > 0$. Similarly, the equation defining $\frac{M_0}{p_0}$ is increasing in h_0 so that $\frac{M_0}{p_0} > \frac{M_0}{p_0^*} > 0$. This shows that any $h_0 \geq h^*$ defines an admissible equilibrium.

If $h_0 > h^*$, then h_t goes to infinity. Also R_t, the rate of return on currency, goes to $\frac{1}{\beta} > 1$. Therefore, for t large enough, an economy with $h_0 > h^*$ will experience deflation, in spite of positive deficit.

In order to rank those equilibria, note first that they share the same consumption stream $c_t = y - g$. Thus the household utility in equilibrium h_0 is:

$$\frac{\ln(y-g)}{1-\beta} + \gamma \sum_{t=0}^{\infty} \beta^t \ln(h_t).$$

It is apparent that $h_t = h^* + \left(\frac{1}{\beta}\right)^t (h_0 - h^*)$ is increasing in h_0 and that the utility is increasing in h_t. Therefore the equilibrium with higher h_0 will yield higher utility to the household.

(f) We rephrase the old time religion as: "Larger sustained government deficit, which leaves g unchanged, implies permanently higher inflation rate". In this event, a larger deficit corresponds to a lower tax rate τ. In the stationary equilibrium, a decrease in τ implies a decrease in h^*. As R^* is an increasing function of h^*, this also implies a lower R^*, i.e. a higher inflation. Consider the non stationary equilibrium with initial money holding $h_0 \neq h^*$:

$$h_t = \left(\frac{1}{\beta}\right)^t h_0 + \left(1 - \left(\frac{1}{\beta}\right)^t\right) h^*.$$

Since $\beta < 1$, h_t is a decreasing function of h^*. In this case a decrease in τ decreases h^*, which increases h_t and thus increase R_t, implying a lower inflation.

Exercise 17.4

(a) Consider the Lagrangian associated to the household's problem:

$$\sum_{t=0}^{\infty}\beta^t\left\{c_t^{\alpha}\left(\frac{A}{c_t^{\eta}}\frac{m_{t+1}}{p_t}\right)^{1-\alpha}+\lambda_t\left[y-\tau_t+\frac{m_t}{p_t}+B_t-c_t-\frac{m_{t+1}}{p_t}-B_{t+1}R_t\right]\right\},$$

where leisure has been substituted out and $\lambda_t > 0$ is the Lagrange multiplier associated to the household's budget constraint.

The first order conditions are:

$$c_t \quad : \qquad \lambda_t = [\alpha-\eta(1-\alpha)]\, c_t^{\alpha}\left(\frac{A}{c_t^{\eta}}\frac{m_{t+1}}{p_t}\right)^{1-\alpha} c_t^{-1} \qquad (12)$$

$$m_{t+1} \quad : \qquad \lambda_t = \beta\lambda_{t+1}R_{mt}+(1-\alpha)\, c_t^{\alpha}\left(\frac{A}{c_t^{\eta}}\frac{m_{t+1}}{p_t}\right)^{1-\alpha}\left(\frac{m_{t+1}}{p_t}\right)^{-1} \qquad (13)$$

$$B_{t+1} \quad : \qquad \lambda_t = \beta\lambda_{t+1}R_t, \qquad (14)$$

where $R_{mt} = p_t/p_{t+1}$ is the rate of return on money.

Substituting for $\beta\lambda_{t+1}$ from (14) into (13) and rearranging we obtain:

$$\lambda_t\left(1-\frac{R_{mt}}{R_t}\right) = (1-\alpha)\, c_t^{\alpha}\left(\frac{A}{c_t^{\eta}}\frac{m_{t+1}}{p_t}\right)^{1-\alpha}\left(\frac{m_{t+1}}{p_t}\right)^{-1}.$$

Substituting for λ_t (12), simplifying and rearranging:

$$\frac{m_{t+1}}{p_t} = \frac{1-\alpha}{\alpha-\eta(1-\alpha)}\frac{c_t}{1-\frac{R_{mt}}{R_t}}.$$

Next, using the result that $R_{mt}/R_t = (1+i_t)^{-1}$, and that $c_t = (y-g)$, one gets the demand for money given by:

$$\frac{m_{t+1}}{p_t} = \gamma\frac{y-g}{1-\frac{1}{(1+i_t)}} = \gamma\left(1+\frac{1}{i_t}\right)(y-g), \qquad (15)$$

where $\gamma \equiv \frac{1-\alpha}{\alpha-\eta(1-\alpha)}$. From (15), the demand for real money balances is clearly decreasing in the nominal interest rate.

(b) Since g is constant, $c_t = (y-g)$ is constant as well. From the money demand equation (15), the demand for real money balances is constant. From (12), since both consumption and the demand for real money balances are constant, also the Lagrange multiplier associated to the household's budget constraint is constant. Hence, from (14), $\beta R = 1$, or $R = 1/\beta$.

(c) The demand for money is:

$$\frac{m_{t+1}}{p_t} = \gamma \frac{y-g}{1-\beta R_m}$$

>From the government's budget constraint it follows that the deficit must equal to the revenue from printing money:

$$\begin{aligned} d &= g + \frac{B}{R}\left(R-1\right) - \tau = \frac{m_{t+1}-m_t}{p_t} = \\ &= \frac{m_{t+1}}{p_t} - \frac{m_t}{p_{t-1}}\frac{p_{t-1}}{p_t} = \frac{m_{t+1}}{p_t}\left(1-R_m\right) \\ d &= \gamma\frac{y-g}{1-\beta R_m}\left(1-R_m\right) \equiv \Gamma\left(R_m\right). \end{aligned} \tag{16}$$

The deficit is decreasing in the of return on money:

$$\Gamma'\left(R_m\right) = \gamma \frac{y-g}{\left(1-\beta R_m\right)^2}\left(\beta - 1\right) < 0.$$

Hence, higher deficits are associated with higher inflation[1].

(d) Money demand at time $t = T-1$ and at time $t = T$ is:

$$\begin{aligned} \frac{m_T}{p_{T-1}} &= \gamma\frac{y-g}{1-\beta R_m} \Rightarrow p_{T-1} = \frac{1}{\gamma}\frac{1-\beta R_m}{y-g}m_T \\ \frac{m_{T+1}}{p_T} &= \gamma\frac{y-g}{1-\beta R_m} \Rightarrow p_T = \frac{1}{\gamma}\frac{1-\beta R_m}{y-g}m_{T+1} \end{aligned}$$

Taking the ratio of the two expressions above:

$$\frac{p_T}{p_{T-1}} = \frac{m_{T+1}}{m_T}.$$

Using the money market clearing condition, $m_t = M_t$, and the fact that $M_{T+1} = \left(1+\mu\right)M_T$:

$$\frac{p_T}{p_{T-1}} = \frac{M_{T+1}}{M_T} = \frac{\left(1+\mu\right)M_T}{M_T} = \left(1+\mu\right)$$

(e) Consider the stationary economy before the announcement. Denote the (stationary) price level by p. It follows that:

$$p = \frac{1-\beta}{\gamma}\frac{M}{y-g}.$$

Consider next what happens after time T. From then on, the economy is stationary. Thus, for $t \geq T$, $R_m = 1$, and $p_t = p^* = \left(1+\mu\right)p$.

[1] Recall that inflation, π, equals R_m^{-1}.

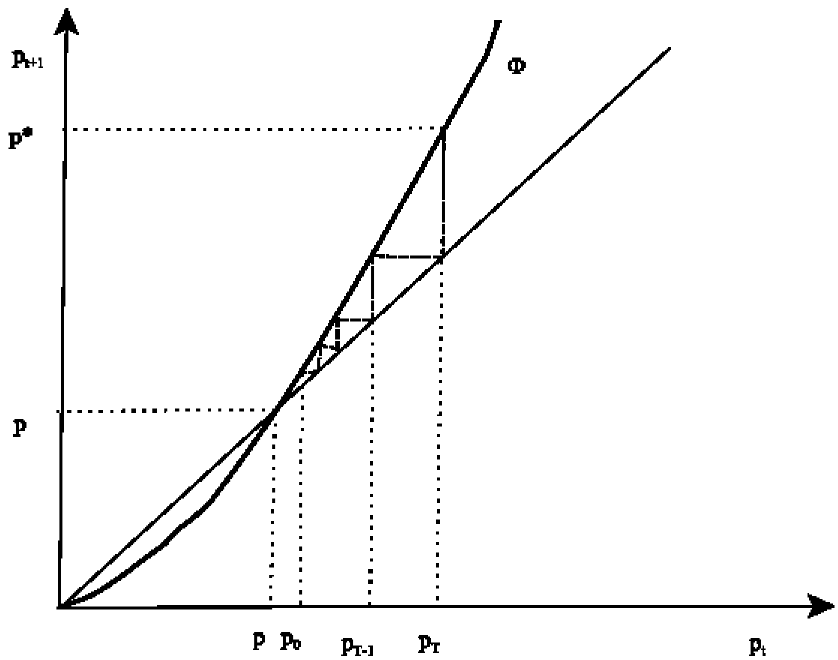


Figure 1: The path of p_t.

>From time 0 to time T, the price level evolution is determined by the money demand equation (15), which can be restated as:

$$p_{t+1} = \frac{\beta}{\frac{1}{p_t} - \gamma \frac{y-g}{M}} = \Phi\left(p_t\right), \tag{17}$$

and by the terminal condition $p_T = p^*$. Clearly, $\Phi\left(0\right) = 0$, $\Phi\left(p\right) = p$, $\Phi' > 0$ and $\Phi'' > 0$. Thus, for a sequence of prices to be an equilibrium, it must be the case that $p_0 > p$. Since $p^*/p = 1 + \mu$, the total rate of inflation p^*/p_0 is less than $(1 + \mu)$. Since the sequence of prices is strictly increasing[2], the inflation rate in every period is smaller than $1 + \mu$.

[2] This implies that inflation is positive in every period from 0 to T, i.e. $\pi_t = \frac{p_{t+1}}{p_{t-}} > 1$.

Question 2.

(a) Market decentralizations.

i. Sequence of markets equilibrium. At each date, t, the household maximizes discounted utility from then on:

$$\sum_{j=t}^{\infty} \beta^{j-t} u(c_j);$$

subject to a sequence of budget constraints:

$$c_j + i_{pj} \le r_j k_{pj} + w_j n - T_j + P_j; \; j \ge t;$$

where w_j and r_j are market prices beyond the control of the household. The household uses its entire endowment of time for labor effort, n; because it does not value leisure. The firms choose n_t and $k_{p;t}$ such that profits are maximized, where profits are defined as follows:

$$Pt = k_{g;t}^{\gamma} n_t^{(1-\alpha)} k_{pt}^{\alpha} - w_t n_t - r_t k_t.$$

A sequence of markets equilibrium is a set of prices and quantities, $\{r_t; w_t; t \ge 0\}$; $\{y_t; c_t; n_t; k_{pt}; k_{gt}; i_{pt}; i_{gt}; t \ge 0\}$ taxes, $\{T_t; t \ge 0\}$; and profits, $\{P_t\}$; such that

- for each t; given taxes, profits and prices, the quantities solve the household problem.
- given the prices and sequence of government capital stocks, the quantities solve the firm problem, for all t.
- given the quantities and a value of s, the government budget constraint is satisfied for all t.
- the resource constraint is satisfied for all t.

ii. Date zero, Arrow-Debreu equilibrium. Let $\{p_t\}$ denote the sequence of date t consumption goods prices, denominated in date 0 consumption units. Let $\{r_t\}$ and $\{w_t\}$ denote the sequences of capital rental rates and wage rates, denominated in date t consumption units. The household's budget constraint is:

$$\sum_{t=0}^{\infty} p_t[c_t + i_{pt}] \le \sum_{t=0}^{\infty} p_t[r_r k_{pt} + w_t n_t - T_t] + P.$$

At date 0, the household enters all markets, and selects quantities to maximize utility.

Consider the firm problem. At date 0, the firm rents factors of production at all dates in order to solve

$$P = \max_{\{y_t; k_{pt}; n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t[y_t - r_t k_{pt} - w_t n_t];$$

subject to the production function, and the given sequences of prices and k_{gt}.

The government's budget constraint is:

$$\sum_{t=0}^{\infty} p_t[i_{gt} - T_t] \le 0.$$

A date 0 Arrow-Debreu competitive equilibrium is a set of prices $\{p_t; r_t; w_t; t \ge 0\}$; quantities, $\{y_t; c_t; n_t; k_{pt}; k_{gt}; i_{pt}; i_{gt}; t \ge 0\}$ taxes, $\{T_t; t \ge 0\}$; and a level of profits, P; such that

- given the prices and level of profits, the quantities solve the household problem.
- given the prices and sequence of government capital stocks, the quantities solve the firm problem
- given the quantities and a value of s, the government budget constraint is satisfied.
- the resource constraint is satisfied for all t.

iii. A recursive competitive. First, define the household problem. To define the problem, for the household to know three numbers: $K_p; K_g$; and k_p; where the first two objects are the economy-wide stocks of private and government capital (note the slight switch in notation regarding government capital), and the individual household's stock of capital. They also need four functions $P(K_p; K_g); r(K_p; K_g); w(K_p; K_g); T(K_p; K_g); I(K_p; K_g);$ which are the level of profits, competitive rental rate and wage rate, the level of taxes and economy-wide average level of investment. The problem is to choose an investment level, $i(K_p; K_g; k_p);$ and a level of employment $n(K_p; K_g; k_p)$ to solve

$$v(K_p; K_g; k_p) = \max_{i_g; \hat{n}} u(c) + \beta v(K_p'; K_g'; k_p');$$

subject to $0 \le \hat{n} \le n; c \ge 0; k_p' \ge 0; k_p' = (1 - \delta)k_p + i_p;$ and

$K_g' = (1 - \delta)K_g + I(K_p; K_g); \; c + i_g \le r(K_p; K_g)k_p + w(K_p; K_g)\hat{n} - T(K_p; K_g):$

The firm problem is to solve

$$P(K_p; K_g) = \max_{\tilde{k}_p; \tilde{n}} y - r(K_p; K_g)\tilde{k}_p - w(K_p; K_g)\tilde{n}:$$

The government has to satisfy a period-by-period budget constraint, $i_g = T(K_p; K_g):$
There are two consistency conditions:

$$k_p = K_p; \text{ and } i(K_p; K_p; K_g) = I(K_p; K_g):$$

The first of these says that everyone's individual stock of capital has to equal the aggregate (economy-wide average) stock. The second says that everyone's individual investment decision has to correspond to the economy wide average.
A recursive competitive equilibrium is a set of functions: $P(K_p; K_g); r(K_p; K_g); w(K_p; K_g); T(K_p; K_g); I(K_p; K_g); i(K_p; K_g; k_p); n(K_p; K_g; k_p); v(K_p; K_g; k_p);$ which satisfy:

- given $P; r; w; T; I;$ the functions $v; i$ and n solve the household problem, for each $k_p; K_p; K_g:$
- for each $K_p; K_g;$ the quantities $K_p;\ P(K_p; K_g)$ and $n(K_p; K_g; K_p)$ solve the firm problem, given $r(K_p; K_g); w(K_p; K_g):$
- the consistency conditions are satisfied
- the resource constraint is satisfied, $c + i_g + i_p \le y;$ for all $K_p; K_g:$

(b) the first order condition for the household is

$$u_{c,t} = \beta u_{c,t+1}[r_{t+1} + 1 - \delta_p],$$

and the firm sets $f_{k_p,t+1} = r_{t+1}$, where $f_{k_p,t+1}$ is the marginal product of private capital. Combining these, and taking functional forms into account:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta\left[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right].$$

Let g_c denote the gross growth rate of consumption in a balanced growth path. Then,

$$(g_c)^{\gamma} = \beta[\alpha(ns)^{(1-\alpha)} + 1 - \delta_p].$$

Suppose g_c corresponds to some given positive net growth rate, ie., $g_c > 1$. Then,

$$s = \frac{1}{n}\left\{\frac{1}{\alpha}\left[\frac{g_c^{\gamma}}{\beta} + \delta_p - 1\right]\right\}^{\frac{1}{1-\alpha}}.$$

The number in square brackets is positive, so that s is well defined. Thus the Euler equation is consistent with constant consumption growth in steady state. To fully answer the question, we need to establish (i) that the other equations - the household budget equation and the resource constraint - are also satisfied with a constant consumption growth rate and (ii) that the other quantity variables display positive growth too. Let g_g and g_p denote the gross growth rates of government and private capital, respectively. Then, the government's policy for choosing $k_{g,t}$ implies:

$$g_g = g_p = g,$$

say. Note that output can be written

$$k_{gt}^{(1-\alpha)} k_{pt}^{\alpha} n^{(1-\alpha)} = k_{gt}(k_{pt}/k_{gt})^{\alpha} n^{(1-\alpha)} = k_{gt} s^{\alpha} n^{(1-\alpha)}.$$

Divide the resource constraint by k_{gt}:

$$\frac{c_t}{k_{gt}} + g_{t+1} - (1 - \delta_g) + g_{t+1} - (1 - \delta_p) = s^{\alpha} n^{(1-\alpha)}.$$

So, in a constant growth steady state (i.e., $g_{t+1} = g$, constant) the consumption to public capital ratio is a constant, equal to the following:

$$s\alpha n^{(1-\alpha)} + (1-\delta_g) + (1-\delta_p) + 2g.$$

But, the consumption to public capital ratio being constant implies:

$$g_c = g.$$

The household budget constraint is trivially satisfied, since it is equivalent with the resource constraint given the first order conditions of firms, linear homogeneity of the production function with respect to firms' choice variables, and the government budget constraint.

(c) The planner's problem is: choose $c_t, k_{g,t+1}, k_{p,t+1}$, $t \geq 0$ to maximize discounted utility. After substituting out consumption using the resource constraint, the problem becomes:

$$\max_{\{k_{g,t+1},k_{p,t+1}\}} \sum_{t=0}^{\infty} \beta^t u[k_{gt}^{(1-\alpha)} n^{(1-\alpha)} k_{pt}^{\alpha} + (1-\delta_g)k_{g,t} + (1-\delta_p)k_{p,t} - k_{p,t+1} - k_{g,t+1}],$$

subject to the object in square brackets (consumption) being nonnegative at all dates, and to $k_{g,t}, k_{p,t} \geq 0$. The planner's first order conditions are:

$$\begin{aligned} u_{c,t} &= \beta u_{c,t+1}[f_{k_{p,t+1}} + 1 - \delta_p] \\ u_{c,t} &= \beta u_{c,t+1}[f_{k_{g,t+1}} + 1 - \delta_g], \end{aligned}$$

for $t = 0, 1, 2, \ldots$. With the functional forms:

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} &= \beta\left[\alpha k_{g,t+1}^{\gamma} \left(\frac{n}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right] \\ \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} &= \beta[\gamma (k_{g,t+1})^{\gamma-1} n^{(1-\alpha)} (k_{p,t+1})^{\alpha} + 1 - \delta_g]. \end{aligned}$$

Substituting out consumption using the resource constraint, these two equations represent a vector difference equation in k, k', k'', where $k = [k_g \; k_p]'$. There are many solutions to this equation that are consistent with the given initial condition, $k_0 = [k_{g,0} \; k_{p,0}]$. One can construct the whole family of solutions by indexing them by k_1: different values of k_1 give rise, by iterating on the euler equation, to different sequences of capital. Not all are optimal. Only the one solution that also satisfies the transversality condition is optimal. Thus, satisfying the Euler equation is not sufficient for an optimum.

(d) Setting $\gamma = 1 - \alpha$ and equating the planner's two first order conditions, we get:

$$\begin{aligned} &\beta\left[\alpha\left(\frac{n k_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right] \\ = \; &\beta\left[(1-\alpha) n^{(1-\alpha)}\left(\frac{k_{p,t+1}}{k_{g,t+1}}\right)^{\alpha} + 1 - \delta_g\right]; \end{aligned}$$

which requires that $\frac{k_{p,t+1}}{k_{g,t+1}}$ be a particular constant for $t = 0, 1, \ldots$. Call this constant s^*. By setting $s = s^*$ the government cannot do better, since this achieves the planner's optimum.