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Practice Problems

Question 1- Equilibria in a Cash-Credit Good Model With Constant Money Growth.

Household preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}, l_t),\ 0 < \beta < 1,$$

where $c_{1t}, c_{2t}, l_t \geq 0$ denote cash goods, credit goods, and employment, respectively. Employment must satisfy the upper bound constraint, $l_t \leq L$, where L denotes the time endowment. The utility function, u, is strictly increasing in c_{1t}, c_{2t}, and $L - l_t$. The typical period is divided into two subperiods, an asset trading period and a goods trading period. At the end of the asset trading period, the household has total financial assets A_t^d. It divides these into money, M_t^d, and bonds, B_t^d, as follows:

$$A_t^d \geq M_t^d + B_t^d, \tag{1}$$

where the superscript 'd' indicates household demand. The household allocates part of its assets to non-interest bearing money in order to satisfy its cash in advance constraint:

$$P_t c_{1t} \leq M_t^d, \tag{2}$$

where P_t denotes the price of goods (cash or credit). The household's assets, A_t^d, reflect a net influx of funds arising from activities in the previous trading period and in the previous goods market period:

$$A_t^d = M_{t-1}^d - P_{t-1}\left(c_{1,t-1} + c_{2,t-1}\right) + W_{t-1} l_{t-1} + B_{t-1}^d (1 + R_{t-1}) + D_{t-1} - P_{t-1}\tau_{t-1}. \tag{3}$$

Here, D_t denotes profits, τ_t denotes real lump sum taxes, and R_t denotes the net nominal rate of interest. Equations (1) and (3) can be written

$$A_t^d \leq (1 + R_{t-1}) A_{t-1}^d + I_{t-1} - S_{t-1}. \tag{4}$$

where household 'income' is:

$$I_t \equiv W_t L - P_t \tau_t + D_t, \tag{5}$$

and household 'spending' is:

$$S_t \equiv R_t M_t^d + P_t\left(c_{1t} + c_{2t}\right) + W_t (L - l_t). \tag{6}$$

The household is required to satisfy the following restriction:

$$\lim_{T\to\infty} q_T A_T^d \geq 0, \tag{7}$$

where

$$q_t = \prod_{j=0}^{t-1} \frac{1}{1+R_j}, \; q_0 \equiv 1. \tag{8}$$

(a) (5 points) Show that (4) and (7) imply, for $t = 0$:

$$\sum_{j=0}^{\infty} \frac{q_{t+j+1}}{q_t} S_{t+j} \leq A_t + \sum_{j=0}^{\infty} \frac{q_{t+j+1}}{q_t} I_{t+j}. \tag{9}$$

$>$From here on, you may treat (9), for each t, as equivalent to (4) and (7).

In period 0 the household takes $A_0^d > 0$ and $\{R_t,\, W_t,\, P_t,\, D_t,\, \tau_t;\, t \geq 0\}$ as given, and it selects $\{M_t^d,\, B_t^d,\, c_{1t},\, c_{2t},\, l_t;\, t \geq 0\}$ to maximize utility subject to, (1)-(3), (7).

(b) (5 points) Explain why prices and rates of return must satisfy the following restrictions in any equilibrium:

$$P_t,\; W_t > 0,\; R_t \geq 0,\; \lim_{T\to\infty} \sum_{j=0}^{T} q_{j+1} I_j \text{ finite}, \tag{10}$$

for all t. (When we discuss the economy's technology below, we shall see that per capita output, y_t, is equal to per capita labor, l_t in this economy.)

(c) (11 points) Suppose $M_t^d,\, B_t^d,\, c_{1t},\, c_{2t},\, l_t;\, t \geq 0$ are strictly positive and solve the household problem. Show that:

$$\frac{u_{1t}}{P_t} = \beta\,(1+R_t)\,\frac{u_{1t+1}}{P_{t+1}}, \tag{11}$$

$$\frac{-u_{3t}}{u_{2t}} = \frac{W_t}{P_t}, \tag{12}$$

$$\frac{u_{1t}}{u_{2t}} = 1 + R_t, \tag{13}$$

$$R_t(M_t^d - P_t c_{1t}) = 0,\; M_t^d - P_t c_{1t} \geq 0. \tag{14}$$

$$\lim_{T\to\infty} q_T A_T^d = 0. \tag{15}$$

In the case of (15), you may simply provide the intuition of the proof of necessity.

>From here on, you may treat (11)-(15) as being both necessary and sufficient for household optimization. Firms have access to the production technology, $y = f(l) = l$. They are perfectly competitive in product and labor markets, so that firm optimization requires:

$$\frac{W_t}{P_t} = 1. \tag{16}$$

The aggregate resource constraint is

$$c_{1t} + c_{2t} + g \leq l_t, \tag{17}$$

where $g \geq 0$ denotes government consumption.

We now describe the activities of the government. The government's sources of funds in the period t asset market are lump sum tax obligations arising in the previous goods market, $P_{t-1}\tau_{t-1}$, money earned by issuing new bonds, B_t, and addition to the aggregate stock of money, $M_t - M_{t-1}$, subject to $M_t \geq 0$. The government's uses of funds include payments for goods, g, purchased in the previous goods market, and interest on debt issued in the previous asset market:

$$M_t - M_{t-1} + B_t + P_{t-1}\tau_{t-1} = P_{t-1}g + (1 + R_{t-1})B_{t-1}. \tag{18}$$

Government policy is composed of fiscal and monetary policy. Fiscal policy refers to the setting of τ_t. We assume that τ_t is set in such a way that, regardless of the realization of prices or money:

$$\lim_{T\to\infty} q_T B_T = 0. \tag{19}$$

(d) (5 points) Equations (18) and (19) are equivalent to the proposition that the government eventually pays off its debt:

$$\sum_{j=0}^{\infty} \frac{q_{t+j+1}}{q_t} P_{t+j} s_{t+j} = B_t, \; t \geq 0, \tag{20}$$

where s_t is the real government surplus, including seignorage revenues:

$$s_t = \tau_t + \frac{M_{t+1} - M_t}{P_t} - g,$$

where seignorage is

$$\frac{M_{t+1} - M_t}{P_t}.$$

Show that (18) and (19) imply (20) for $t = 0$.

Monetary policy refers to the government's choice about the evolution of M_t. We suppose the money supply evolves as follows:

$$\frac{M_{t+1}}{M_t} = \mu > 1. \tag{21}$$

(e) (5 points) Provide a definition of equilibrium for this economy.

(f) Suppose

$$u(c_1, c_2, l) = \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\delta}}{1-\delta} - l, \ \sigma, \delta > 0.$$

Let $m_t = M_t/P_t$ denote real money balances.

A- (16 points) Show that a sequence of m_t's and c_{1t}'s corresponds to an equilibrium if, and only if, for $t = 0, 1, 2, ...,$

(i) it satisfies

$$m_t = \frac{\beta}{\mu} \frac{m_{t+1}}{c_{1t+1}^{\sigma}} \tag{22}$$

Hint: combine the household's and the firm's first order conditions.

(ii) $m_t \geq c_{1t}$, $c_{1t} < 1 \Longrightarrow m_t = c_{1,t}$,

(iii) $0 \leq c_{1t} \leq 1$

(iv) it satisfies

$$\lim_{T\to\infty} \beta^T \frac{m_T}{c_{1T}^{\sigma}} = 0. \tag{23}$$

where u_{it} denotes the derivative of $u(c_{1t}, c_{2t}, l_t)$ with respect to its i^{th} argument. Also, the phrase 'x corresponds to an equilibrium' means that the remaining objects in an equilibrium can be found such that these and x together constitute an equilibrium.

B- (5 points) Show that $c_{1t} = m_t = (\beta/\mu)^{1/\sigma}$, for all t corresponds to an equilibrium. Display formulas for R_t, P_{t+1}/P_t and l_t in that equilibrium.

C- (16 points) Show that there exists an exploding hyperinflation equilibrium in which $m_t \to 0$, if $\sigma < 1$. (**Hint**: show that when m_t is small, then $c_{1t} = m_t$ and (22) reduces to a first order difference equation in m_t and m_{t+1}.) Display what happens to the rate of inflation in this equilibrium. How many such equilibria are there?

D- (16 points) Show that there exists a candidate deflation equilibrium in which $R_t = 0$ for all t, and the corresponding c_{1t}'s and m_t's satisfy (i)-(iii) above. What is the rate of inflation in this candidate equilibrium? What values of m_0 are consistent with this candidate equilibrium? What is q_t in this candidate equilibrium? Show that this candidate equilibrium is not an actual equilibrium because it fails to satisfy (23). Would it be an equilibrium if instead monetary policy set $\beta < \mu < 1$? Explain.

E- (5 points) Under a 'Ricardian' fiscal policy, fiscal policy satisfies a version of (19) with B_T replaced by $B_T + M_T$. Explain why the candidate deflation equilibrium in **D-** is an actual equilibrium under a Ricardian fiscal policy.

F- (11 points) Show that seignorage revenues are exploding in the candidate deflation equilibrium. Compare what is done with these revenues under the Ricardian fiscal policy and under the solvency condition, (19).

Question 2

Consider the following economy. Preferences of the representative household are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where

$$u(c) = \frac{c^{1-\nu} - 1}{1-\nu},\ \ \nu > 0.$$

The economy accumulates private capital according to the following capital accumulation technology:

$$k_{p,t+1} = (1-\delta_p) k_{p,t} + i_{p,t},$$

where $k_{p,t}$ is the beginning of period stock of private capital, $i_{p,t}$ is period t gross investment, and $0 < \delta_p < 1$. The initial stock of private capital $k_{p,0}$ is given. The household is endowed with $n = 1$ units of time, and must satisfy: $c_t \geq 0$, $k_{p,t} \geq 0$.

The technology for producing output is:

$$y_t = k_{g,t}^{\gamma} k_{p,t}^{\alpha} n_t^{1-\alpha},\ \ \alpha \in (0,1), \gamma \geq 0,$$

where $k_{g,t}$ is the per-capita stock of government capital. The households and the firm take the sequence $\{k_{g,t}\}$ as given and out of their control.

The technology for accumulating government capital is:

$$k_{g,t+1} = (1-\delta_g) k_{g,t} + i_{g,t},$$

where $i_{g,t}$ denotes per capita gross investment by the government, and $0 < \delta_g < 1$. The government finances investment by levying lump sum taxes, T_t. Taxes are lump sum in the sense that each household must pay the amount T_t, regardless of their choice of consumption, investment, or labor. The government's budget constraint in each period is:

$$i_{g,t} = T_t.$$

The resource constraint for this economy is:

$$c_t + i_{p,t} + i_{g,t} \leq y_t.$$

a) Suppose the government chooses its sequence of public investment according to the rule:

$$k_{g,t} = s k_{p,t},\ \ t \geq 1,\ \ s > 0.$$

Describe the household problem and the firm problem, assuming that the households own and accumulate private capital, as well as the firms.

i) Define a sequence of markets equilibrium.

ii) Define an Arrow Debreu equilibrium.

b) Suppose $\gamma = 1 - \alpha$. Show that s can be chosen so that there is a steady-state balanced growth path for this economy, in which all quantity variables but employment display the same positive growth rate. Explain intuitively why steady state growth is possible in this case, but not when $\gamma = 0$.

c) Display an optimization problem, the solution of which gives the efficient allocation for this economy. Write out the Euler equations for this problem. Are they sufficient for a maximum of the problem? Explain your answer.

d) Suppose $\gamma = 1 - \alpha$. Assume that the economy is in a sequence of markets equilibrium, and that the government must optimally choose a sequence $i_{g,t}$, $t = 0, 1, 2, ...$ Would that sequence be consistent with the specification of public investment assumed in part a)? Explain your answer.

Other Questions on Monetary Models

Question 24.1, 24.2, 24.4 in Sargent and Ljunqvist.