G 6222 ADV. MACRO ANALYSIS II
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PROF. STEFANIA ALBANESI

Assignment 1, SUGGESTED ANSWERS.

Question 1- Commodity Taxation

Resource constraint

$$F\left(c_1+g_1, c_2+g_2, c_3+g_3, l\right)=0 \tag{1}$$

where c_i and g_i denote private and government consumption of each good i, l denotes labor, and F denotes a production process which satisfies constant returns to scale.

1. The consumer's problem is to maximize utility:

   $$\max U\left(c_1, c_2, c_3, l\right) \tag{2}$$

   subject to

   $$\sum_i p_i\left(1+\tau_i\right) c_i=l \tag{3}$$

   where p_i is the price of good i, and τ_i is an ad valorem tax rate on good i. The wage is normalized to 1.

   A competitive representative firm operates the constant returns to scale technology F and solves:

   $$\max_{(x,l)} \sum_i p_i x_i-l \tag{4}$$

   subject to

   $$F\left(x_1, x_2, x_3, l\right)=0 \tag{5}$$

   where x_i denotes output of good i.

   i) Taking government expenditures $[g_1, g_2, g_3]$ as given, a competitive equilibrium for this economy is a policy $\pi=\left(\tau_i\right)_{i=1}^3$; allocations c, l, x, and a price system p such that:

   (a) The allocations c and l maximize household's utility (2) subject to (3)

   (b) The allocations x and l solve the problem of the firm (4)

   (c) The government constraint holds

   $$\sum_i p_i g_i=\sum_i p_i \tau_i c_i \tag{6}$$

(d) Goods market clears

$$c_i + g_i = x_i \tag{7}$$

Assuming strict concavity for $U(\cdot)$ and imposing the Inada conditions for the utility function and the firm's technology, we can guarantee convexity of the optimization problem and interiority of the solution. Hence, the first-order conditions for firms and households are necessary and sufficient.

ii) Condition (1) follows from substituting the market-clearing condition (7) into (5).

To derive

$$\sum_i U_i c_i + U_l l = 0 \tag{8}$$

we solve the household's problem:

$$L = U(c_1, c_2, c_3, l) - \lambda \left[\sum_i p_i (1+\tau_i) c_i - l \right]$$

$$\begin{array}{rcll} [c_i] & : & U_i = \lambda p_i (1+\tau_i) & \text{for } i = 1,2,3 \\ [l] & : & -U_l = \lambda & \\ [\lambda] & : & \sum_i p_i (1+\tau_i) c_i = l & \end{array}$$

Combining $[c_i]$ and $[l]$

$$-\frac{U_i}{U_l} = p_i (1+\tau_i) \tag{9}$$

back into $[\lambda]$and we obtain (8)

$$\begin{aligned} \sum_i -\frac{U_i}{U_l} c_i &= l \\ \sum_i U_i c_i &= U_l l \\ \sum_i U_i c_i + U_l l &= 0 \end{aligned}$$

To show that for allocations that satisfy (1) and (8), we can find policies and prices such that together with the given allocation constitute a competitive equilibrium, we use the f.o.n.c for the firm:

$$L = \sum_i p_i x_i - l - \theta (F(x_1, x_2, x_3, l))$$

$$\begin{array}{rcll} [x_i] & : & p_i - \theta F_i = 0 & \text{for } i = 1,2,3 \\ [l] & : & -1 = \theta F_l & \end{array}$$

Combining $[x_i]$ and $[l]$ we obtain an expression for prices

$$p_i = -\frac{F_i}{F_l} \qquad \text{for } i = 1, 2, 3 \tag{10}$$

Similarly, we obtain an expression for the policy by combining (9) and (10)

$$\begin{aligned} -\frac{U_i}{U_l} &= \left[-\frac{F_i}{F_l}\right](1+\tau_i) \\ (1+\tau_i) &= \frac{U_i}{U_l}\frac{F_l}{F_i} \qquad \text{for } i = 1, 2, 3 \end{aligned} \tag{11}$$

Notice that by defining p_i and τ_i as we just did, we satisfy household's and firm's maximization problems, so together with the given allocation, they constitute a competitive equilibrium.

iii) A Ramsey equilibrium is a policy $\pi = (\tau_i)_{i=1}^3$ in Π; allocation rules $c(\cdot)$, $l(\cdot)$, $x(\cdot)$; and a price function $p(\cdot)$ that satisfy:

(a) The policy π solves

$$\max_{\pi'} U\left(c\left(\pi'\right), l\left(\pi'\right)\right)$$

subject to

$$\sum_i p_i\left(\pi'\right) g_i = \sum_i p_i\left(\pi'\right) \tau_i' c_i\left(\pi'\right)$$

(b) for every π', the allocations $c(\pi')$, $l(\pi')$, $x(\pi')$, the price system $p(\pi')$ and the policy π' constitute a competitive equilibrium.

iv) Ramsey allocation problem: Choose c and l to maximize $U(c, l)$ subject to (1) and (8)

$$L = U(c_1, c_2, c_3, l) + \mu\left[F\left(c_1 \underset{-}{+} g_1, c_2 \underset{-}{+} g_2, c_3 \underset{-}{+} g_3, \underset{+}{l}\right)\right] + \gamma\left[\sum_i U_i c_i + U_l l\right]$$

$$\begin{aligned} [c_i] \quad &: \quad U_i - \mu F_i + \gamma\left(U_i + U_{ii}c_i + U_{ji}c_j + U_{ki}c_k + U_{il}l\right) = 0 \\ &: \quad (1+\gamma)U_i + U_i\gamma\left(\frac{U_{ii}}{U_i}c_i + \frac{U_{ji}}{U_i}c_j + \frac{U_{ki}}{U_i}c_k + \frac{U_{il}}{U_i}l\right) = \mu F_i \\ &: \quad (1+\gamma)U_i + \gamma U_i\left(\textstyle\sum_{j=1}^3 \frac{U_{ji}}{U_i}c_j + \frac{U_{il}}{U_i}l\right) = \mu F_i \end{aligned}$$

$$(1+\gamma)U_i - \gamma\varepsilon_i U_i = \mu F_i \tag{12}$$

where

$$\varepsilon_i = -\left(\textstyle\sum_{j=1}^3 \frac{U_{ji}}{U_i}c_j + \frac{U_{il}}{U_i}l\right) \qquad \text{for } i = 1, 2, 3$$

$$\begin{aligned}
[l] \quad &: \quad -U_l + \mu F_l - \gamma \left(U_{1l} c_1 + U_{2l} c_2 + U_{3l} c_3 + U_l + U_{ll} l \right) = 0 \\
&: \quad (1+\gamma) U_l + U_l \gamma \left(\frac{U_{1l}}{U_l} c_1 + \frac{U_{2l}}{U_l} c_2 + \frac{U_{3l}}{U_l} c_3 + \frac{U_{ll}}{U_l} l \right) = \mu F_l \\
&: \quad (1+\gamma) U_l + U_l \gamma \left(\textstyle\sum_{j=1}^{3} \dfrac{U_{jl}}{U_l} c_j + \dfrac{U_{ll}}{U_l} l \right) = \mu F_l
\end{aligned}$$

$$(1+\gamma) U_l - U_l \gamma \varepsilon_l = \mu F_l \tag{13}$$

where

$$\varepsilon_l = -\left(\textstyle\sum_{j=1}^{3} \dfrac{U_{jl}}{U_l} c_j + \dfrac{U_{ll}}{U_l} l \right) \qquad \text{for } i = 1,2,3$$

Notice that ε_i and ε_l can be interpreted as the inverse of the income elasticities.

Rearranging (12) and making use of (11)

$$\begin{aligned}
\left[(1+\gamma) - \gamma \varepsilon_i \right] \frac{U_i}{F_i} &= \mu \\
\left[(1+\gamma) - \gamma \varepsilon_i \right] (1+\tau_i) \frac{U_l}{F_l} &= \mu
\end{aligned}$$

Rearranging (13)

$$\left[(1+\gamma) - \gamma \varepsilon_l \right] \frac{U_l}{F_l} = \mu$$

Combining this equation with the former,

$$\begin{aligned}
(1+\gamma) - \gamma \varepsilon_l &= \left[(1+\gamma) - \gamma \varepsilon_i \right] (1+\tau_i) \\
(1+\gamma) - \gamma \varepsilon_l &= (1+\gamma) + \tau_i (1+\gamma) - \gamma \varepsilon_i (1+\tau_i) \\
-\gamma \varepsilon_l &= \tau_i (1+\gamma) - \gamma \varepsilon_i (1+\tau_i) \\
-\gamma \varepsilon_l - \tau_i \gamma \varepsilon_l + \tau_i \gamma \varepsilon_l &= \tau_i (1+\gamma) - \gamma \varepsilon_i (1+\tau_i) \\
-\gamma \varepsilon_l (1+\tau_i) + \tau_i \gamma \varepsilon_l &= \tau_i (1+\gamma) - \gamma \varepsilon_i (1+\tau_i) \\
-(1+\tau_i)(\gamma \varepsilon_l - \gamma \varepsilon_i) &= \tau_i (1+\gamma - \gamma \varepsilon_l)
\end{aligned}$$

$$\frac{\tau_i}{(1+\tau_i)} = \frac{\gamma (\varepsilon_i - \varepsilon_l)}{(1+\gamma-\gamma\varepsilon_l)} \qquad \text{for } i = 1,2,3 \tag{14}$$

Hence, for any i, $j = 1,2,3$

$$\begin{aligned}
\frac{\frac{\tau_i}{(1+\tau_i)}}{\frac{\tau_j}{(1+\tau_j)}} &= \frac{\frac{\gamma(\varepsilon_i - \varepsilon_l)}{(1+\gamma-\gamma\varepsilon_l)}}{\frac{\gamma(\varepsilon_j - \varepsilon_l)}{(1+\gamma-\gamma\varepsilon_l)}} = \frac{(\varepsilon_i - \varepsilon_l)}{(\varepsilon_j - \varepsilon_l)} \\
\frac{\tau_i}{(1+\tau_i)} \frac{1}{(\varepsilon_i - \varepsilon_l)} &= \frac{\tau_j}{(1+\tau_j)} \frac{1}{(\varepsilon_j - \varepsilon_l)}
\end{aligned}$$

If $\varepsilon_i > \varepsilon j \implies \frac{1}{(\varepsilon_i - \varepsilon_l)} < \frac{1}{(\varepsilon j - \varepsilon_l)} \implies \frac{\tau_i}{(1+\tau_i)} > \frac{\tau_j}{(1+\tau_j)} \implies \tau_i > \tau_j$. This implies that goods that exhibit higher ε or lower income elasticity (i.e. necessities) should be taxed more than those that show lower ε, or higher elasticity (i.e. luxuries).

To see why ε_i and ε_l can be interpreted as the inverse of the income elasticities, suppose that preferences are additively separable. Then

$$\varepsilon_i = -\frac{U_{ii}}{U_i} c_i$$

Let $c_i(p, m)$ be the demand function for good i and $l(p, m)$ be the labor supply, when households maximize utility with respect to $\sum p_i c_i + m$, where m is nonlabor income. Hence, the f.o.n.c. with respect to.c_i would be

$$U_i\left(c_i(p, m)\right) = \lambda(p, m) p_i$$

where $\lambda(p, m)$ is the Lagrange multiplier. Differentiating this equation with respect to m,

$$U_{ii} \frac{\partial c_i(p, m)}{\partial m} = \frac{\partial \lambda(p, m)}{\partial m} p_i$$

and combining with the former,

$$\begin{aligned} U_{ii} \frac{\partial c_i(p, m)}{\partial m} &= \frac{\partial \lambda(p, m)}{\partial m} \frac{U_i\left(c_i(p, m)\right)}{\lambda(p, m)} \\ \frac{U_{ii}}{U_i} \frac{\partial c_i}{\partial m} &= \frac{1}{\lambda} \frac{\partial \lambda}{\partial m} \end{aligned}$$

From the definition of ε_i we know that $\varepsilon_i = -\left(\frac{U_{ii}}{U_i} c_i\right) \Longrightarrow \frac{U_{ii}}{U_i} = -\frac{\varepsilon_i}{c_i}$. Substituting in the latter expression

$$\begin{aligned} \frac{\varepsilon_i}{c_i} \frac{\partial c_i}{\partial m} &= -\frac{1}{\lambda} \frac{\partial \lambda}{\partial m} \\ \varepsilon_i &= -\frac{c_i}{\lambda} \frac{\frac{\partial \lambda}{\partial m}}{\frac{\partial c_i}{\partial m}} = \frac{1}{\eta_i} \end{aligned}$$

where η_i denotes the income elasticity of demand for good i.

(v) The first order condition for consumers is

$$U_i(c_i(p, m)) = \lambda(p, m) p_i$$

where λ is the Lagrange multiplier before the constraint: $\sum_{i=1}^{3} p_i c_i = l + m$. To derive the income elasticity of demand, totally differentiate this first order condition with respect to m:

$$\begin{aligned}
U_{ii}\frac{\partial c_i(p,m)}{\partial m} &= p_i\frac{\partial \lambda(p,m)}{\partial m} && (15)\\
U_{ii}\frac{\partial c_i(p,m)}{\partial m} &= \frac{U_i}{\lambda(p,m)}\frac{\partial \lambda(p,m)}{\partial m} && (16)\\
-\frac{U_{ii}c_i}{U_i} &= -\frac{c_i}{\lambda(p,m)}\frac{\partial \lambda(p,m)}{\partial m}\frac{1}{\frac{\partial c_i(p,m)}{\partial m}} && (17)\\
\varepsilon_i &= -\frac{c_i}{m}\frac{1}{\frac{\partial c_i(p,m)}{\partial m}}\frac{m}{\lambda(p,m)}\frac{\partial \lambda(p,m)}{\partial m} && (18)\\
\varepsilon_i &= \frac{1}{\eta_i}\frac{m}{\lambda(p,m)}\frac{\partial \lambda(p,m)}{\partial m} && (19)\\
\varepsilon_i\eta_i &= \frac{m}{\lambda(p,m)}\frac{\partial \lambda(p,m)}{\partial m} && (20)\\
\varepsilon_i\eta_i &= \varepsilon_j\eta_j && (21)\\
\frac{\varepsilon_i}{\varepsilon_j} &= \frac{\eta_j}{\eta_i}. && (22)
\end{aligned}$$

Hence, ε_i is inversely related with the income elasticity of demand. One interpretation for this result is that goods with relatively low income elasticities should be taxed more heavily, given the preference are additively separable.

(vi) Now the first order condition is rewritten as

$$U_i(c_i(p,m)) = \lambda p_i$$

where λ is the Lagrange multiplier before the constraint: $\sum_{i=1}^{3} p_i c_i = l + m$. We can write the condition in this way because there is no income effect, which can be seen from the optimal condition $U_l = 1 = \lambda$. Take the first order derivative with respect to p_i,

$$\begin{aligned}
U_{ii}\frac{\partial c_i(p,m)}{\partial p_i} &= \lambda && (23)\\
U_{ii}\frac{\partial c_i(p,m)}{\partial p_i} &= \frac{U_i}{p_i} && (24)\\
-\frac{U_{ii}c_i}{U_i} &= -\frac{c_i}{p_i}\frac{1}{\frac{\partial c_i(p,m)}{\partial p_i}} && (25)\\
\varepsilon_i &= \left[-\frac{\partial c_i(p,m)}{\partial p_i}\frac{p_i}{c_i}\right]^{-1} && (26)
\end{aligned}$$

This result shows that ε_i is inversely related with the price elasticity of demand. One interpretation for this result is that goods with relatively low price elasticities should be taxed more heavily. This is a standard result in public finance which is usually derive in partial equilibrium. In general equilibrium models, it only holds if preferences do not admit income effects and are additively separable in consumption and labor.

(vii) The Ramsey allocation problem, if the government can only tax good 1 and good 2, is to choose c and l to maximize

$$U(c_1, c_2, c_3, l)$$

subject to

$$\begin{aligned} 0 &= F(c_1 + g_1, c_2 + g_2, c_3 + g_3, l) && (27) \\ 0 &= \sum_i U_i c_i + U_l l && (28) \\ p_1 \tau_1 c_1 + p_2 \tau_2 c_2 &\geq p_1 g_1 + p_2 g_2 + p_3 g_3 && (29) \end{aligned}$$

Let μ, γ, and θ be the Lagrange multipliers associated with the resource constraint, the implementability constraint (following the same logic, we can show that this constraint is the same as before), and the $\tau_3 = 0$ restriction, respectively, then the first order conditions are:

$$\begin{aligned} (1 + \gamma - \gamma \varepsilon_1) - \mu \frac{F_1}{U_1} &= \theta \tau_1 \frac{p_1}{U_1} && (30) \\ (1 + \gamma - \gamma \varepsilon_2) - \mu \frac{F_2}{U_2} &= \theta \tau_2 \frac{p_2}{U_2} && (31) \\ (1 + \gamma - \gamma \varepsilon_3) U_3 &= \mu F_3 && (32) \\ (1 + \gamma - \gamma \varepsilon_l) U_l &= \mu F_l && (33) \end{aligned}$$

From (19) and (20), it can be seen that the distortion introduced by the $\tau_3 = 0$ restriction, is $\theta p_i \tau_i, i = 1, 2$. This distortion will push consumer to consumer less c_1 and c_2. From the first order conditions, we have

$$\frac{(1 + \gamma - \gamma \varepsilon_1) - \mu \frac{F_1}{U_1}}{(1 + \gamma - \gamma \varepsilon_2) - \mu \frac{F_2}{U_2}} = \frac{\tau_1}{\tau_2} \frac{p_1}{U_1} \frac{U_2}{p_2}$$

Express μ as a function of γ, U_1, and F_1: $\mu = (1 + \gamma - \gamma \varepsilon_l) \frac{U_l}{F_l}$. Plug the expression of μ into the equation (23):

$$\frac{(1 + \gamma - \gamma \varepsilon_1) - (1 + \gamma - \gamma \varepsilon_l) \frac{U_l}{F_l} \frac{F_1}{U_1}}{(1 + \gamma - \gamma \varepsilon_2) - (1 + \gamma - \gamma \varepsilon_l) \frac{U_l}{F_l} \frac{F_2}{U_2}} = \frac{\tau_1}{\tau_2} \frac{p_1}{U_1} \frac{U_2}{p_2}$$

Since the Ramsey equilibrium outcome is competitive equilibrium first, all the first order conditions for the consumer and firm still hold (by construction). From these conditions, it can be shown that

$$\frac{U_l}{F_l}\frac{F_1}{U_1} = \frac{1}{1+\tau_1} \tag{34}$$

$$\frac{U_l}{F_l}\frac{F_2}{U_2} = \frac{1}{1+\tau_2} \tag{35}$$

$$\frac{p_1}{U_1}\frac{U_2}{p_2} = \frac{1+\tau_2}{1+\tau_1} \tag{36}$$

Thus

$$\frac{(1+\gamma-\gamma\varepsilon_1)-(1+\gamma-\gamma\varepsilon_l)\frac{1}{1+\tau_1}}{(1+\gamma-\gamma\varepsilon_2)-(1+\gamma-\gamma\varepsilon_l)\frac{1}{1+\tau_2}} = \frac{\tau_1}{\tau_2}\frac{1+\tau_2}{1+\tau_1} \tag{37}$$

$$\frac{(1+\tau_1)(1+\gamma-\gamma\varepsilon_1)-(1+\gamma-\gamma\varepsilon_l)}{(1+\tau_2)(1+\gamma-\gamma\varepsilon_2)-(1+\gamma-\gamma\varepsilon_l)} = \frac{\tau_1}{\tau_2} \tag{38}$$

$$\frac{\tau_2\left[(1+\tau_1)(1+\gamma-\gamma\varepsilon_1)-(1+\gamma-\gamma\varepsilon_l)\right]}{\tau_1\left[(1+\tau_2)(1+\gamma-\gamma\varepsilon_2)-(1+\gamma-\gamma\varepsilon_l)\right]} = 1 \tag{39}$$

Rearrange

$$(\tau_2-\tau_1)(1-\varepsilon_l) = \tau_2(1+\tau_1)(1-\varepsilon_1) - \tau_1(1+\tau_2)(1-\varepsilon_2)$$

That's the relationship among ε_1, ε_2, and ε_l.

(viii) Now the utility function becomes:

$$U(c,l) = W(G(c),l)$$

where $c = (c_1, \cdots, c_n)$ and G is homothetic. From the homoeticity of $G(c)$, it follows that

$$\frac{U_i(\phi c,l)}{U_k(\phi c,l)} = \frac{U_i(c,l)}{U_k(c,l)}, \qquad \text{for } i \text{ and } k = 1,2,3$$

Differentiating (33) with respect to ϕ, we get

$$\sum_{j=1}^{3}\frac{c_j U_{ij}}{U_i} = \sum_{j=1}^{3}\frac{c_j U_{kj}}{U_k} \tag{40}$$

$$\sum_{j=1}^{3} c_j U_{ij} = AU_i, \qquad \text{for some constant } A \tag{41}$$

From the part (iv), we have

$$(1+\gamma)U_i + \gamma U_i\left(\sum_{j=1}^{3}\frac{U_{ji}}{U_i}c_j + \frac{U_{il}}{U_i}l\right) = \mu F_i$$

Using the utility function, we have

$$\begin{aligned} U_i &= W_1 G_i && (42) \\ U_{il} &= W_{12} G_i && (43) \end{aligned}$$

Combining (35), (36), (37), and (38), we get

$$(1+\gamma) W_1 G_i + \gamma W_1 G_i \left(A + \frac{W_{12}}{W_1} l\right) = \mu F_i$$

Thus $\frac{W_1 G_i}{F_i}$ is a constant. Since $\frac{U_i}{F_i} = \frac{W_1 G_i}{F_i}$, it follows that $\frac{U_i}{F_i}$ is a constant too. Given the result we get in part (ii):

$$(1+\tau_i) = \frac{U_i}{U_l}\frac{F_l}{F_i} \qquad \text{for } i = 1,2,3$$

It follows that τ_i is constant across goods. This is the famous "uniform commodity taxation" result. This is at the basis of the optimality of the Friedman rule in monetary economies.

Question 2- Optimal Price Volatility

A) The household problem is:

$$\max_{B', c, n, c^h, c^l, n^h, n^l} U(x)$$

where

$$U(x) = c - \frac{n^2}{2} + \frac{\beta}{2}\left\{\left[c^h - \frac{\left(n^h\right)^2}{2}\right] + \left[c^l - \frac{\left(n^l\right)^2}{2}\right]\right\}, \tag{44}$$

subject to the budget constraints:

$$\begin{aligned} \frac{B'}{R} + Pc &\leq B + P(1-\tau) n \\ P^h c^h &\leq B' + P^h\left(1-\tau^h\right) n^h \\ P^l c^l &\leq B' + P^l\left(1-\tau^l\right) n^l. \end{aligned}$$

Since (44) is strictly increasing in c, c^h, c^l, the constraints (45) – (47) will be binding:

$$\begin{aligned} c &= \frac{B}{P} - \frac{B'}{RP} + (1-\tau) n && (45) \\ c^h &= \frac{B'}{P^h} + \left(1-\tau^h\right) n^h && (46) \\ c^l &= \frac{B'}{P^l} + \left(1-\tau^l\right) n^l && (47) \end{aligned}$$

Hence the household problem can be restated as:

$$\max_{B',n,n^h,n^l} \frac{B}{P}-\frac{B'}{RP}+(1-\tau)\,n-\frac{n^2}{2}+$$
$$+\frac{\beta}{2}\left\{\left[\frac{B'}{P^h}+\left(1-\tau^h\right)n^h-\frac{\left(n^h\right)^2}{2}\right]+\left[\frac{B'}{P^l}+\left(1-\tau^l\right)n^l-\frac{\left(n^l\right)^2}{2}\right]\right\} \tag{48}$$

The first order necessary conditions are:

$$B' \quad : \quad \frac{1}{RP}=\frac{\beta}{2}\left(\frac{1}{P^h}+\frac{1}{P^l}\right) \tag{49}$$
$$n \quad : \quad n=1-\tau \tag{50}$$
$$n^h \quad : \quad n^h=1-\tau^h \tag{51}$$
$$n^l \quad : \quad n^l=1-\tau^l \tag{52}$$

B) With the normalization $R=1$, there are 10 variables to be determined in equilibrium: P, P^h, P^l, B', c, c^h, c^l, n, n^h and n^l. They are determined by the three household budget constraint evaluated with equality ((45) – (47)), by the four household first order necessary conditions ((49)-(52)), and by the three resource constraints:

$$c+g\leq n \tag{53}$$
$$c^i+g^i\leq n^i \qquad i=h,l. \tag{54}$$

These 10 equations, together with the restrictions $P,P^i>0$ $i=h,l$, characterized the equilibrium for a given government policy[1].

C.1) The resource constraints (53) – (54) with equalities imply:

$$c=n-g$$
$$c^i=n^i-g^i \qquad i=h,l.$$

Hence,

$$c-\frac{n^2}{2}=n-\frac{n^2}{2}-g$$
$$c^i-\frac{\left(n^i\right)^2}{2}=n^i-\frac{\left(n^i\right)^2}{2}-g^i \qquad i=h,l.$$

Substituting for n, n^h, n^l from the household first order conditions (50) – (52) :

$$c-\frac{n^2}{2}=(1-\tau)-\frac{(1-\tau)^2}{2}-g=\frac{2(1-\tau)-1+2\tau-\tau^2}{2}-g=\frac{1-\tau^2}{2}-g$$
$$c^i-\frac{\left(n^i\right)^2}{2}=n^i-\frac{\left(n^i\right)^2}{2}-g^i=\left(1-\left(\tau^i\right)\right)-\frac{\left(1-\left(\tau^i\right)\right)^2}{2}-g^i=\frac{1-\left(\tau^i\right)^2}{2}-g^i \qquad i=h,l.$$

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[1] If one such equilibrium exists.

Substituting the expressions above into (44):

$$U\left(x\left(\pi\right)\right)=\frac{1-\tau^{2}}{2}-g+\frac{\beta}{2}\left[\frac{1-\left(\tau^{h}\right)^{2}}{2}-g^{h}+\frac{1-\left(\tau^{l}\right)^{2}}{2}-g^{l}\right]$$

Multiplying by 2 does not change the problem:

$$\begin{aligned}U\left(x\left(\pi\right)\right) &= 1-\tau^{2}-2g+\frac{\beta}{2}\left[1-\left(\tau^{h}\right)^{2}-2g^{h}+1-\left(\tau^{l}\right)^{2}-2g^{l}\right]= \\ &= -\tau^{2}-\frac{\beta}{2}\left[\left(\tau^{h}\right)^{2}+\left(\tau^{l}\right)^{2}\right]+\underbrace{2\left[\frac{1}{2}-g+\frac{\beta}{2}\left(1-g^{h}-g^{l}\right)\right]}_{\equiv\kappa}= \\ &= -\tau^{2}-\frac{\beta}{2}\left[\left(\tau^{h}\right)^{2}+\left(\tau^{l}\right)^{2}\right]+\kappa\end{aligned}$$

C.2) Consider the Government budget constraints:

$$\frac{B'}{R}+P\tau n\geq B+Pg \tag{55}$$

$$P^{i}\tau^{i}n^{i}\geq B'+P^{i}g^{i}\qquad i=h,l \tag{56}$$

Divide (55) by P and (56) by P^{i}:

$$\begin{aligned}\frac{B'}{PR}+\tau n &\geq \frac{B}{P}+g \\ \tau^{i}n^{i} &\geq \frac{B'}{P^{i}}+g^{i}\qquad i=h,l\end{aligned}$$

Add to the first line the second times $\beta/2$ for every state, $i=h,l$:

$$\frac{B'}{PR}+\tau n+\frac{\beta}{2}\left[\tau^{h}n^{h}+\tau^{l}n^{l}\right]\geq\frac{B}{P}+g+\frac{\beta}{2}\left[\frac{B'}{P^{h}}+\frac{B'}{P^{l}}+g^{h}+g^{l}\right].$$

Substitute for n, n^{h}, n^{l} from the household first order conditions $(50)-(52)$, simplify the term involving B' using (49), impose $P=1$, and rearrange:

$$\tau\left(1-\tau\right)+\frac{\beta}{2}\left[\tau^{h}\left(1-\tau^{h}\right)+\tau^{l}\left(1-\tau^{l}\right)\right]\geq\frac{B}{P}+g+\frac{\beta}{2}\left[g^{h}+g^{l}\right]$$

$$B\leq\tau\left(1-\tau\right)-g+\frac{\beta}{2}\left[\tau^{h}\left(1-\tau^{h}\right)-g^{h}+\tau^{l}\left(1-\tau^{l}\right)-g^{l}\right] \tag{57}$$

>From (56), substituting for n^{i}:

$$P^{i}\left[\tau^{i}\left(1-\tau^{i}\right)-g^{i}\right]\geq B'\geq 0\qquad i=h,l.$$

Thus $P^{i}>0$ implies:

$$\tau^{i}\left(1-\tau^{i}\right)-g^{i}\geq 0\qquad i=h,l \tag{58}$$

C.3) The Ramsey problem, incorporating the restriction $P = 1$, in Lagrangian form is:

$$\begin{gathered}\max_{\tau,\tau^h,\tau^l} -\tau^2 - \frac{\beta}{2}\left[\left(\tau^h\right)^2 + \left(\tau^l\right)^2\right] + \\ +\lambda\left\{\tau\left(1-\tau\right) - g + \frac{\beta}{2}\left[\tau^h\left(1-\tau^h\right) - g^h + \tau^l\left(1-\tau^l\right) - g^l\right] - B\right\} + \\ +\mu^h\left[\tau^h\left(1-\tau^h\right) - g^h\right] + \mu^l\left[\tau^l\left(1-\tau^l\right) - g^l\right],\end{gathered}$$

where λ, μ^h, $\mu^l \geq 0$ are the Lagrange multipliers.

C.4) The necessary and sufficient condition for the optimum are the constraints $(57)-(58)$, the non-negativity constraints on the Lagrange multipliers, λ, μ^h, $\mu^l \geq 0$, the complementary slackness conditions:

$$\lambda\left\{\tau\left(1-\tau\right) - g + \frac{\beta}{2}\left[\tau^h\left(1-\tau^h\right) - g^h + \tau^l\left(1-\tau^l\right) - g^l\right] - B\right\} = 0 \tag{59}$$

$$\mu^h\left[\tau^h\left(1-\tau^h\right) - g^h\right] = 0 \tag{60}$$

$$\mu^l\left[\tau^l\left(1-\tau^l\right) - g^l\right] = 0, \tag{61}$$

and the first order conditions:

$$\tau \quad : \quad -2\tau + \lambda - 2\lambda\tau = 0 \tag{62}$$

$$\tau^i \quad : \quad \frac{\beta}{2}\left(-2\tau^i + \lambda - 2\lambda\tau^i\right) + \mu^i - 2\mu^i\tau^i = 0 \qquad i = h, l. \tag{63}$$

Rearranging (62):

$$\lambda = \frac{2\tau}{1-2\tau} \tag{64}$$

Rearranging (63), and substituting for λ from (64) :

$$\begin{aligned}\mu^i &= \beta\frac{\tau^i}{1-2\tau^i} - \frac{\beta}{2}\lambda \\ \mu^i &= \beta\left(\frac{\tau^i}{1-2\tau^i} - \frac{\tau}{1-2\tau}\right).\end{aligned} \tag{65}$$

C.5) Since (57) is always binding, as discussed in **(D.1)** below, there are three cases to examine:

1. both constraints in (58) are binding;
2. one of the constraints in (58) is binding, the other is not;
3. none of the constraints in (58) is binding.

The second and third possibility are considered in **(D.2)** and **(D.3)** below.

If both constraints in (58) are binding, then[2]:

$$\begin{aligned}\tau^i &= \frac{1}{2} - \left(1 - 4g^i\right)^{1/2} \qquad i = h, l \\ \tau &= \frac{1}{2} - \left[1 - 4\left(B + g\right)\right]^{1/2}\end{aligned}$$

The Government budget constraint (56) imply $P^h = P^l = \infty$. This is consistent with (49) only if[3] $B' = 0$. In words, if the Government will renege its debt in all the states of the world, only an equilibrium with no debt is possible.

C.6) With the Ramsey policies at hand n, n^h and n^l can be computed from $(50) - (52)$. The resource constraints $(53) - (54)$ with equality determine c, c^h and c^l. Finally, the Government budget constraints $(55) - (56)$ can be used to compute B', P^h and P^l.

D.1) We can show that (57) must hold with a strict equality by contradiction. Suppose (57) holds with a strict inequality. Then, (59) implies $\lambda = 0$. From (64), $\tau = 0$. The strict inequality of (57), implies that at least one of the constraints in (58) is strict. The complementary conditions associated to such a constraint (either (60) or (61)) implies that the corresponding Lagrange multiplier is equal to zero. (63) and $\tau = 0$ require that the correspondent tax rate is zero, but this contradicts the non-negativity of the primary surplus (58).

In words, since the Government objective function is decreasing in the tax rates it would be suboptimal to raise more taxes than the minimum needed to satisfy (57) with equality.

Without imposing $P = 1$, the period-1 intertemporal budget constraint of the Government, with equality, would have been:

$$\frac{B}{P} = \tau\left(1 - \tau\right) - g + \frac{\beta}{2}\left[\tau^h\left(1 - \tau^h\right) - g^h + \tau^l\left(1 - \tau^l\right) - g^l\right].$$

It would have been optimal to inflate away any initial debt by setting $P = \infty$.

D.2) If none of the constraints in (58) is binding, the complementary slackness conditions $(60) - (61)$ imply $\mu^h = \mu^l = 0$. From the first order conditions (63), $\tau^i = \tau$, $i = h, l$. The Government budget constraints for period 2 can be rewritten as:

$$\tau\left(1 - \tau\right) = \frac{B'}{P^i} + g^i \qquad i = h, l$$

This implies $P^h > P^l$ whenever $g^h > g^l$.

D.3) Suppose (58) is binding in state h. Then, $\mu^h > 0$, $\tau^h\left(1 - \tau^h\right) - g^h = 0$, and, from (63), $\tau^h > \tau$. The Government budget constraints for period 2, state h can be rewritten as:

$$\underbrace{\tau^h\left(1 - \tau^h\right) - g^h}_{=0} = \frac{B'}{P^h}.$$

---

[2] Clearly, for the problem to be meaningful, $g^h, g^l, (B + g) < 1/4$.

[3] (49) holds only for $B' > 0$.

The Government inflates away the debt in state h by setting $P^h = \infty$.

The household first order condition associated to debt holdings can be restated as:

$$\frac{\beta}{2}\left(\underbrace{\frac{1}{P^h}}_{=0} + \frac{1}{P^l}\right) = 1$$

$$\Rightarrow P^l = \frac{\beta}{2}$$

To ensure that households will have an incentive to accumulate debt in the first period the real rate of return on debt must be high in the state in which the Government pays the debt back, i.e. P^l must be low.

Question 3- Optimal Capital Taxation (Ljungqvist-Sargent 15.4)

(a) The consumer's budget constraint is:

$$\sum_{t=0}^{\infty} q_t^0 c_t = \sum_{t=0}^{\infty} q_t^0 \left(1 - \tau_{1t}^n\right) w_{1t} n_{1t} + \sum_{t=0}^{\infty} q_t^0 \left(1 - \tau_{2t}^n\right) w_{2t} n_{2t} + \left[\left(1 - \tau_0^k\right) r_0\right] k_0 + b_0, \tag{1}$$

where $q_t^0 = \prod_{i=1}^{t} R_i^{-1}$.

Step 1. Let λ be the Lagrange multiplier associated to the household's budget constraint. The FONC for the household's problem are:

$$\begin{aligned} c_t &: \quad \beta^t u_c(t) - \lambda q_t^0 = 0 \\ n_{it} &: \quad -\beta^t u_l(t) + \lambda q_t^0 \left(1 - \tau_{it}^n\right) w_{it} = 0, \ i = 1, 2 \end{aligned}$$

With the numeraire $q_0^0 = 1$, these conditions imply:

$$q_t^0 = \beta^t \frac{u_c(t)}{u_c(0)} \tag{2a}$$

$$\left(1 - \tau_{it}^n\right) w_{it} = \frac{u_l(t)}{u_c(t)} \tag{2b}$$

$$\frac{q_t^0}{q_{t+1}^0} = \left(1 - \tau_{t+1}^k\right) r_{t+1} + 1 - \delta \tag{3}$$

Firm maximization and factor market equilibrium imply:

$$r_t = F_k(t) \tag{4a}$$

$$w_{it} = F_{n_i}(t), \ i = 1, 2 \tag{4b}$$

Step 2. Substitute equations (2) and $r_0 = F_k(0)$ into equation (1):

$$\sum_{t=0}^{\infty} \beta^t \left[u_c(t) c_t - u_l(t)\left(n_{1t} + n_{2t}\right)\right] - A = 0 \tag{5}$$

$$A = A\left(c_0, n_{10}, n_{20}, \tau_0^k\right) = u_c(0)\left\{\left[\left(1 - \tau_0^k\right) F_k(0) + 1 - \delta\right] k_0 + b_0\right\} \tag{6}$$

Step 3. The Ramsey problem is:

$$
\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-n_{1t}-n_{2t}\right) \\
\text{s.t. } & c_t + g_t + k_{t+1} = F\left(k_t, n_{1t}, n_{2t}\right) + (1-\delta) k_t \qquad (7) \\
& \sum_{t=0}^{\infty} \beta^t \left[u_c(t) c_t - u_l(t)\left(n_{1t}+n_{2t}\right)\right] - A = 0.
\end{aligned}
$$

Assume that government expenditures are small enough that the problem has a convex constraint set. Let Φ be the Lagrange multiplier on (6) and define:

$$
V\left(c_t, n_{1t}, n_{2t}, \Phi\right) = u\left(c_t, 1-n_{1t}-n_{2t}\right) + \Phi\left[u_c(t) c_t - u_l(t)\left(n_{1t}+n_{2t}\right)\right]. \qquad (8)
$$

Notice that n_{1t} and n_{2t} enter in a symmetric way in $V(.)$, which depends only on their sum. The Lagrangean associated to the government's problem is:

$$
\begin{aligned}
J = \sum_{t=0}^{\infty} \beta^t \{ & V\left(c_t, n_{1t}, n_{2t}, \Phi\right) + \theta_t [F\left(k_t, n_{1t}, n_{2t}\right) + \\
& (1-\delta) k_t - c_t - g_t - k_{t+1}]\} - \Phi A, \qquad (9)
\end{aligned}
$$

where $\{\theta_t\}_{t=0}^{\infty}$ is the sequence of Lagrange multipliers on the resource constraints. Given b_0 and k_0, fix τ_0^k and maximize J with respect to $\{c_t, n_{1t}, n_{2t}, k_{t+1}\}_{t=0}^{\infty}$. The FONC's are:

$$
\begin{aligned}
c_t &: V_c(t) = \theta_t, \ t \geq 1 \\
n_{it} &: V_n(t) = -\theta_t F_{n_i}(t), \ i = 1, 2, \ t \geq 1 \\
k_{t+1} &: \theta_t = \beta \theta_{t+1}\left[F_k(t+1) + 1 - \delta\right], \ t \geq 0 \\
c_0 &: V_c(0) = \theta_0 + \Phi A_c \\
n_{i0} &: V_n(0) = -\theta_0 F_n(0) + \Phi A_{n_i}, \ i = 1, 2, \ t \geq 1
\end{aligned}
$$

These conditions imply:

$$
\begin{aligned}
V_c(t) &= \beta V_c(t+1)\left[F_k(t+1) + 1 - \delta\right] && \text{(10a)} \\
\frac{-V_n(t)}{V_c(t)} &= F_{n_i}(t), \ i = 1, 2, \ t \geq 1 && \text{(10b)} \\
V_n(0) &= \left[\Phi A_c - V_c(0)\right] F_{n_i}(0) + \Phi A_{n_i}, \ i = 1, 2 && \text{(10c)}
\end{aligned}
$$

The allocation $\{c_t, n_{1t}, n_{2t}, k_{t+1}\}_{t=0}^{\infty}$ and the multiplier Φ satisfy $(10a) - (10c)$, (5) and (7).

Step 4. Having computed an allocation, q_t^0, r_t, w_{1t}, w_{2t}, τ_{1t}^n, τ_{2t}^n, and τ_t^k can be obtained by equations $(2a)$, $(4a)$, $(4b)$, $(2b)$, and (3) respectively.

Equations $(2b)$, $(4b)$ imply that $\left(1-\tau_{1t}^n\right) F_{n_1}(t) = \left(1-\tau_{2t}^n\right) F_{n_2}(t)$. Equation $(10b)$ implies $F_{n_1}(t) = F_{n_2}(t)$, for $t \geq 1$. Hence $\tau_{1t}^n = \tau_{2t}^n$ for $t \geq 1$.

Equation ($10c$) impies:

$$\left[V_c(0) - \Phi A_c\right]\left[F_{n_1}(0) - F_{n_2}(0)\right] = \Phi\left(A_{n_1} - A_{n_2}\right) \tag{11}$$

Differentiating equation (6) with respect to n_i, $i = 1, 2$:

$$A_{n_i} = -u_{cl}(0)\left\{\left[\left(1 - \tau_0^k\right) F_k(0) + 1 - \delta\right] k_0 + b_0\right\} + u_c(0)\left(1 - \tau_0^k\right) F_{kn_i}(0) \tag{12}$$

Substituting equation (12) into equation (11):

$$F_{n_1}(0) - F_{n_2}(0) = \frac{\Phi u_c(0)\left(1 - \tau_0^k\right)}{V_c(0) - \Phi A_c}\left[F_{kn_1}(0) - F_{kn_2}(0)\right] \tag{13}$$

If we assume that n_1 and k are complements, i.e. $F_{kn_1} > 0$, and that n_2 and k are substitutes, i.e. $F_{kn_2} < 0$, equation (13) implies:

$$F_{n_1}(0) - F_{n_2}(0) \geq 0,$$

since $\frac{\Phi u_c(0)\left(1-\tau_0^k\right)}{V_c(0)-\Phi A_c} \geq 0$.

On the other hand, equations ($2b$) and ($4b$) imply that $\left(1 - \tau_{10}^n\right) F_{n_1}(0) = \left(1 - \tau_{20}^n\right) F_{n_2}(0)$. Thus, $\tau_{10}^n \geq \tau_{20}^n$.

(b) The Ramsey problem is:

$$\max \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - n_{1t}, 1 - n_{2t}\right)$$
$$\text{s.t. } c_t + g_t + k_{t+1} = F\left(k_t, n_{1t}, n_{2t}\right) + (1 - \delta) k_t$$
$$\sum_{t=0}^{\infty} \beta^t \left[u_c(t) c_t - u_{n_1}(t) n_{1t} - u_{n_2}(t) n_{2t}\right] - A = 0$$
$$\xi(t) \equiv u_{n_1}(t) F_{n_1}(t) - u_{n_2}(t) F_{n_2}(t) = 0$$

Assume that government expenditures are small enough that the problem has a convex constraint set. Let Φ be the Lagrange multiplier on the second constraint and define:

$$V\left(c_t, n_{1t}, n_{2t}, \Phi\right) = u\left(c_t, 1 - n_{1t}, 1 - n_{2t}\right) + \Phi\left[u_c(t) c_t - u_{n_1}(t) n_{1t} - u_{n_2}(t) n_{2t}\right].$$

The Lagrangean associated to the planner's problem is:

$$J = \sum_{t=0}^{\infty} \beta^t \{V\left(c_t, n_{1t}, n_{2t}, \Phi\right) + \theta_t [F\left(k_t, n_{1t}, n_{2t}\right) +$$
$$(1 - \delta) k_t - c_t - g_t - k_{t+1}] + \eta_t \xi(t)\} - \Phi A,$$

where $\{\theta_t\}_{t=0}^{\infty}$ and $\{\eta_t\}_{t=0}^{\infty}$ are the sequences of Lagrange multipliers on the resource constraints and the tax rate equality constraint respectively.

Given b_0 and k_0, fix τ_0^k and maximize J with respect to $\{c_t, n_{1t}, n_{2t}, k_{t+1}\}_{t=0}^{\infty}$. The FONC's are:

$$\begin{array}{rcl} c_t & : & V_c(t) = \theta_t - \eta_t \xi_c(t), \ t \geq 1 \\ n_{it} & : & V_{n_i}(t) = -\theta_t F_{n_i}(t) + \eta_t \xi_{n_i}(t), \ i = 1, 2, \ t \geq 1 \\ k_{t+1} & : & \theta_t = \beta \theta_{t+1} \left[F_k(t+1) + 1 - \delta \right] + \beta \eta_{t+1} \xi_k(t+1), \ t \geq 0 \\ c_0 & : & V_c(0) = \theta_0 + \Phi A_c - \eta_0 \xi_c(0) \\ n_{i0} & : & V_{n_i}(0) = -\theta_0 F_{n_i}(0) + \Phi A_{n_i} + \eta_0 \xi_{n_i}(0), \ i = 1, 2, \ t \geq 1 \end{array}$$

(c) If the solution to the Ramsey problem converges to a steady state, the FONC's derived in **(b)** become:

$$\begin{array}{rcl} V_c & = & \theta - \eta \xi_c \\ V_{n_i} & = & -\theta F_{n_i} + \eta \xi_{n_i}, \ i = 1, 2 \\ \theta & = & \beta \theta (F_k + 1 - \delta) + \beta \eta \xi_k \end{array}$$

The equations above indicate that the limiting tax on capital will be zero if either $\eta = 0$ or $\xi_k = 0$. Assuming that the constraint that the two labor taxes should be equal is binding means that $\eta \neq 0$.

>From the definition of ξ, it follows:

$$\xi_k = \frac{u_{n_1}}{F_{n_1}} \left(\frac{F_{n_1 k}}{F_{n_1}} - \frac{F_{n_2 k}}{F_{n_2}} \right).$$

It follows that $\xi_k = 0$ if and only if:

$$\frac{F_{n_1 k}}{F_{n_1}} = \frac{F_{n_2 k}}{F_{n_2}}.$$

This condition defines a special class of production functions which include Cobb-Douglas, but not the general CES case.

Question 4- See LS solution manual page 190 at http://pages.stern.nyu.edu/~svnieuwe/. Note that on page 192, point b. the arbitrage condition that should hold is $R_t = F_k(t+1) + 1 - \delta$. Moreover, note that the same allocation can be implemented with a capital tax. In that case, the arbitrage condition is $R_t = 1 + (1 - \theta(t+1))(F_k(t+1) - \delta)$. Note that $\theta(t+1) > 1$ is equivalent to an implementation with consumption taxes in which $\tau_{t+1}^c > \tau_t^c$.