G 6222 Adv. Macro Analysis II
Columbia University, Spring 2006
Prof. Stefania Albanesi

Assignment 3- SUGGESTED ANSWERS.

Question 1: Idiosyncratic capital returns

i) Consider the model in LS exercise 17.4, without government insurance-that is assume $\tau = 0$ and $g = 0$.

The household's decision problem in sequence form is:

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} \leq y + x_t k_t^{\alpha}, \; c_t, k_{t+1} \geq 0, \; t \geq 0.$$

The corresponding Bellman equation for the household problem is:

$$v(k, x) = \max_{k'} \left\{ u(y + xk^{\alpha} - k') + \beta \sum_{x'} v(k', x') P(x, x') \right\}. \tag{1}$$

The solution of this problem is attained by the decision rule:

$$k' = G(k, x),$$

that induces the distribution of agents across states $\lambda(k, x)$.

The average physical capital in the economy is:

$$Y = \sum_k \sum_x (x \times k^{\alpha}) \lambda(k, x).$$

A **stationary equilibrium** is a policy function $G(k, x)$, a probability distribution $\lambda(k, x)$, and a positive number Y, such that:

i) the policy function solves the problem in (1);

ii) the stationary distribution $\lambda(k, x)$ is induced by the policy function $G(\cdot)$ and by the transition function $P(x, x')$:

$$\lambda(k', x') = \sum_{\{k: k' = G(k,x)\}} \sum_{x'} \lambda(k, x) P(x', x);$$

iii) the average value of output is induced by the policy function $G(\cdot)$:

$$Y = \sum_k \sum_x (x \times k^{\alpha}) \lambda(k, x)$$

ii) Assume that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$. Show that if if k^* is the steady state level of capital and $\beta\alpha (k^*)^{\alpha-1} x_2 = 1$, then the marginal utility of consumption converges to a strictly positive constant.[1]

To show this, consider the Euler equation for problem (1):

$$-u'(y + xk^\alpha - k') + \beta \sum_{x'} v_k(k', x') P(x, x') \leq 0, \tag{2}$$

with equality for $k' > 0$. The envelope condition for problem (1) is:

$$v'(k) = u'(y + xk^\alpha - k') x\alpha k^{\alpha-1}.$$

We can rewrite (2) as follows:

$$v'(k) = \beta \left(x\alpha k^{\alpha-1}\right) \sum_{x'} v_k(k', x') P(x, x'), \tag{3}$$

for $x = x_1, x_2$, using the fact that the non-negativity constraint on capital is never going to be binding since the marginal product of capital goes to ∞ for k tending to 0. Using $x_1 > x_2$, this implies:

$$v'(k) \geq \beta \left(x_2\alpha k^{\alpha-1}\right) \sum_{x'} v_k(k', x') P_\infty(x'), \tag{4}$$

where $P_\infty(x')$ denotes the invariant distribution of the transition matrix $P(x, x')$.

If we assume that $\beta\alpha (k^*)^{\alpha-1} x_2 = 1$ at a conjectured steady state capital level, then (4) implies that $v'(k)$ is a supermartingale. By the margingale convergence theorem (see LS page 560), $v'(k^*)$ converges to a non-negative number. Using our assumption on u and the envelope condition, it follows that:

$$\lim_{k \to k^*} v'(k) = \lim_{k \to k^*} (y + xk^\alpha - k)^{-\sigma} x\alpha k^{\alpha-1}.$$

Assume that the limiting value of $v'(k)$ is equal to 0. But this requires $k^* \to 0$, which is not consistent with $\beta\alpha (k^*)^{\alpha-1} x_2 = 1$. Then, it must be that $\lim_{k \to k^*} v'(k) > 0$.

The intuition for this result stems from the fact that by increasing capital holdings increase their exposure to idiosyncratic risk. Hence, despite their precautionary saving motive induce by the unavailibility of insurance to off-set the fluctuations in their capital returns, they do not have an incentive to accumulate infinite amounts of capital.

iii) Consider the same model and assume that agents can also trade risk-free bonds, b_t, with interest rate r_t, subject to the borrowing constraint $b_t \geq 0$. The agents' budget constraint is:

$$c_t + b_{t+1} + k_{t+1} \leq b_t (1 + r_t) + k_t^\alpha x_t + y.$$

[1] For this problem, you may assume that the economy converges to the steady state.

iii.A) Derive a Bellman equation for the agents and define a stationary equilibrium.

The household's decision problem in sequence form is:

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + b_{t+1} + k_{t+1} &\leq b_t(1+r_t) + k_t^{\alpha} x_t + y, \ t \geq 0, \\ b_t &\geq \bar{B}, \end{aligned}$$

where $\bar{B} = 0$ is the natural borrowing limit here, since agents could run their capital down to 0.

The corresponding Bellman equation for the household problem is:

$$v(b,k;x,r) = \max_{k',b' \geq 0} \left\{ u\left(y + xk^{\alpha} + (1+r)b - k' - b'\right) + \beta \sum_{x'} v(b',k';x',r') P(x,x') \right\}, \tag{5}$$

subject to $r = R(K)$ and $K' = \varkappa(K)$, where $\varkappa$ is the law of motion for capital and $R(K)$ is the aggregate pricing function.

The solution of this problem is attained by the decision rule:

$$\begin{aligned} k' &= G(b,k;x,r), \\ b' &= H(b,k;x,r). \end{aligned}$$

A **stationary equilibrium** is given by policy functions $G(b,k;x,r)$, $H(b,k;x,r)$, a pricing function $R(K)$, a probability distribution $\lambda(b,k,x,r)$, and positive numbers Y, K' such that:

i) the policy functions solve the problem in (5);

ii) the stationary distribution $\lambda(b,k;x,r)$ is induced by the policy functions $G(\cdot)$ and $H(\cdot)$ and by the transition function $P(x,x')$:

$$\lambda(b',k',x',r) = \sum_{\{b:b'=H(b,k;x,r)\}} \sum_{\{k:k'=G(b,k;x,r)\}} \sum_{x'} \lambda(b,k;x,r) P(x',x);$$

iii) the law of motion for capital is induced by the policy function $G(\cdot)$:

$$K' = \sum_b \sum_k \sum_x G(b,k;x,R(K)) \lambda(b,k;x,r) = \varkappa(K),$$

iv) $r = R(K)$ is such that the bond market clears:

$$B' = \sum_b \sum_k \sum_x H(b,k;x,r) \lambda(b,k;x,r) = 0.$$

iii.B) Characterize the steady state of the competitive equilibrium. Show that it must be that $1+r^{*}<1/\beta$ in a steady state, where r^{*} is the steady state interest rate. Also show that $E(x)(k^{*})^{\alpha-1}>1+r^{*}$, where k^{*} is the steady state capital level. Provide some intuition for this finding.

In a steady state, the interest rate, r^{*}, is constant, the aggregate capital level, K^{*}, is constant and there is an invariant probability distribution of agents in the population, $\lambda^{*}(\cdot)$, generated by $\lambda(\cdot)$ and $P_{\infty}(x)$, the invariant distribution of the transition probability $P(x,x')$.

The first order necessary conditions for the agents' problem evaluated at the steady state are:

$$
\begin{aligned}
u'(c) &= \beta\sum_{x'} v_k(b',k';x',r)\,P_{\infty}(x'), && (6)\\
u'(c) &\geq \beta\sum_{x'} v_b(b',k';x',r)\,P_{\infty}(x'), && (7)
\end{aligned}
$$

where by the envelope condition:

$$
\begin{aligned}
v_b(b,k;x,r) &= (1+r)\,u'(y+xk^{\alpha}+(1+r)\,b-k'-b'), && (8)\\
v_k(b,k;x,r) &= E\left[\alpha x k^{\alpha-1}u'(y+xk^{\alpha}+(1+r)\,b-k'-b')\right]. && (9)
\end{aligned}
$$

Consider steady states where both capital and bonds are held by the agents in equilibrium. By (7) and (8), following the usual arguments, we have that for $1+r^{*}\rightarrow 1/\beta$, $v_b(\cdot)$ is a supermartingale and thus converges almost surely to a non-negative number. Given that we are assuming that agents hold positive quantities of capital, their consumption is stochastic if the invariant distribution of x is non-degenerate. Hence, the limit of $v_b(\cdot)$ must be zero which implies that total assets converge to infinity.

Let c^{*} and k^{*} denote optimal individual steady state consumption and r^{*} denote the steady state equilibrium interest rate. Rewriting (6) and (**??**):

$$
\begin{aligned}
u'(c^{*}) &= \beta(1+r^{*})\,E_{\infty}u'(c'^{*}), && (10)\\
u'(c^{*}) &= \beta E_{\infty}\alpha x\left(k'^{*}\right)^{\alpha-1}u'(c'^{*}), && (11)
\end{aligned}
$$

where E_{∞} denotes the expectation with respect tothe distribution $P_{\infty}(x)$. Then:

$$
\begin{aligned}
1 &= \beta\frac{E_{\infty}\alpha x\left(k'^{*}\right)^{\alpha-1}u'(c'^{*})}{u'(c^{*})}\\
&= E_{\infty}\left[\alpha x\left(k'^{*}\right)^{\alpha-1}\right]\frac{\beta E_{\infty}u'(c'^{*})}{u'(c^{*})}+\beta\frac{Cov_{\infty}\left(\alpha x\left(k'^{*}\right)^{\alpha-1},u'(c'^{*})\right)}{u'(c^{*})}\\
&= E_{\infty}\left[\alpha x\left(k'^{*}\right)^{\alpha-1}\right]\frac{1}{(1+r^{*})}+\frac{Cov_{\infty}\left(\alpha x\left(k'^{*}\right)^{\alpha-1},u'(c'^{*})\right)}{u'(c^{*})}.
\end{aligned}
$$

Given incomplete markets and the fact that bond holdings are finite for $r^* < 1/\beta - 1$, $Cov_\infty\left(\alpha x\left(k^{'*}\right)^{\alpha-1}, u'\left(c'^*\right)\right) < 0$, which implies:

$$1 < E_\infty\left[\alpha x\left(k^{'*}\right)^{\alpha-1}\right]\frac{1}{(1+r^*)}. \tag{12}$$

Agents requires a risk premium for holding capital vs bonds, given that holding capital exposes them to idiosyncratic risk.

iii.C) What are the implications for optimal capital taxation for the results in part B? (You need not solve the full optimal capital taxation problem. Your findings in part B should be sufficient to make informed conjectures on this issue.)

Using (12), we can write:

$$(1+r^*) < E_\infty(x)\,\alpha\,(K^*)^{\alpha-1},$$

where K^* is the aggregate steady state capital, generated from the steady state probability distribution λ^*. The expected marginal product of aggregate capital is then greater than the gross interest rates on bonds. This leaves open the possibility that $E_\infty(x)\,\alpha\,(K^*)^{\alpha-1} > 1/\beta$. That is, unlike in Aiyagari (1994, 1995), the impact of incomplete markets on the capital stock is ambiguous in general.

In Aiyagari (1995), the Ramsey equilibrium prescribes a positive capital tax since incomplete markets and borrowing constraints imply that the steady state aggregate capital level is too high. In this version of the model with idiosyncratic capital risk, this result need not hold.

For a more formal treatment of this analysis, see Angeletos, Uninsured Idiosyncratic Risk and Aggregate Savings, 2005, manuscript MIT.

Question 2: Optimal Capital Taxation with Enforcement Constraints

This problem is based on the following paper: "Why Tax Capital?", by Chien and Lee, manuscript, UCLA, 2005.

D) To show that in any competitive equilibrium:

$$q_t = \beta \max \frac{u_{c,t+1}\left(a_0, s^{t+1}\right)}{u_{c,t}\left(a_0, s^t\right)}, \tag{13}$$

note that from the households' Euler equation:

$$q_t u_{c,t}\left(a_0, s^t\right)\xi\left(s^t\right) = \beta u_{c,t+1}\left(a_0, s^{t+1}\right)\xi\left(s^{t+1}\right).$$

For households that do not have a binding enforcement constraint $\xi\left(s^{t}\right)=\xi\left(s^{t+1}\right)$. For households that switch to a state with a binding enforcement constraint, $\xi\left(s^{t}\right)<\xi\left(s^{t+1}\right)$. It follows that the intertemporal marginal rate of substitution for households with a non-binding enforcement constraint is greater than for households with a non-binding enforcement constraint.

Then:

$$q_{t}u_{c,t}\left(a_{0},s^{t}\right)=\beta u_{c,t+1}\left(a_{0},s^{t+1}\right),$$

holds for household with a non-binding constraint and:

$$q_{t}u_{c,t}\left(a_{0},s^{t}\right)>\beta u_{c,t+1}\left(a_{0},s^{t+1}\right),$$

holds for households that switch to a binding enforcement constraint. (13) follows directly from this.

Note that this property implies that in this economy, assets are priced by the non-constrained households. Moreover, the intertemporal Euler equation of households who switch to states with a binding enforcement constraint looks like the Euler equation with a binding borrowing constraint. This points to the interpretation of the enforcement constraint as an endogenous borrowing constraint.

F.ii) Derivation of equilibrium risk sharing rule. From the households' Euler equation:

$$\beta^{t}\xi_{t}\left(s^{t}\right)u_{c}\left(s^{t}\right)=\left(\prod_{j=1}^{t}\frac{1}{R_{j-1}}\right). \tag{14}$$

Using $u\left(c\right)=c^{1-\gamma}/\left(1-\gamma\right):$

$$c\left(s^{t}\right)=\xi_{t}\left(s^{t}\right)^{1/\gamma}\left[\left(\prod_{j=1}^{t}\frac{1}{R_{j-1}}\right)/\beta^{t}\right]^{-1/\gamma}. \tag{15}$$

Integrating across agents:

$$\begin{aligned}C_{t} &= \sum_{s^{t}}\int c\left(s^{t}\right)d\Phi_{0}=\left[\left(\prod_{j=1}^{t}\frac{1}{R_{j-1}}\right)/\beta^{t}\right]^{-1/\gamma}\sum_{s^{t}}\int\xi_{t}\left(s^{t}\right)^{1/\gamma}d\Phi_{0}\\ &= \left[\left(\prod_{j=1}^{t}\frac{1}{R_{j-1}}\right)/\beta^{t}\right]^{-1/\gamma}h_{t}.\end{aligned}$$

Substituting in (14):

$$c\left(s^{t}\right)=\xi_{t}\left(s^{t}\right)^{1/\gamma}\frac{C_{t}}{h_{t}}.$$

Further reading on economies with enforcement constraints:

References

[1] Dirk Krueger and Fabrizio Perri. 2005. Public versus Private Risk Sharing. Manuscript, NYU.

[2] Dirk Krueger and Fabrizio Perri. 2006. Does Income Inequality lead to Consumption inequality? Evidence and theory. The Review of Economic Studies.

[3] Patrick Kehoe and Fabrizio Perri. 2004. Competitive Equilibria with Limited Enforcement. Journal of Economic Theory.

[4] Patrick Kehoe and Fabrizio Perri. 2000. International Business Cycles with Endogenous Incomplete Markets. Econometrica.

[5] Fernando Alvarez and Urban Jerman. 2000. EFFICIENCY, EQUILIBRIUM, AND ASSET PRICING WITH RISK OF DEFAULT. Econometrica.

[6] Fernando Alvarez and Urban Jermann. 2001. Quantitative Asset Pricing Implications of Endogenous Solvency Constraints. Review of Financial Studies.

[7] Kehoe, Timothy J & Levine, David K, 1993. "Debt-Constrained Asset Markets," Review of Economic Studies, Blackwell Publishing, vol. 60(4), pages 865-88.