G 6222 ADV. MACRO ANALYSIS II
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Assignment 2- Due on February 20.

Question 1: Optimal fiscal policy with heterogeneous agents.

Consider an economy populated by two types of infinitely lived consumers, indexed $i = 1, 2$. The preferences of the representative consumer of type i are given by

$$\sum_{t=0}^{\infty} \beta^t U^i (c_{it}, l_{it})$$

where c_{it} and l_{it} denote the consumption and labor supply of consumer i. Assume that the discount factors (β) are the same for both types of consumers. Each type faces the following budget constraint:

$$c_{i,t}+k_{i,t+1}+b_{i,t+1} \leq (1 - \tau_{i,t}) w_{i,t}l_{i,t}+[1 + (1 - \theta_{i,t}) (r_t - \delta)] k_{i,t}+R_t b_{i,t},\ i = 1, 2,$$

where δ is the depreciation rate on capital; $\tau_{i,t}$ and $\theta_{i,t}$ and are the tax rates on the labor and capital income of agent i at time t, respectively; $b_{i,t+1}$ is the government debt held by the representative type i agent at t and R_t is the return on it.

There is a representative firm that produces aggregate output using a constant returns to scale production function, $F (k_t, l_{1,t}, l_{2,t})$, where $k_t = k_{1,t} + k_{2,t}$ denotes the aggregate capital stock in t.

The government must finance an exogenous stream of public spending $\{g_t\}_{t=0}^{\infty}$, satisfying its budget constraint

$$\tau_{1,t}w_{1,t}l_{1,t} + \tau_{2,t}w_{2,t}l_{2,t} + \theta_{1,t} (r_t - \delta) k_{1,t} + \theta_{2,t} (r_t - \delta) k_{2,t} + b_{t+1} \geq g_t + R_t b_t$$

Let $\pi = \{\tau_{1t}, \tau_{2t}, \theta_t\}_{t=0}^{+\infty}$ be a tax policy, Ψ be the set of all such policies, and $\Pi \subset \Psi$ be the subset of tax policies for which a competitive equilibrium exists. Assume that the government has access to a commitment technology that allows it in period 0 to bind itself to choosing a one shot sequence of policies $\pi \in \Pi$ once and for all, and never deviate from it. Finally assume that the tax rates on capital income at time 0, $\theta_{1,0}$ and $\theta_{2,0}$ are given numbers.

(a) Define a competitive equilibrium for this economy. Derive the first order necessary conditions for the households' and firms' problems.

(b) The government is benevolent, and seeks to optimize a weighted sum of consumers' utilities of the form

$$\sum_{i=1}^{2} \omega_i \sum_{t=0}^{\infty} \beta^t U^i (c_{it}, l_{it}),$$

where the welfare weights $\omega_i \in [0, 1]$ satisfy $\omega_1 + \omega_2 = 1$. State the government's problem for the optimal choice of π. Define a Ramsey equilibrium.

(c) Using the first order conditions derived in part **(a)**, derive the implementability constraints for the representative consumer of type i, $i = 1, 2$. State the Ramsey allocation problem, and derive the first order conditions.

(d) Assume that there is a $T > 0$ for which $g_t = g \ \forall t \geq T$, that there exists a solution to the Ramsey problem, and that such a solution converges to a steady-state with constant allocations. What are the steady-state tax rates on capital? How do they depend on the welfare weights ω_1 and ω_2?

(e) Assume that the government is forced to set the same tax rate on capital income across types, that is $\theta_{1,t} = \theta_{2,t} \ \forall t$. Combine the *intertemporal* Euler equations for the two types of consumers and the firm's first order conditions to derive the additional restriction on the Ramsey allocation problem implied by $\theta_{1,t} = \theta_{2,t} \ \forall t$. Compute the new steady-state tax rates on capital income.

Hint: you do not need to solve a new Ramsey allocation problem to be able to answer.

(f) Assume now that the government is free to set different tax rates on the capital income for the two consumers, but it is constrained to set the same tax rate on labor income, that is $\tau_{1,t} = \tau_{2,t} \ \forall t$. Combine the *intratemporal* Euler equations for the two types of consumers and the firm's first order conditions to derive the additional restriction on the Ramsey allocation problem implied by $\tau_{1,t} = \tau_{2,t} \ \forall t$. Assume that the constraint $\tau_{1,t} = \tau_{2,t}$ $\forall t$ is binding. Do the optimal tax rates on capital income differ from the ones computed in **(e)**? Provide intuition to support your answer.

Hint: you do not need to solve a new Ramsey allocation problem to be able to answer.

Question 2: Optimality of the Friedman Rule

Consider the following economy. There are a continuum of measure 1 of identical, infinitely lived, households, with preferences defined over consumption, c, real balances, m, and labor, n:

$$\sum_{t=0} \beta^t u\left(c_t, m_t, n_t\right).$$

In each period $t \geq 0$, households chose $\{M_t, B_{t,s}, b_{t,s}, c_t, n_t\}$. Here, $m_t = M_t/P_t$, where M_t denotes end-of period currency holdings and P_t is the price of one unit of consumption in terms of currency. $B_{t,s}$, $b_{t,s}$ denote end of period holdings of nominal and real bonds, respectively, issued at t and paying of at time $s > t$. Nominal bonds pay off in units of currency while real bonds pay off in units of consumption. They face the following budget constraint:

$$P_t\left(1+\tau_t\right) c_t + M_t - M_{t-1} - W_t n_t + \sum_{s=t+1}^{\infty} Q_{t,s} B_{t,s} + P_t \sum_{s=t+1}^{\infty} q_{t,s} b_{t,s} - \sum_{s=-1}^{t-1} B_{s,t} - P_t \sum_{s=-1}^{t-1} b_{s,t} \leq 0,$$

where W_t is the nominal wage, τ_t is the consumption tax rate and $Q_{t,s}$ ($q_{t,s}$) is the price of a nominal (real) bond issued at t with maturity $t-s$. $M_{-1}, \{B_{-1},t, b_{-1,t}\}_{t\geq 0}$ are given.

Firms are competitive and technology is linear. The resource constraint is: $c_t + g_t = n_t$, where g_t is government consumption at time t.

The government budget constraint is:

$$P_t g_t - P_t \tau_t c_t = M_t^s - M_{t-1}^s + \sum_{j=t+1}^{\infty} Q_{t,j} B_{t,j}^s + P_t \sum_{j=t+1}^{\infty} q_{t,s} b_{t,s}^s - \sum_{j=-1}^{t-1} B_{j,t}^s - P_t \sum_{j=-1}^{t-1} b_{j,t}^s.$$

A. Define and characterize a competitive equilibrium for this economy for an arbitrary policy $\{\tau_t, M_t^s, [B_{t,j}^s, b_{t,j}^s]_{j\geq t+1}\}_{t\geq 0}$.

B. Assume that the government is benevolent and chooses policy $\{\tau_t, M_t^s, [B_{t,j}^s, b_{t,j}^s]_{j\geq t+1}\}_{t\geq 0}$ at time 0 to maximize the present discounted value of utility of the representative agent in a private sector equilibrium.

i) Set up the government problem and formally define a Ramsey equilibrium.

ii) Derive the Ramsey allocation problem for this economy. In this problem, the government chooses an allocation $\{c_t, n_t, m_t\}_{t\geq 0}$ subject to the constraint that it must be a competitive equilibrium for this economy. Describe the set of competitive equilibrium allocations for this economy and prove that the Ramsey allocation problem is equivalent to the Ramsey problem in i).

iii) Present a condition on outstanding government liabilities at time 0 that ensures the government will be indifferent between different values of P_0.

C. Assume that:

$$u(c,m,n) = u(w(c,m), n), \tag{1}$$

where $w(c,m)$ is homothetic i.e.

$$\frac{u_m(\alpha c, \alpha m, n)}{u_c(\alpha c, \alpha m, n)} = \frac{u_m(c,m,n)}{u_c(c,m,n)}, \tag{2}$$

where α is a positive scalar.

Derive the first order conditions for the Ramsey allocation problem. Show that the Friedman rule is optimal.

D. Assume instead of using a proportional tax on consumption, the government taxes labor income at a proportional rate. The households' budget constraint is:

$$P_t c_t + M_t - M_{t-1} - W_t n_t (1-\tau_t) + \sum_{s=t+1}^{\infty} Q_{t,s} B_{t,s} + P_t \sum_{s=t+1}^{\infty} q_{t,s} b_{t,s} - \sum_{s=-1}^{t-1} B_{s,t} - P_t \sum_{s=-1}^{t-1} b_{s,t} \leq 0,$$

and the government budget constraint is:

$$P_t g_t - W_t \tau_t n_t = M_t^s - M_{t-1}^s + \sum_{j=t+1}^{\infty} Q_{t,j} B_{t,j}^s + P_t \sum_{j=t+1}^{\infty} q_{t,s} b_{t,s}^s - \sum_{j=-1}^{t-1} B_{j,t}^s - P_t \sum_{j=-1}^{t-1} b_{j,t}^s.$$

i) Derive the implementability constraint and the conditions that ensure that an allocation $\{c_t, n_t, m_t\}_{t\geq 0}$ is a competitive equilibrium for this specification of government policy.

ii) Assume that 1 and 2 hold. Derive the first order conditions for the Ramsey allocation problem in this case. Under which conditions is the Friedman rule a Ramsey equilibrium outcome? What is the intuition for this finding?

Question 3: Optimal Capital Taxation with Overlapping Generations

Consider the following infinitely lived economy populated by overlapping generations of identical individuals. Each individual lives for $J+1$ periods, from age 0 to age J, where J is arbitrary but finite. At each period a new generation is born and indexed by its date of birth. Each new generation has the same size, so that the total population is constant and the fraction of agents of a given age is also constant for $t \geq 0$. At date 0, the generations alive at $-J$, $-J+1$, ..., 0. Agents make a labor leisure choice in each period. They are endowed with one unit of time and their labor productivity at age j is denote with z_j. Letting $c_{t,j}$ and $l_{t,j}$ denote consumption and labor at time $t+j$, of an age j individual born at t.

The government collects taxes on consumption, labor and capital income to finance an exogenous stream of spending. The government can also issue debt. We assume that all taxes can be age specific. Hence, the after tax price of consumption and the after tax returns on labor capital and bonds will also be age specific. Hence, we can write an agent's problem in the following way:

$$U^t = \max_{\{c_{t,j}, l_{t,j}, a_{t,j+1}\}} \sum_{j=j_0(t)}^{J} \beta^j u\left(c_{t,j}, l_{t,j}\right)$$

subject to

$$q_{t,j} c_{t,j} + a_{t,j+1} = w_{t,j} z_j l_{t,j} + \left(1 + r_{t,j}\right) a_{t,j}, \tag{3}$$

for $j = j_{0(t)}, ...J$, where $j_0(t)$ is the agent of an individual alive at time 0. Hence: $j_0(t) = 0$ for $t \geq 0$ and $j_{0(t)} = \{J, J-1, ..., 1\}$ for $t = \{-J, -J+1, .. -1\}$. Outstanding wealth at age j, denoted with $a_{t,j}$, is given by the sum of government bond holdings and capital holdings. The tax system equalizes after tax returns on government bonds and capital, hence, the formulation of the budget constraint in (3). The initial endowment of capital and bonds, k_0 and b_0 are taken as given. Initial wealth $a_{t,j_{0(t)}}$ is given and we assume $a_{t,j_{0(t)}} = 0$ for $t \geq 0$. The function U is strictly increasing in both arguments, strictly concave and satifies the Inada conditions. The parameter β represents the discount factor with $\beta \in (0,1)$.

At each date there is a single produced good that can be used as capital or for consumption. The production technology is given by:

$$y_t = f\left(k_t, n_t\right),$$

where y, k, and n denote aggregate output, capital and effective labor. Capital and labor are paid at their marginal product so that: $\hat{r}_t = f_k\left(k_t, n_t\right) - \delta$, $\hat{w}_t =$

$f_n(k_t, n_t)$, where $\delta \in (0,1)$ is the depreciation rate. The resource constraint is given by:

$$c_t + k_{t+1} - (1-\delta)\, k_t + g_t \le y_t, \tag{4}$$

where c_t is aggregate consumption and g_t is government consumption. Note that: $c_t = \sum_{j=0}^{J} \mu_{t-j,j} c_{t-j,j}$, $n_t = \sum_{j=0}^{J} \mu_{t-j,j} z_j l_{t-j,j}$, where $\mu_{t-j,j}$ is the fraction of agent of age j alive at time t.

The government budget constraint is given by:

$$(1+\hat{r}_t)\, b_t + g_t = b_{t+1} + \sum_{j=0}^{J} (q_{t-j,j} - 1)\, \mu_{t-j,j} c_{t-j,j} + \sum_{j=0}^{J} (\hat{r}_t - r_{t-j,j})\, \mu_{t-j,j} a_{t-j,j} + \sum_{j=0}^{J} (\hat{w}_t - w_{t-j,j})\, \mu_{t-j,j} z_j (1 - \tag{5}$$

where $q_{t-j,j} = 1 + \tau^C_{t,j}$, $w_{t,j} = \left(1 - \tau^w_{t,j}\right) \hat{w}_{t,j}$ and $r_{t,j} = \left(1 - \tau^k_{t,j}\right) \hat{r}_{t+j}$, where $\tau^C_{t,j}$, $\tau^w_{t,j}$, $\tau^k_{t,j}$ are the consumption, wage and capital taxes for individuals born at time t at age j, and b_t is government debt.

i) Define a competitive equilibrium for this economy.

The government's objective function is:

$$\max \sum_{t=-J}^{\infty} \gamma^t U^t, \tag{6}$$

where U^t is lifetime utility of agents born at date t and $\gamma \in (0,1)$ is the rate at which the lifetime utility of different generations is discounted by the government.

ii) State the direct Ramsey problem for this economy and define a Ramsey equilibrium.

iii) Show that the direct Ramsey problem is equivalent to the following Ramsey allocation problem:

$$\max \sum_{t=-J}^{\infty} \gamma^t U^t, \tag{RAP}$$

s.t.

$$\sum_{j=j_0(t)}^{J} \beta^{j-j_0(t)} \left(u_{c_{t,j}} c_{t,j} + u_{l_{t,j}} l_{t,j}\right) = u_{c_{t,j_0(t)}} \left(1 + r_{t,j_0(t)}\right) a_{t,j_0(t)}, \ \forall t \ge -J, \tag{7}$$

and (4), where $u_{c_{t,j}}$ is the marginal utility of consumption at age j for an agent born at time t, and similarly for $u_{l_{t,j}}$.

iv) Derive the first order necessary conditions for the Ramsey allocation problem (RAP). For this purpose, it may be useful to define the following pseudo-welfare function of the government:

$$\begin{aligned} W_t &= U^t + \lambda_t \sum_{j=j_0(t)}^{J} \left(u_{c_{t,j}} c_{t,j} + u_{l_{t,j}} l_{t,j}\right) \\ &\quad - \lambda_t u_{c_{t,j_0(t)}} \left(1 + r_{t,j_0(t)}\right) a_{t,j_0(t)}, \end{aligned}$$

that incorporates the implementability constraint for generation t, where λ_t is the multiplier for such a constraint.

v) Derive the necessary conditions under which the Ramsey allocation features zero taxes on labor, capital or both, by comparing the optimality conditions of the government problem to those of the household problem.

vi) Show that it is not optimal to tax capital in the long run if $z_j = z$ for all j and $\gamma = \beta$.

vii) Show that for $U(c,l) = V(G(c), l)$ where $G(\cdot)$ is homothetic, the capital tax is zero from period 1 onward. Interpret this result in relation to the uniform commodity taxation principle.

viii) Suppose all taxes were restricted to be age independent. Identify the additional constraints on the government in the corresponding Ramsey allocation problem. Are optimal capital taxes zero in this case? (You need not fully charatcerize the Ramsey allocation in this case. Instead, just point out why the arguments above may not hold.)