G 6222 Adv. Macro Analysis II
Columbia University, Spring 2006
Prof. Stefania Albanesi

Assignment 2- SUGGESTED ANSWERS

Question 1: Optimal fiscal policy with heterogeneous agents.

(a) Given a tax policy π, and a sequence of public consumption $\{g_t\}_{t=0}^{\infty}$, a competitive equilibrium in this economy is a sequence of individual allocations, $\{c_{i,t}, l_{i,t}, k_{i,t+1}; i = 1, 2\}_{t=0}^{\infty}$, government debt $\{b_{i,t+1}; i = 1, 2\}_{t=0}^{\infty}$, and relative prices $\{r_t, w_{1,t}, w_{2,t}, R_t\}_{t=0}^{\infty}$, such that:

- the allocations solve the following problem for consumers of type i, $i = 1, 2$:

$$\max \sum_{t=0}^{\infty} \beta^t U^i(c_{it}, l_{it})$$

s.t. $c_{i,t} + k_{i,t+1} + b_{i,t+1} \leq (1 - \tau_{i,t}) w_{i,t} l_{i,t} + [1 + (1 - \theta_{i,t})(r_t - \delta)] k_{i,t} + R_t b_{i,t}$

given prices and government policy;

- the prices $r_t, w_{1,t}, w_{2,t}$ are consistent with the representative firm's optimization problem:

$$\max F(k_t, l_{1,t}, l_{2,t}) - w_{1,t} l_{1,t} - w_{2,t} l_{2,t} - r_t k_t$$

- the government budget constraint holds:

$$\tau_{1,t} w_{1,t} l_{1,t} + \tau_{2,t} w_{2,t} l_{2,t} + \theta_{1,t} r_t k_{1,t} + \theta_{2,t} r_t k_{2,t} + (b_{1,t+1} + b_{2,t+1}) \geq g_t + R_t (b_{1,t} + b_{2,t})$$

- the resource constraint is satisfied:

$$c_{1,t} + c_{2,t} + (k_{1,t+1} + k_{2,t+1}) - (1 - \delta)(k_{1,t} + k_{2,t}) + g_t \leq F(k_{1,t} + k_{2,t}, l_{1,t}, l_{2,t})$$

Let $\beta^t \alpha_{i,t}$ be the Lagrange multiplier associated with the period t budget constraint for consumer of type i. The FONC's are given by:

$$c_{i,t} \;:\; U_c^i(c_{i,t}, l_{i,t}) - \alpha_{i,t} = 0 \tag{1}$$
$$l_{i,t} \;:\; \alpha_{i,t}(1 - \tau_{i,t}) w_{i,t} + U_l^i(c_{i,t}, l_{i,t}) = 0 \tag{2}$$
$$k_{i,t+1} \;:\; \alpha_{i,t} - \beta [1 + (1 - \theta_{i,t+1})(r_{t+1} - \delta)] \alpha_{i,t+1} = 0 \tag{3}$$
$$b_{i,t+1} \;:\; \alpha_{i,t} - \beta R_{t+1} \alpha_{i,t+1} = 0. \tag{4}$$

The FONC's for the representative firm are:

$$k_t \;:\; F_{k_t} = r_t \tag{5}$$
$$l_{1,t} \;:\; F_{l_1}(t) = w_{1,t} \tag{6}$$
$$l_{2,t} \;:\; F_{l_2}(t) = w_{2,t}. \tag{7}$$

(b) The government problem is:

$$\max_{\pi\in\Pi}\sum_{i=1}^{2}\omega_i\sum_{t=0}^{\infty}\beta^t U^i\left(c_{it}\left(\pi\right),l_{it}\left(\pi\right)\right)$$

$$\text{s.t. } \tau_{1,t}w_{1,t}\left(\pi\right)l_{1,t}\left(\pi\right)+\tau_{2,t}w_{2,t}\left(\pi\right)l_{2,t}\left(\pi\right)+r_t\left(\pi\right)\left(\theta_{1,t}k_{1,t}\left(\pi\right)+\theta_{2,t}k_{2,t}\left(\pi\right)\right)$$
$$+b_{1,t+1}\left(\pi\right)+b_{2,t+1}\left(\pi\right)\geq g_t+R_t\left(\pi\right)\left(b_{1,t}\left(\pi\right)+b_{2,t}\left(\pi\right)\right)\ \forall t$$

Let $\{x_\pi, p_\pi\}$ be a competitive equilibrium tax policy rule which associates to each policy $\pi\in\Pi$, the corresponding competitive equilibrium allocations $x_\pi=\{x_\pi(t)\}_{t=0}^{\infty}$, and prices $p_\pi=\{p_\pi(t)\}_{t=0}^{\infty}$.

A Ramsey equilibrium is a policy $\hat{\pi}$ and a competitive equilibrium tax policy rule $\{x_\pi, p_\pi\}$ such that, given $\{x_\pi, p_\pi\}$, the government's policy solves the problem stated above.

(c) Multiply each of the equations $(1)-(2)$ by its respective control variable:

$$c_{i,t}U_c^i\left(c_{i,t},l_{i,t}\right)=\alpha_{i,t}c_{i,t}$$
$$l_{i,t}U_l^i\left(c_{i,t},l_{i,t}\right)=-\alpha_{i,t}\left(1-\tau_{i,t}\right)w_{i,t}l_{i,t}.$$

Multiply by β^t, some over $t=0,1,2,...$, and add up:

$$\sum_{t=0}^{\infty}\beta^t\left[c_{i,t}U_c^i\left(c_{i,t},l_{i,t}\right)+l_{i,t}U_l^i\left(c_{i,t},l_{i,t}\right)\right]=\sum_{t=0}^{\infty}\beta^t\alpha_{i,t}\left[c_{i,t}-\left(1-\tau_{i,t}\right)w_{i,t}l_{i,t}\right]$$

Substitute for the term in square brackets in the RHS from the budget constraints, collect terms, use equations (3) and (4), and finally substitute for $\alpha_{i,0}$ from (1):

$$\sum_{t=0}^{\infty}\beta^t\left[c_{i,t}U_c^i\left(c_{i,t},l_{i,t}\right)+l_{i,t}U_l^i\left(c_{i,t},l_{i,t}\right)\right]=$$
$$=\sum_{t=0}^{\infty}\beta^t\alpha_{i,t}\left\{\left[1+\left(1-\theta_{i,t}\right)\left(r_t-\delta\right)\right]k_{i,t}+R_tb_{i,t}-k_{i,t+1}-b_{i,t+1}\right\}=$$

$$=\alpha_{i,0}\left\{\left[1+\left(1-\theta_{i,0}\right)\left(r_0-\delta\right)\right]k_{i,0}+R_0b_{i,0}\right\}+$$
$$+\sum_{t=0}^{\infty}\left\{\alpha_{i,t}\left[1+\left(1-\theta_{i,t+1}\right)\left(r_{t+1}-\delta\right)\right]-\beta\alpha_{i,t+1}\right\}k_{i,t+1}+$$
$$+\sum_{t=0}^{\infty}\left\{\alpha_{i,t}R_{t+1}-\beta\alpha_{i,t+1}\right\}b_{i,t+1}=$$
$$=U_c^i\left(c_{i,0},l_{i,0}\right)\left\{\left[1+\left(1-\theta_{i,0}\right)\left(r_0-\delta\right)\right]k_{i,0}+R_0b_{i,0}\right\}$$

The Ramsey allocation problem is:

$$\max \sum_{i=1}^{2} \omega_i \sum_{t=0}^{\infty} \beta^t U^i (c_{it}, l_{it})$$

s.t. $c_{1,t} + c_{2,t} + (k_{1,t+1} + k_{2,t+1}) + g_t = F(k_{1,t} + k_{2,t}, l_{1,t}, l_{2,t}) + (1-\delta)(k_{1,t} + k_{2,t})$

$$\sum_{t=0}^{\infty} \beta^t \left[c_{i,t} U_c^i (c_{i,t}, l_{i,t}) + l_{i,t} U_l^i (c_{i,t}, l_{i,t}) \right] = A_i, \ i = 1, 2,$$

where $A_i = U_c^i (c_{i,0}, l_{i,0}) \{[1 + (1-\theta_{i,0})(r_0 - \delta)] k_{i,0} + R_0 b_{i,0}\}$. Let η_i be the Lagrange multiplier on the implementability constraint for the consumer of type i, $i = 1, 2$. Let us define:

$$V(c_{1,t}, c_{2,t}, l_{1,t}, l_{2,t}, \eta_1, \eta_2) = \sum_{i=1,2} \left\{ \omega_i U^i (c_{i,t}, l_{i,t}) + \eta_i \left[c_{i,t} U_c^i (c_{i,t}, l_{i,t}) + l_{i,t} U_l^i (c_{i,t}, l_{i,t}) \right] \right\}$$

The Ramsey allocation problem can be restated as:

$$\max \sum_{t=0}^{\infty} \beta^t V(c_{1,t}, c_{2,t}, l_{1,t}, l_{2,t}, \eta_1, \eta_2) - \sum_{i=1,2} \eta_i A_i$$

s.t. $c_{1,t} + c_{2,t} + (k_{1,t+1} + k_{2,t+1}) + g_t = F(k_{1,t} + k_{2,t}, l_{1,t}, l_{2,t}) + (1-\delta)(k_{1,t} + k_{2,t})$,

The FONC's associated to the previous problem are:

$$c_{i,t} : V_{c_{i,t}} = \mu_t \tag{8}$$

$$l_{i,t} : -V_{l_{i,t}} = \mu_t F_{l_{i,t}} \tag{9}$$

$$k_{i,t+1} : \mu_t = \beta \mu_{t+1} \left[F_{k_{t+1}} + (1-\delta) \right] \tag{10}$$

where μ_t is the Lagrange multiplier associated to the resource constraint.

(d) Substituting for $\alpha_{i,t}$ from (1), and for r_{t+1} from (5) into (3) :

$$U_c^i (c_{i,t}, l_{i,t}) = \beta \left[1 + (1-\theta_{i,t+1}) \left(F_{k_{t+1}} - \delta \right) \right] U_c^i (c_{i,t+1}, l_{i,t+1}). \tag{11}$$

In steady state the previous equation simplifies to:

$$\frac{1}{\beta} = 1 + (1-\theta_i)(F_k - \delta). \tag{12}$$

Substituting for μ_t and for from (8) into (10):

$$V_{c_{i,t}} = \beta V_{c_{i,t}} \left[F_{k_{t+1}} + (1-\delta) \right].$$

In steady state:

$$\frac{1}{\beta} = F_k + 1 - \delta.$$

Hence, in a steady state, the optimal tax on capital is zero for both type of agents, independently of the welfare weights the government places on the two types of agents.

(e) The intertemporal Euler equation for consumer i, equation (11) above, can be written as:

$$\frac{U_c^i\left(c_{i,t}, l_{i,t}\right)}{U_c^i\left(c_{i,t+1}, l_{i,t+1}\right)} = \beta\left[1 + \left(1 - \theta_{t+1}\right)\left(F_{k_{t+1}} + 1 - \delta\right)\right],$$

where we used $\theta_{1,t+1} = \theta_{2,t+1} = \theta_{t+1}$. Since the RHS does not depend on i, the restriction

$$\frac{U_c^1\left(c_{1,t}, l_{1,t}\right)}{U_c^1\left(c_{1,t+1}, l_{1,t+1}\right)} = \frac{U_c^2\left(c_{2,t}, l_{2,t}\right)}{U_c^2\left(c_{2,t+1}, l_{2,t+1}\right)}.$$

holds in any competitive equilibrium.

This constraint does not pose any additional restriction in steady-state, since the government was already choosing $\theta_1 = \theta_2 = 0$. Thus the optimal choice is the same as in **(d)**.

(f) Combining equation (1), (2), (6) and (7) we can write the intratemporal Euler equations as:

$$\frac{-U_l^i\left(c_{i,t}, l_{i,t}\right)}{U_c^i\left(c_{i,t}, l_{i,t}\right) F_{l_{i,t}}} = 1 - \tau_t$$

where we have used $\tau_{1,t} = \tau_{2,t} = \tau_t$.

Since the RHS does not depend on i, the restriction

$$\frac{-U_l^1\left(c_{1,t}, l_{1,t}\right) U_c^2\left(c_{2,t}, l_{2,t}\right)}{U_c^1\left(c_{1,t}, l_{1,t}\right) U_l^2\left(c_{2,t}, l_{2,t}\right)} = \frac{F_{l_{1,t}}}{F_{l_{2,t}}}.$$

holds in any competitive equilibrium. Notice that this additional constraint does, in general, depend on the level of capital k_t if and only if the ratio $\frac{F_{l_{1,t}}}{F_{l_{2,t}}}$ depends on k_t. If this is the case, the steady-state tax rates on capital do not need to be equal to zero.

Zero capital income taxation in the steady-state is optimal if the extra constraints do not depend on the capital stock, and it is not optimal if these constraints depend on the capital stock, and if they are binding.

Question 2: Optimality of the Friedman Rule

For the answers to parts A, B, and C please refer to Alvarez, Kehoe and Neumeyer, 2002, "The Time Consistency of Fiscal and Monetary Policies", Minneapolis Fed Staff Report 305, or the published version, *Econometrica*, March 2004 - Volume 72 Issue 2 Page 541 - 567.

For part D, the first order conditions for the household problem are given by:

$$u_{ct} - \lambda_t P_t = 0, \tag{13}$$

$$\frac{1}{P_t} u_{m,t} - \lambda_t + \beta \lambda_{t+1} = 0, \tag{14}$$

$$u_{nt} + W_t (1 - \tau_t) \lambda_t = 0, \tag{15}$$

$$\lambda_t Q_{t,s} = \beta^{s-t} \lambda_s, \tag{16}$$

$$P_t \lambda_t q_{t,s} = \beta^{s-t} P_s \lambda_s, \tag{17}$$

where λ_s is the discounted multiplier on the dynamic budget constraint. The trasversality conditions are:

$$\lim_{T\to\infty} \sum_{s=T+1}^{\infty} \lambda_s \, {}_T B_s \beta^s = \lim_{T\to\infty} \sum_{s=T+1}^{\infty} P_s \lambda_s \, {}_T b_s \beta^s = \lim_{T\to\infty} \beta^T \lambda_T M_T = 0. \tag{18}$$

The first step in solving the Ramsey problem is to derive the implementability constraint. To do this, multiply the dynamic budget constraint at t with equality by λ_t, impose (13)-(17), multiply by β^t and sum from 0 to T. Finally, let $T \to \infty$ and apply the trasversality conditions. This yields:

$$\sum_{t=0}^{\infty} \left[u_{ct} \left(c_t - \, {}_{-1} b_t \right) + u_{mt} \frac{M_t}{P_t} + u_{nt} n_t \right] = \lambda_0 M_{-1} + \sum_{t=0}^{\infty} \lambda_t \beta^t \, {}_{-1} B_t. \tag{19}$$

Additional constraints on the allocation, beyond the resource constraint, are:

$$u_{mt} \geq 0, \tag{20}$$

since

$$\begin{aligned} u_{mt} &= P_t \lambda_t - \beta P_t \lambda_{t+1} \\ &= u_{ct} \left(1 - {}_t Q_{t+1} \right), \end{aligned}$$

and no arbitrage requires ${}_t Q_{t+1} \leq 1$.

The Ramsey allocation problem can be set up as usual. Let ω_t denote the multipler on the the resource constraint, ξ the multiplier on the implementability constraint, and η_t the multiplier on constraint (20). The lagrangian for the Ramsey problem is:

$$\Lambda = \sum_{t=0}^{\infty} \beta^t \left\{ u\left(c_t, \frac{M_t}{P_t}, n_t \right) + \xi \left[u_{ct} \left(c_t - \, {}_{-1} b_t \right) + u_{mt} \frac{M_t}{P_t} + u_{nt} n_t \right] - \omega_t \left(c_t + g_t - n_t \right) + \eta_t u_{mt} \right\}.$$

The condition $M_{-1} + \sum_{t=0}^{\infty} {}_{-1} B_t = 0$ has been imposed to avoid trivial solutions.

The first order conditions for the RAP are:

$$u_{ct} (1 + \xi) + \xi \left[u_{cct} \left(c_t - \, {}_{-1} b_t \right) + u_{cm,t} \frac{M_t}{P_t} + u_{nc,t} n_t \right] + \eta_t u_{mct} = \omega_t, \tag{21}$$

$$u_{mt}\left(1+\xi\right)+\xi\left[u_{cmt}\left(c_t-\ _{-1}b_t\right)+u_{mm,t}\frac{M_t}{P_t}+u_{nm,t}n_t\right]+\eta_t u_{mmt}=\omega_t, \quad (22)$$

$$u_{nt}\left(1+\xi\right)+\xi\left[u_{cnt}\left(c_t-\ _{-1}b_t\right)+u_{mn,t}\frac{M_t}{P_t}+u_{nn,t}n_t\right]+\eta_t u_{mnt}=-\omega_t, \quad (23)$$

$$\eta_t u_{m,t}=0,\ \eta_t\geq 0,\ u_{m,t}\geq 0, \quad (24)$$

Optimality of the Friedman rule corresponds to $u_{m,t}=0$. It is easy to see that the homotheticity and separability conditions on utility are not sufficient for optimality of the Friedman rule. To see this, notice that to prove the sufficiency of homotheticity and separability for the optimality of the FR, one way to procede is by contradiction. Hence, assume that $u_{m,t}>0$, so that the FR is not optimal. Then, it must be that $u_{m,t}\leq u_{c,t}$. Subtract (22) devided by $u_{c,t}$ from (21) devided by u_{mt}. This yields:

$$\begin{aligned} 0 \quad = \quad & \xi\left[\frac{u_{cct}c_t+u_{cm,t}\frac{M_t}{P_t}}{u_{c,t}}-\frac{u_{cmt}c_t+u_{mm,t}\frac{M_t}{P_t}}{u_{m,t}}-\left(u_{cct}-u_{cm,t}\right)\ _{-1}b_t+\left(\frac{u_{nc,t}}{u_{c,t}}-\frac{u_{nm,t}}{u_{c,t}}\right)n_t\right] \\ & +\eta_t\left(u_{mct}-u_{mmt}\right)-\omega_t\left(\frac{1}{u_{c,t}}-\frac{1}{u_{m,t}}\right). \end{aligned}$$

This equality simplifies to:

$$-\xi\left(u_{cct}-u_{cm,t}\right)\ _{-1}b_t=\omega_t\left(\frac{1}{u_{c,t}}-\frac{1}{u_{m,t}}\right),$$

using homotheticity and separability, and (24). The right hand side of this equation is negative, by $\omega_t\geq 0$, and $\xi\geq 0$. Hence, to obtain a contradiction and to guarantee that the Friedman rule is optimal, the additional condition

$$\left(u_{cc,t}-u_{cm,t}\right)_{-1}b_t\geq 0 \quad (25)$$

must be imposed. If real balances are a normal good, $u_{cct}-u_{cm,t}<0$, so that $_{-1}b_t\leq 0$ is the additional condition required to guarantee optimality of the Friedman rule with income taxes.

Condition (25) must be imposed because the government does not have access to a tax on interest income. Since interest income is a function of outstanding bond holdings which are given when the government chooses policy, a tax on interest income would be non-distortionary. A departure from the Friedman rule amounts to a tax on interest income, which accrues at the end of the period and can only be used for consumption in the following period.

This example illustrates the principle of *incomplete taxation* which often applies to Ramsey problems. When the government is endowed with a limited set of fiscal instruments, each fiscal instrument plays multiple roles. Therefore, a policy which is optimal for a Ramsey problem with a larger set of fiscal instruments may not be optimal if the government has access to a smaller set of instruments. A tax on consumption taxes all sources of income. If the government can impose consumption taxes (or, equivalently in this model, a total income tax), it does not have an incentive to distort additional margins to tap unexploited tax bases.

Question 3- Optimal Capital Taxation with Overlapping Generations

Please see the following paper:

Erosa, Andres, and Martin Gervais. 2002. Optimal Taxation in Lifecycle Economies. *Journal of Economic Theory* 105, 338-369.