G 6222 ADV. MACRO ANALYSIS II
COLUMBIA UNIVERSITY, SPRING 2006
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Assignment 1, due February 6, 2006 in class.

You are encouraged to work in groups on the assignment, however, you must turn in an individual solution.

Question 1- Commodity Taxation

Consider a model economy in which 3 types of consumption goods are produced with labor. The resource constraint is given by:

$$F\left(c_1 + g_1, c_2 + g_2, c_3 + g_3, l\right) = 0,$$

where c_i and g_i denote private and government consumption of each good i, l denotes labor, and F denotes a production process which satisfies constant returns to scale.

The consumer's problem is to maximize utility:

$$\max U\left(c_1, c_2, c_3, l\right)$$

subject to

$$\sum p_i \left(1 + \tau_i\right) c_i = l,$$

where p_i is the price of good i, and τ_i is an ad valorem tax rate on good i. The wage is normalized to 1.

A competitive representative firm operates the constant returns to scale technology F and solves:

$$\max_{(x,l)} \sum_i p_i x_i - l,$$

subject to

$$F\left(x_1, x_2, x_3, l\right) = 0,$$

where x_i denotes output of good i.

i) Define a competitive equilibrium for this economy. Impose conditions on U and F to ensure that necessary conditions for optimality are sufficient and that the optima for households and firms are interior.

Assume that $[g_1, g_2, g_3]$ are exogenously given.

ii) Show that the allocations in a competitive equilibrium satisfy:

$$F\left(c_1 + g_1, c_2 + g_2, c_3 + g_3, l\right) = 0, \tag{1}$$

and

$$\sum_i U_i c_i + U_l l = 0, \tag{2}$$

where U_i is the marginal utility of good i and U_l is the marginal utility of leisure.

In addition, show that, for allocations that satisfy (1) and (2), policies and prices can be found that, together with the given allocation, constitute a competitive equilibrium.

iii) Let Π denote the set of policies for which a competitive equilibrium exists. Define a Ramsey equilibrium for this economy.

iv) Write the Ramsey allocation problem and derive the first order conditions for this problem. Let:

$$\begin{aligned} \varepsilon_i &= \frac{-\left(\sum_{j=1}^{3} U_{ij} c_j + U_{li} l\right)}{U_i}, \; i = 1, 2, 3, \\ \varepsilon_l &= \frac{-\left(\sum_{j=1}^{3} U_{lj} c_j + U_{ll} l\right)}{U_l}. \end{aligned}$$

Provide an interpretation for ε_i and ε_l. Using the first order conditions for the Ramsey allocation problem, show that if $\varepsilon_i > \varepsilon_j$, then $\tau_i < \tau_j$.

v) Assume that U is additively separable. Then, $\varepsilon_i = -U_{ii} c_i / U_i$. Let $c(p, m)$ and $l(p, m)$ denote the solution to the problem of maximizing utility subject to $\sum_{i=1}^{3} p_i c_i = l + m$, where m is non-labor income and $c(p, m) = \{c_i(p, m)\}_{i=1,2,3}$.

Let η_i be the income elasticity of the demand for good i, i.e. the elasticity of the demand for good i with respect to m. Show that:

$$\frac{\varepsilon_i}{\varepsilon_j} = \frac{\eta_j}{\eta_i}.$$

Interpret the result in part iv) based on this finding.

vi) Assume that utility is given by:

$$\sum_{i=1}^{3} V^i(c_i) - l.$$

Show that in this case:

$$\varepsilon_i = \left[-\frac{\partial c_i}{\partial p_i} \frac{p_i}{c_i}\right]^{-1}.$$

Interpret the result in part iv) based on this finding.

vii) Now assume that the government can only tax good 1 and good 2, i.e. $\tau_3 = 0$. Write the Ramsey allocation problem with this restriction on government policies. Derive the relation between ε_1, ε_2 and ε_l in this case.

viii) Assume that that utility satisfies:

$$U(c, l) = W(G(c), l),$$

where $c = (c_1, ..., c_n)$ and G is homothetic. Show that Ramsey policy implies:

$$\frac{U_i}{U_j} = \frac{F_i}{F_j},$$

in other words $\tau_i = \tau_j$ for $i, j = 1, 2, 3$.

Question 2- - Optimal Price Volatility

Consider the following model economy composed of firms, households and a government, which live for two periods. Households and firms interact in competitive markets. The government must finance an exogenously given level of expenditures by issuing distortionary proportional taxes on labor and possibly issuing debt. There is no uncertainty in the first period but there is uncertainty over the level of government spending in the second period. Spending could be high, state h, or low, state l, with equal probability. The uncertainty is resolved at the beginning of the second period. The government commits to its policies before the first period. Trade occurs by barter and there is no money in the model.

Firms produce according to the technology:

$$y = n,\ y^i = n^i \text{ for } i = l, h,$$

in the first and second period. This implies that the real wage is unity in equilibrium, which we impose henceforth.

Household preferences over consumption and labor over the two periods take the form:

$$U(x) = c - \frac{1}{2}n^2 + \frac{1}{2}\beta\left\{[c^h - \frac{1}{2}\left(n^h\right)^2] + [c^l - \frac{1}{2}\left(n^l\right)^2]\right\},$$

where c, n are consumption, labor in the first period, β is the discount factor and c^i, n^i are consumption, labor in the second period in state $i = l, h$. An allocation is $x = \left(c,\ c^h,\ c^l,\ n,\ n^h,\ n^l\right)$.

The household's period 1 budget constraint is given by:

$$\frac{B'}{R} + Pc \leq B + P(1 - \tau)n,$$

where P is the period 1 price level, B is the nominal bonds that the household inherits from the past and R is the gross nominal rate of interest. τ is the tax rate on labor income and B' denotes bonds that the household acquires in period 1.

The household budget constraint in the second period, conditional on the realization of uncertainty, is given by:

$$P^i c^i \leq B' + P^i\left(1 - \tau^i\right)n^i,\ \text{for } i = h, l.$$

The household maximizes utility by choice of non-negative values of B', c, n, c^i, $n^{\ i}$ for $i = h, l$, subject to the budget constraints in each period and for each state, taking prices and government policy as given.

A) Show that the first order conditions associated with the optimal choice of labor and bonds are:

$$n = 1 - \tau,\ n^i = 1 - \tau^i,\ i = l, h,\ \frac{1}{RP} = \frac{1}{2}\beta\left(\frac{1}{P^h} + \frac{1}{P^l}\right).$$

The government's budget constraint in the first and second period are:

$$\frac{B'}{R} + P\tau n \geq B + Pg,$$

$$P^i\tau^i n^i \geq B' + P^i g^i,\ \text{for } i = l, h.$$

Here, g denotes government consumption in the first period and g^i denotes government consumption in the second period in state $i = h, l$.

Inspection of the equations in this model reveals that R only appears as R/P^i or B'/R. Thus, we cannot pin down B', P^i and R separately. So it is convenient to adopt the normalization $R = 1$, from here on.

Government policy is a vector $\pi = \left(\tau, \tau^h, \tau^l\right)$. This policy is non-Ricardian. There is no set of values of π for which the government's intertemporal budget constraint is satisfied for all prices.

Combining household and government budget equations yields the economy's resource constraints:

$$c + g \leq n,\ c^i + g^i \leq n^i \text{ for } i = h, l.$$

B) Characterize the private sector equilibrium. There are 10 variables to be determined in equilibrium:

$$P,\ P^i, B', c, n, c^i, n^i \text{ for } i = h, l.$$

Therefore, you must produce 10 equations that can pin down the value of these variables for a given government policy, as well as nay additional restrictions that must hold in equilibrium e.g. non-negative prices.

Denote the mapping from π to x in a private sector equilibrium with $x(\pi)$.

C) A Ramsey equilibrium for this economy is a private sector equilibrium associated with the policy π that solves:

$$\max_{\pi} U(x(\pi)),$$

subject to the requirement that the corresponding prices are strictly positive, $B' \geq 0$ and the elements in x are non-negative.

C.1) Show that:

$$U(x(\pi)) = -\tau^2 - \frac{1}{2}\beta[\left(\tau^h\right)^2 + \left(\tau^l\right)^2] + \kappa,$$

where κ is constant which depends on exogenous variables only.

Before writing down the Ramsey problem, we must confront the issue related to the choice of P. In this economy, the government has an incentive to set P high if $B > 0$. To remove this incentive, usually the Ramsey problem is studied under $B = 0$. For this problem we will instead impose that $P = 1$. No restriction will instead be placed on P^h, P^l or B, B'.

C.2) Show that under $P = 1$, the government's intertemporal budget constraint is:

$$B \leq \tau (1-\tau) - g + \frac{1}{2}\beta[\tau^h \left(1-\tau^h\right) - g^h + \tau^l \left(1-\tau^l\right) - g^l]. \tag{3}$$

Show that the restrictions $P^i > 0$ imply:

$$\tau^i \left(1-\tau^i\right) - g^i \geq 0, \tag{4}$$

for $i = h, l$.

Set up the (direct) Ramsey problem in Lagrangian form. Denote with λ, μ^i for $i = h, l$ the multipliers on constraints (3) and (4).

C.4) Show that the first order conditions for τ, τ^h, τ^l can be written as:

$$\lambda = \frac{2\tau}{1-2\tau},$$

$$\mu^i = \beta \left(\frac{\tau^i}{1-2\tau^i} - \frac{\tau}{1-2\tau} \right) \text{ for } i = h, l.$$

C.5) Solve the Ramsey under $P = 1$ by finding multipliers λ, μ^h, μ^l and policies τ, τ^h, τ^l that satisfy these conditions and the constraints (3)-(4).

C.6) Solve for n, n^i, c, c^i, B', P^i for $i = h, l$ in the Ramsey equilibrium.

D) Show that the solution has the following qualitative features:

1. The weak inequality in (3) is satisfied as a strict equality. What would this imply for the optimal P if the restriction $P = 1$ were not imposed?
2. Show that if (4) are not binding $\tau = \tau^h = \tau^l$ is optimal. Show that this implies $P^h > P^l$ as long as $g^h > g^l$.
3. Show that if (4) is binding for at least one i than the price fluctuations across states in period 2 are bigger. Provide economic intuition for your finding.

Question 3- Optimal Capital Taxation with Heterogeneous Labor Inputs

LS Exercise 15.4.

The nonstochastic model referred to in the question in the one sector growth model with no uncertainty.

Representative agent preferences are:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

where $\beta \in (0,1)$ is the discount factor. Technology is described in the problem. Capital depreciates at rate $\delta \in (0,1)$.

Question 4- Optimal Consumption Taxes

LS Exercise 15.2