G 6222 Adv. Macro Analysis II
Columbia University, Spring 2006
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Assignment 6- due on May 1

The economy is populated by a continuum of unit measure of agents. Each agent lives for two periods and her lifetime utility is given by:

$$U = u(c_0) - v(e) + u(c_1),$$

where, c_t denotes consumption in period $t = 0, 1$ and e denotes effort exerted at time 0, with $e \in \{0, 1\}$. Assume $u' > 0$, $u'' < 0$, $v' > 0$, $v'' > 0$, and $\lim_{c \to 0} u'(c) = \infty$.

Agents are endowed with K_0 units of the consumption good at time 0. Assume for simplicity that the distribution of initial capital is degenerate at K_0. Agents can operate an investment technology. If K_1 is the amount invested at time 0, the return on investment at time 1 is $R(K_1)$, where:

$$R(K_1) = K_1(1 + x),$$

and x is the random net return on capital. The stochastic process for x is:

$$x = \begin{cases} \overline{x} \text{ with probability } \pi(e), \\ \underline{x} \text{ with probability } 1 - \pi(e), \end{cases} \quad (1)$$

with $\bar{x} > \underline{x}$ and $\pi(1) > \pi(0)$. The first assumption implies that $E_1(x) > E_0(x)$, where E_e denotes the expectation operator for probability distribution $\pi(e)$. Hence, the expected returns on capital is increasing in effort.

Assume effort is *private information*. The realized value of x, as well as its distribution, and K_0 and K_1 are *public information*.

The constrained-efficient allocation for this economy is the solution to the following problem:

$$\{e^*, K_1^*, c_0^*, c_1^*(\underline{x}), c_1^*(\bar{x})\} = \arg \max_{e \in \{0,1\}, K_1 \in [0, K_0],\ c_0, c_1(x) \geq 0} u(c_0) - v(e) + E_e u(c_1(x))$$

subject to

$$c_0 + K_1 \leq K_0 + G_0,\ E_e c_1(x) \leq K_1 E_e(1 + x) + G_1, \quad (2)$$

$$E_1 u(c_1(x)) - E_0 u(c_1(x)) \geq v(1) - v(0), \quad (3)$$

where E_e denotes the expectation operator with respect to the probability distribution $\pi(e)$. The constraints in (2) stem from resource feasibility, while (3) is the incentive compatibility constraint, arising from the unobservability of effort. Here, G_t for $t = 0, 1$ is exogenous planner consumption.

i) Characterize the constrained-efficient allocation for this economy. Write the first order necessary conditions for the planner's problem, and characterize any wedges. Is the intertemporal wedge positive or negative? Provide intuition for your answer.

ii) Consider fiscal implementation of the constrained-efficient allocation for this economy. Assume that agents can trade capital and risk-free bonds, which provide a net return of r. Bonds can be issued by the government or by private agents. Bond holdings and capital holdings as well as capital returns are observable in the market economy. The government collects taxes at time 0 and time 1, conditional on observables, and finances the sequence $\{G_t\}_{t=0,1}$. Hence, the tax system can be represented as a couple of functions $T_0(K_0, B_0)$ and $T_1(K_1, B_1, x)$, where B_t denotes bond holdings at the beginning of time $t = 0, 1$.

ii.a) Define a competitive equilibrium for this economy

ii.b) Assume that $T_0(B_0) = G_0$ and $T_1(B_1, x) = G_1$, that is taxes are lump sum and the government budget constraint is balanced in each period. Analyze the agents problem in the competitive equilibrium. What is the relation between the optimal level of effort and the level of B_1 and K_1? How do bonds and capital influence the agents' incentives? Explain.

Hint: Analyze the second order conditions for the agents' problem.

ii.c) Consider candidate tax systems in the class $T_1(K_1, B_1, x) = \rho(x) + \tau_K(x) K_1 + \tau_B(x) B_1$ and assume $T_0(B_0, K_0) = 0$. Construct a tax system that implements the constrained-efficient allocation. What is the equilibrium level of agents' bond holdings B_1 in the resulting competitive equilibrium? What are the properties of the optimal marginal tax rates on the bonds and on capital? What is the equilibrium interest rate r on the bond? Is the assumption $T_0(B_0, K_0) = 0$ restrictive? Explain.

iii) Assume the following functional forms: $u(c) = \left[c^{1-\sigma} - 1\right] / (1-\sigma)$, $v(e) = \gamma e$, $\gamma > 0$, $\pi(e) = a + be$, with $0 < a + b < 1$ and $a \geq 0$. In addition, assume $K_0 = 1$, $G_0 = 0.10$, $\underline{x} = 0.25$, $\bar{x} = 0.75$, $a = 0$, $b = 0.5$, $\gamma = 0.10$.

iii.a) Compute the constrained-efficient allocation as a function of σ, for $0.2 < \sigma < 8$, assuming $G_1 = 0.10$. Compute the optimal tax system and equilibrium bond holdings and the rate of interest as a function of σ.

iii.b) Fix $\sigma = 1.5$. Compute the constrained-efficient allocation for $G_1 \in [0, 0.25]$. Compute the corresponding optimal tax system, the equilibrium quantity of bonds held and the equilibrium interest rate. How do marginal taxes depend on G_1?

iii.c) Fix $\sigma = 1.5$ and set $G_1 = 0.10$. Compute the constrained-efficient allocation for $G_0 \in [0, 0.25]$. Compute the corresponding optimal tax system, the equilibrium quantity of bonds held and the equilibrium interest rate. How do marginal taxes depend on G_0?