G 6222 Adv. Macro Analysis II
Columbia University, Spring 2006
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**Assignment 5- Solution**

**Question 1- Non-Separable Utility**

i) Each agent is characterized by their history of $\theta$ realizations, $\{\theta_1, \theta_2\}$. Formally then, an allocation for an agent with history $\{\theta_1, \theta_2\}$, is a quadruple $\{c_1(\theta_1, \theta_2), S(\theta_1, \theta_2), c_2(\theta_1, \theta_2), y_2(\theta_1, \theta_2),\}$, where $c_1$ and $S$ are measurable with respect to $\theta_1$ and $c_2$ and $y_2$ are measurable with respect to the history $(\theta_1, \theta_2)$. For simplicity, we will drop the dependence on $\theta_2$ of $c_1$ and $S$.The feasibility constraints for this economy can the be written as:

$$E_1 c_1(\theta_1) + S(\theta_1) \leq E_1 y_1(\theta_1),$$

$$E_2 c_2(\theta_1, \theta_2) \leq S(\theta_1) + E_2 y_2(\theta_1, \theta_2),$$

where $s_2$ denotes storage and the operator $E_1$ denotes expectations with respect to $\theta_1$, while the operator $E_2$ denotes expectations with respect to $(\theta_1, \theta_2)$.

To write the incentive compatibility constraints, denote with $W(\theta_1', \theta_2'; \theta_1, \theta_2)$ the lifetime utility of an agent who reports history $\theta_1', \theta_2'$ and is characterized by history $\theta_1, \theta_2$. Then, incentive compatibility requires:

$$W(\theta_1, \theta_2; \theta_1, \theta_2) \geq W(\theta_1', \theta_2'; \theta_1, \theta_2) \text{ for all } \{\theta_1', \theta_2'\}, \{\theta_1, \theta_2\}.$$

ii) If $p(\theta_2|\theta_1) = 1$ if $\theta_2 = \theta_1$, so that agents learn their skill in period 1, we can characterize the allocation as being a function of $\theta_1$ only.

ii.a) A *constrained-efficient allocation* is a feasible and incentive compatible allocation $(c_1^*, S^*, c_2^*, y_2^*)$ that attains the maximum of $E_1[u(c_1(\theta_1)) + u(c_2(\theta_1)) - v(y_2(\theta_1)/\theta_1)]$.

Hence, such an allocation solves the problem:

$$\max_{S, c_2 y_2} E_1[u(y_1 - S(\theta_1)) + u(c_2(\theta_1)) - v(y_2(\theta_1)/\theta_1)]$$

subject to

$$E_1 c_2(\theta_1) \leq E_1 S(\theta_1) + E_1 y_2(\theta_1),$$

$$\begin{aligned} & u(y_1 - S(\theta_H)) + u(c_2(\theta_H)) - v(y_2(\theta_H)/\theta_H) \\ \geq\ & u(y_1 - S(\theta_L)) + u(c_2(\theta_L)) - v(y_2(\theta_L)/\theta_H), \end{aligned}$$

since the incentive compatibility constraint will only be binding for agents with high ability.

The first order necessary conditions for this problem are:

$$-0.5u'(c_1(\theta_L)) + 0.5\lambda - \mu(-u'(c_1(\theta_L))) = 0, \quad (1)$$
$$-0.5u'(c_1(\theta_H)) + 0.5\lambda - \mu(u'(c_1(\theta_H))) = 0, \quad (2)$$

$$
\begin{aligned}
0.5u'\left(c_2\left(\theta_L\right)\right)-\lambda 0.5-\mu\left(u'\left(c_2\left(\theta_L\right)\right)\right) &= 0, \qquad (3)\\
0.5u'\left(c_2\left(\theta_H\right)\right)-\lambda 0.5-\mu\left(-u'\left(c_2\left(\theta_H\right)\right)\right) &= 0, \qquad (4)\\
-0.5v'\left(y_2\left(\theta_L\right)/\theta_L\right)/\theta_L+\lambda 0.5-\mu\left(-v'\left(y_2\left(\theta_L\right)/\theta_H\right)/\theta_H\right) &= 0, \qquad (5)\\
-0.5v'\left(y_2\left(\theta_H\right)/\theta_H\right)/\theta_H+\lambda 0.5-\mu\left(v'\left(y_2\left(\theta_H\right)/\theta_H\right)/\theta_H\right) &= 0, \qquad (6)
\end{aligned}
$$

where $\lambda$ is the multiplier on the time 2 feasibility constraint, while $\mu$ is the multiplier on the incentive c. constraint.

To characterize the intratemporal wedge at time 2, combine 3 and 5, and 2 and 4, to obtain:

$$
\begin{aligned}
u'\left(c_2\left(\theta_L\right)\right)-v'\left(y_2\left(\theta_L\right)/\theta_L\right)/\theta_L &= \frac{\mu}{0.5}\left(u'\left(c_2\left(\theta_L\right)\right)-v'\left(y_2\left(\theta_L\right)/\theta_H\right)/\theta_H\right)\\
u'\left(c_2\left(\theta_H\right)\right)-v'\left(y_2\left(\theta_H\right)/\theta_H\right)/\theta_H &= 0.
\end{aligned}
$$

Hence, there are no intratemporal distortions for $\theta_H$, while the labor supply of $\theta_L$ types are distorted. Note that combining 1 and 3, and 2 and 4:

$$
\begin{aligned}
u'\left(c_1\left(\theta_L\right)\right) &= u'\left(c_2\left(\theta_L\right)\right),\\
u'\left(c_1\left(\theta_H\right)\right) &= u'\left(c_2\left(\theta_H\right)\right).
\end{aligned}
$$

This implies that there is no intertemporal wedge for this economy.

In addition, combining 1 and 2:

$$
u'\left(c_1\left(\theta_H\right)\right)-u'\left(c_1\left(\theta_L\right)\right)+\frac{\mu}{0.5}\left[u'\left(c_1\left(\theta_H\right)\right)+u'\left(c_1\left(\theta_L\right)\right)\right]=0,
$$

which implies $c_t\left(\theta_H\right)>c_t\left(\theta_L\right)$ for $t=1,2$. Hence, this economy features limited insurance.

ii.b) A *tax-equilibrium* is a feasible allocation $\left(\hat{c}_1,\hat{S},\hat{c}_2,\hat{y}_2\right)$ that is individually optimal for agents facing the following budget constraints:

$$
c_1\left(\theta_1\right)+S\left(\theta_1\right)=y_1,\ \forall\theta_1,
$$

$$
c_2\left(\theta_1\right)=y_2\left(\theta_1\right)+S\left(\theta_1\right)-T\left(S\left(\theta_1\right),y_2\left(\theta_1\right)\right),\ \forall\theta_1.
$$

Denote the constrained-efficient values of all variables with $*$. Given that there no intertemporal wedge in this economy, there is no need to tax storage. However, the tax on effective labor must still depend on $S$. The agents' Euler equation is:

$$
u'\left(\hat{c}_1\left(\theta_j\right)\right)=u'\left(\hat{c}_2\left(\theta_j\right)\right),
$$

for $j=L,H$ in any tax equilibrium, which guarantees $\hat{c}_1\left(\theta_j\right)=\hat{c}_2\left(\theta_j\right)$. Hence, we can define $T^*\left(S,y_2\right)$ as follows;

$$
c_2^*\left(\theta_j\right)=y_2^*\left(\theta_j\right)+S^*\left(\theta_j\right)-T^*\left(S^*\left(\theta_j\right),y_2^*\left(\theta_j\right)\right),\ \forall\theta_j \qquad \text{(T1)}
$$

and

$$
2u\left(y_2+S-T^*\left(S,y_2\right)\right)-v\left(y_2/\theta_j\right)\leq\min_j\left\{2u\left(c_1^*\left(\theta_j\right)\right)-v\left(y_2^*\left(\theta_j\right)/\theta_j\right)\right\}, \qquad \text{(T2)}
$$

for $j = L, H$ for $\{S, y_2\} \neq \{S^*(\theta_j), y_2^*(\theta_j)\}_j$.

The tax system defined by $T^*(\cdot)$ implements the constrained-efficient allocation. By T2, $\{S, y_2\} \neq \{S^*(\theta_j), y_2^*(\theta_j)\}_j$ will not be optimal. Hence, the incentive compatibility of the constrained-efficient allocation implies that under $T^*$, $\left(\hat{c}_1, \hat{S}, \hat{c}_2, \hat{y}_2\right)(\theta_j) = (c_1, S, c_2, y_2)^*(\theta_j)$ for $j = L, H$.

iii) Assume $U$ is non-separable and that $U_{cl} < 0$ everywhere, so that consumption and leisure are substitutes in the utility function.

iii.a) The constrained-efficient allocation solves the following problem:

$$\max_{S, c_2 y_2} E_1 \left[ u(y_1 - S(\theta_1)) + U(c_2(\theta_1), -(y_2(\theta_1)/\theta_1)) \right]$$

subject to

$$E_1 c_2(\theta_1) \leq E_1 S(\theta_1) + E_1 y_2(\theta_1),$$

$$\begin{aligned} & u(y_1 - S(\theta_H)) + U(c_2(\theta_H), -(y_2(\theta_H)/\theta_H)) \\ \geq \; & u(y_1 - S(\theta_L)) + U(c_2(\theta_L), -(y_2(\theta_L)/\theta_H)), \end{aligned}$$

assuming the incentive compatibility constraint is binding for the $\theta_H$ type.

The first order necessary conditions for this problem are:

$$-0.5u'(c_1(\theta_L)) + 0.5\lambda - \mu(-u'(c_1(\theta_L))) = 0, \tag{1}$$

$$-0.5u'(c_1(\theta_H)) + 0.5\lambda - \mu(u'(c_1(\theta_H))) = 0, \tag{2}$$

$$0.5U_c(c_2(\theta_L), -(y_2(\theta_L)/\theta_L)) - \lambda 0.5 - \mu(U_c(c_2(\theta_L), -(y_2(\theta_L)/\theta_H))) = 0, \tag{3}$$

$$0.5U_c(c_2(\theta_H), -y_2(\theta_H)/\theta_H) - \lambda 0.5 - \mu(-U_c(c_2(\theta_H), -y_2(\theta_H)/\theta_H)) = 0, \tag{4}$$

$$-0.5U_l(c_2(\theta_L), -y_2(\theta_L)/\theta_L)/\theta_L + \lambda 0.5 - \mu(-U_l(c_2(\theta_L), -y_2(\theta_L)/\theta_H)/\theta_H) = 0, \tag{5}$$

$$-0.5U_l(c_2(\theta_H), -y_2(\theta_H)/\theta_H)/\theta_H + \lambda 0.5 - \mu(U_l(c_2(\theta_H), -y_2(\theta_H)/\theta_H)/\theta_H) = 0, \tag{6}$$

plus the two constraints holding with equality.

Equations 1 and 2 can be combined to show that $c_1^*(\theta_L) < c_1^*(\theta_H)$. Similarly, 4 and 6 can be combined to obtain:

$$U_l(c_2^*(\theta_H), -y_2^*(\theta_H)/\theta_H)/\theta_H = U_c(c_2^*(\theta_H), -y_2^*(\theta_H)/\theta_H),$$

so that there is no distortion on the labor wedge for $\theta_H$. Instead, there is a distortion on the labor wedge for $\theta_L$, which can be seen by combining 3 and 5:

$$\begin{aligned} & U_c(c_2(\theta_L), -(y_2(\theta_L)/\theta_L)) - U_l(c_2(\theta_L), -y_2(\theta_L)/\theta_L)/\theta_L \\ = \; & \frac{\mu}{0.5}\left[U_c(c_2(\theta_L), -(y_2(\theta_L)/\theta_L)) - U_l(c_2(\theta_L), -y_2(\theta_L)/\theta_H)/\theta_H\right]. \end{aligned}$$

The intertemporal conditions are as follows:

$$U_c(c_2(\theta_L), -(y_2(\theta_L)/\theta_L)) - u'(c_1(\theta_L)) = \frac{\mu}{0.5}\left[U_c(c_2(\theta_L), -(y_2(\theta_L)/\theta_H)) - u'(c_1(\theta_L))\right] \neq 0. \tag{IWL}$$

$$U_c\left(c_2\left(\theta_H\right), -y_2\left(\theta_H\right)/\theta_H\right) - u'\left(c_1\left(\theta_H\right)\right) = \frac{\mu}{0.5}\left(u'\left(c_1\left(\theta_H\right)\right) - U_c\left(c_2\left(\theta_H\right), -y_2\left(\theta_H\right)/\theta_H\right)\right).$$

The latter implies $U_c\left(c_2\left(\theta_H\right), -y_2\left(\theta_H\right)/\theta_H\right) - u'\left(c_1\left(\theta_H\right)\right) = 0$.

Hence, the constrained-efficient allocation features an intertemporal distortion for $\theta_L$ when the incentive compatibility constraint is binding. Note that IWL can be rewritten as:

$$\begin{aligned} & \left[U_c\left(c_2\left(\theta_L\right), -\left(y_2\left(\theta_L\right)/\theta_L\right)\right) - u'\left(c_1\left(\theta_L\right)\right)\right]\left(1 - \frac{\mu}{0.5}\right) \\ = & -\frac{\mu}{0.5}\left[U_c\left(c_2\left(\theta_L\right), -\left(y_2\left(\theta_L\right)/\theta_L\right)\right) - U_c\left(c_2\left(\theta_L\right), -\left(y_2\left(\theta_L\right)/\theta_H\right)\right)\right]. \end{aligned}$$

Since $U_c\left(c_2\left(\theta_L\right), -\left(y_2\left(\theta_L\right)/\theta_H\right)\right) < U_c\left(c_2\left(\theta_L\right), -\left(y_2\left(\theta_L\right)/\theta_L\right)\right)$, by $U_{cl} < 0$, so that $U_c\left(c_2\left(\theta_L\right), -\left(y_2\left(\theta_L\right)/\theta_L\right)\right) - u'\left(c_1\left(\theta_L\right)\right) > 0$. This conclusion stems from $1 - \mu/0.5 > 0$, as can be verified by combining 1 and 2 to yield $1 - \mu/0.5 = 2u'\left(c_1\left(\theta_H\right)\right)/\left(u'\left(c_1\left(\theta_L\right)\right) + u'\left(c_1\left(\theta_H\right)\right)\right)$.

The consumption and leisure are substitutes there is a positive intertemporal wedge on type $\theta_L$ if the incentive compatibility constraint on the $\theta_H$ type is binding. If consumption and labor are substitutes, this means that consumption and leisure are complements. Hence, if an agent has higher consumption at time 2, she prefers to have more leisure. Then, higher consumption has an adverse effect on incentives. This explains the presence of the wedge at the c-e allocation.

ii.b) To construct a tax system that implements the constrained-efficient allocation as a tax-equilibrium, we can proceed as usual by setting the value of the tax at time 2 so that the constrained-efficient allocation is affordable. Let $T^*\left(S, y_2\right)$ satisfy:

$$c_2^*\left(\theta_j\right) = y_2^*\left(\theta_j\right) + S^*\left(\theta_j\right) - T^*\left(S^*\left(\theta_j\right), y_2^*\left(\theta_j\right)\right), \ \forall \theta_j \tag{T1}$$

and

$$u\left(y_1 - S\right) + U\left(y_2 + S - T^*\left(S, y_2\right), -y_2/\theta_j\right) \leq \min_j \left\{u\left(c_1^*\left(\theta_j\right)\right) + U\left(c_2\left(\theta_j^*\right), -y_2^*\left(\theta_j\right)/\theta_j\right)\right\}, \tag{T2}$$

for $\left(S, y_2\right) \neq \left\{S^*\left(\theta_j\right), y_2^*\left(\theta_j\right)\right\}_{j=L,H}$.

Then, if an agent of type $\theta_j$ chooses $S^*\left(\theta_j\right)$ by incentive compatibility and by T2, she will find it optimal to choose $c_2^*\left(\theta_j\right), y_2^*\left(\theta_j\right)$, for $j = L, H$. However, this does not guarantee that agents will choose the constrained-efficient allocation. In particular, $\theta_H$ agents could contemplate joint deviations in which they choose $S \neq S^*\left(\theta_H\right)$ and $y_2 = y_2^*\left(\theta_L\right)$. Assuming that $T^*$ is differentiable in the first argument, the Euler equation for the $\theta_L$ type is:

$$\frac{u'\left(c_1^*\left(\theta_L\right)\right)}{U_c\left(c_2^*\left(\theta_L\right), -y_2^*\left(\theta_L\right)/\theta_L\right)} = 1 - T_S^*\left(S^*\left(\theta_L\right), y_2^*\left(\theta_L\right)\right).$$

An agent of type $\theta_H$ who deviates by choosing $y_2^*\left(\theta_L\right)$ will find it optimal to store $S^*\left(\theta_L\right)$ if the following condition holds:

$$\frac{u'\left(c_1^*\left(\theta_L\right)\right)}{U_c\left(c_2^*\left(\theta_L\right), -y_2^*\left(\theta_L\right)/\theta_H\right)} = 1 - T_S^*\left(S^*\left(\theta_L\right), y_2^*\left(\theta_L\right)\right).$$

Clearly, these conditions cannot hold at the same time, since $U_{cl} < 0$. Hence, the tax system must be non-differentiable in $S$ at $S^*(\theta_L)$. In particular, since $U_c(c_2^*(\theta_L), -y_2^*(\theta_L)/\theta_H) > U_c(c_2^*(\theta_L), -y_2^*(\theta_L)/\theta_L)$, $\lim_{S\uparrow S^*(\theta_L)} T_S^*(S, y_2^*(\theta_L)) < T_S^*(S^*(\theta_L), y_2^*(\theta_L))$. There are a large number of tax systems that ensure this condition holds. Hence, the optimal tax system is not unique.

This question is base on the following article:
Kocherlakota, Narayana. 2004. Wedges and Taxes. *AER, Papers and Proceedings.*

**Question 2- Disability Insurance**
i) The problem that define the optimal allocation is given by:

$$\max_{c_1, c_2^w, c_2^n} u(c_1) + \pi u(c_2^w) + (1-\pi) u(c_2^n) - \pi\gamma, \tag{1}$$

subject to

$$Rc_1 + \pi c_2^w + (1-\pi) c_2^n \le R + \pi, \tag{2}$$

$$u(c_2^w) - \gamma \ge u(c_2^n). \tag{3}$$

The first order necessary conditions for this problem are:

$$u'(c_1) - R\lambda = 0, \tag{1}$$

$$\begin{aligned} \pi u'(c_2^w) - \lambda\pi - \mu(-u'(c_2^w)) &= 0, && (2) \\ (1-\pi) u'(c_2^n) - \lambda(1-\pi) - \mu(u'(c_2^n)) &= 0, && (3) \end{aligned}$$

plus the two constraints with equality.
Summing 2 and 3 and combining with 1 obtains:

$$REu'(c_2) - u'(c_1) = \mu R(u'(c_2^n) - u'(c_2^w)). \tag{IW}$$

Combining 3 and 2:

$$u'(c_2^n) - u'(c_2^w) = \mu\left(\frac{u'(c_2^n)}{1-\pi} + \frac{u'(c_2^w)}{\pi}\right) > 0. \tag{PI}$$

Hence, the optimal allocation is characterized by partial insurance and a positive intertemporal wedge.

ii) An insurance-equilibrium is an allocation $(c_1, S, c_2, \text{work status})$ and an insurance scheme $T_s$, such that the allocation is individually optimal given $R$ and the insurance scheme, feasible, and such that $\pi T_w + (1-\pi) T_n = 0$.
The agents solve the following problem in an insurance equilibrium:

$$\max_{c_1, S, c_2^w, c_2^n, w} u(c_1) + \pi u(c_2^w) + (1-\pi) u(c_2^n) - \pi\gamma w, \tag{4}$$

subject to

$$
\begin{aligned}
c_1 + S &\leq 1 &(5)\\
c_2^w &\leq RS + 1 + T_w &(6)\\
c_2^n &\leq RS + T_S, &(7)
\end{aligned}
$$

where $w$ is indicator variable which is equal to 1 if an agent works, and zero otherwise.

ii.a) If individual storage is not observable, the insurance transfers cannot be conditioned on storage. Hence, in any insurance equilibrium with private storage, the individual optimality condition for $S$ is:

$$-u'(c_1) + REu'(c_2) = 0.$$

This is not compatible with the optimal allocation being chosen, since $REu'(c_2^*) > u'(c_1^*)$.

ii.b) If individual storage is observable, the insurance scheme can be conditioned on $S$. Let $S^*$ denote the optimal level of storage in the planner's problem. To ensure incentive compatibility, the insurance scheme must satisfy:

$$
\begin{aligned}
c_2^{w*} &= RS^* + 1 + T_w(S^*), &(8)\\
c_2^{n*} &= RS^* + T_n(S^*). &(9)
\end{aligned}
$$

This will ensure that if agents choose $S^*$, then they will find it optimal to work if they are able, by the incentive compatibility of the optimal allocation. We can then set the insurance transfer as a function of $S$ so that agents do not find it optimal to set $S \neq S^*$ and then not work, even if they are able, in the second period. Note that if the insurance scheme does not depend on $S$, then, an able agent choosing not to work would choose a level of $S > S^*$ since $1 + T_w(S^*) > T_n(S^*)$, by PI. In addition, budget feasibility of the insurance scheme implies that $T_w(S^*) < 0$. Let $T_n(S) = 0$ for any $S > S^*$. Then, if the agent chooses $S > S^*$, her budget line shifts down and the allocation $\{c_1, c_2^{n*}$, not work$\}$ is no longer affordable. Hence, the agent will choose $S^*$.

This assignment is base on the following paper:

Golosov, Mikhail, and Aleh Tsyvinski. 2006. Designing Optimal Disability Insurance: A Case for Asset Testing. Forthcoming, *Journal of Political Economy.*