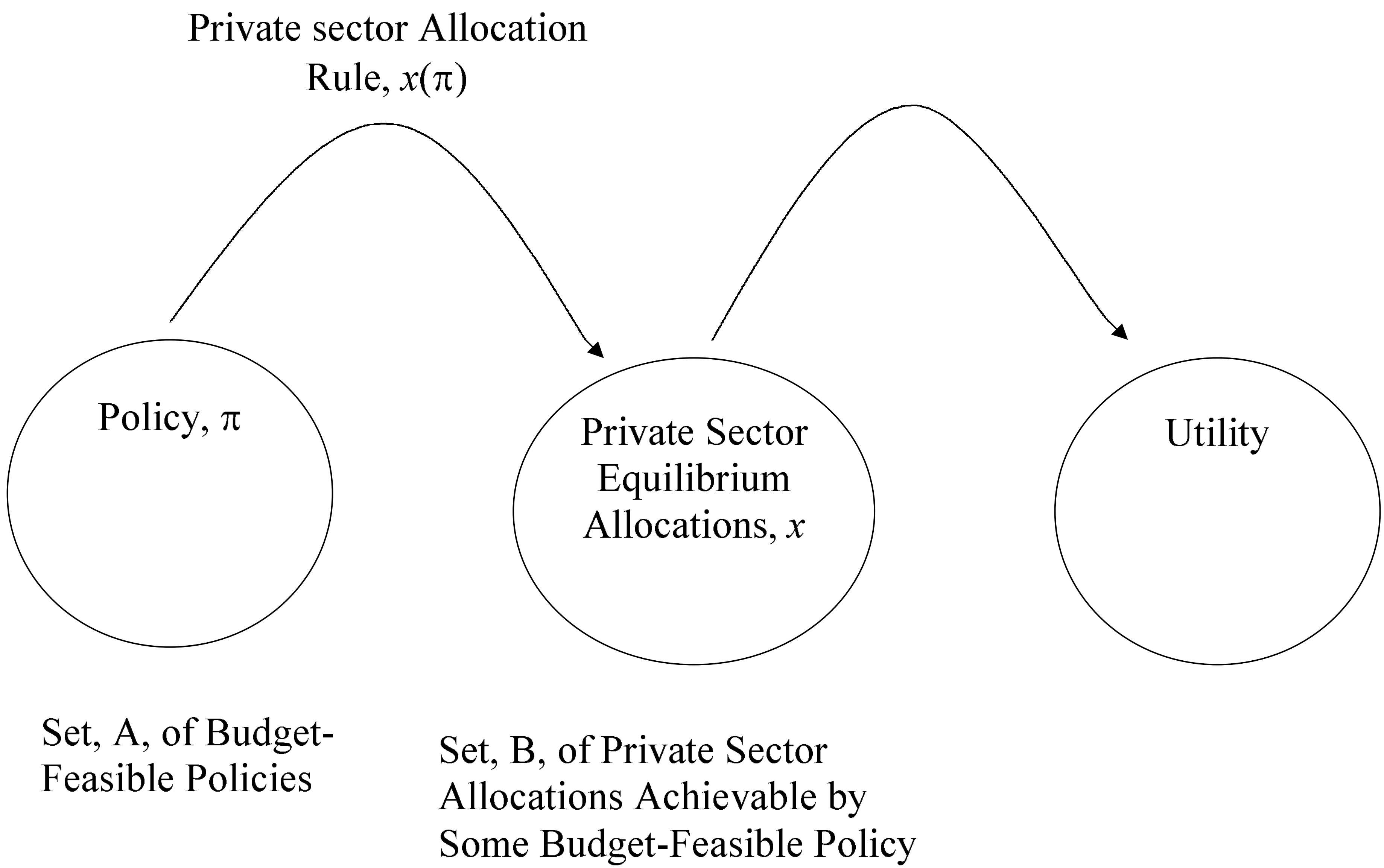
Some Basic Ideas from Ramsey Theory

- **Policy**, π, Belonging to the Set of 'Budget Feasible' Policies, A.
- **Private Sector Equilibrium Allocations**, Equilibrium Allocations, x, Associated with a Given π; $x \in B$.
- **Private Sector Allocation Rule**, mapping from π to x (i.e., $\pi : A \to B$).
- **Ramsey Problem**: Maximize, w.r.t. π, $U(x(\pi))$.
- **Ramsey Equilibrium**: $\pi^* \in A$ and x^*, such that π^* solves Ramsey Problem and $x^* = x(\pi^*)$. 'Best Private Sector Equilibrium'.

- **Ramsey Allocation Problem**: Solve, $\tilde{x} = \arg max\, U(x)$ for $x \in B$
- Alternative Strategy for Solving the Ramsey Problem:
  (a) Solve Ramsey Allocation Problem, to Find $\tilde{x}$.
  (b) Execute the Inverse Mapping, $\tilde{\pi} = x^{-1}(\tilde{x})$.
  (c) $\tilde{\pi}$ and $\tilde{x}$ Represent a Ramsey Equilibrium.
- **Implementability Constraint**: Equations that Summarize Restrictions on Achievable Allocations, B, Due to Distortionary Tax System.


Private sector Allocation Rule, $x(\pi)$
Policy, π
Private Sector Equilibrium Allocations, x
Utility
Set, A, of Budget-Feasible Policies
Set, B, of Private Sector Allocations Achievable by Some Budget-Feasible Policy

Example

- Households:

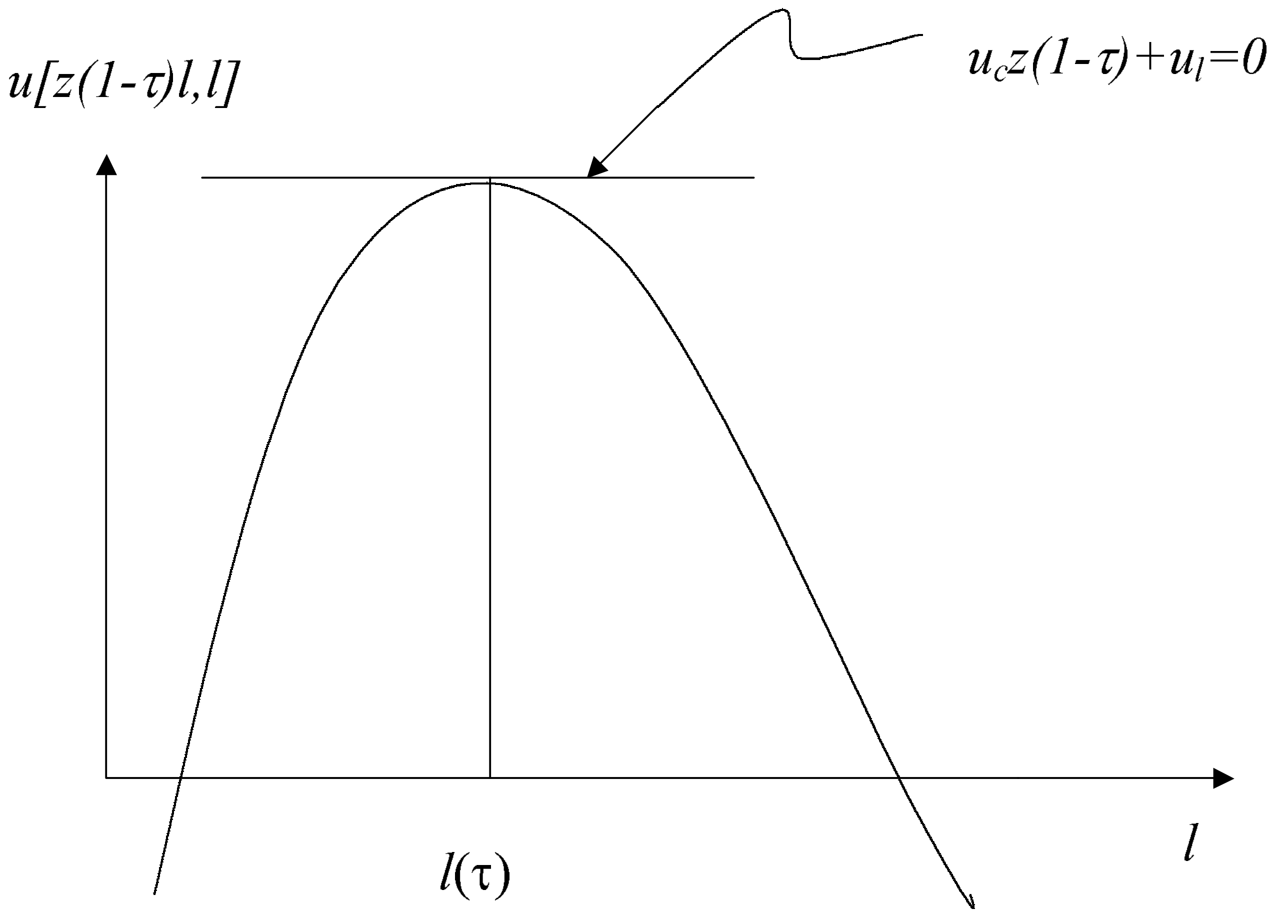
$$\max_{c,l} u(c,l)$$
$$c \leq z(1-\tau)l,$$
$$z \sim \text{wage rate}$$
$$\tau \sim \text{labor tax rate}$$

- Household Problem Implies Private Sector Allocation Rules:

$$l(\tau),\ c(\tau)$$

- Ramsey Problem:

$$\max_{\tau} u(c(\tau), l(\tau))$$
$$\text{subject to } g \leq z l(\tau)\tau$$



Private Sector Allocation Rules:

$l(\tau), \; c(\tau) = z(1-\tau)l$

- Ramsey Equilibrium: τ^*, c^*, l^* such that

  (a) $c^* = c(\tau^*)$, $l^* = l(\tau^*)$
  'Private Sector Allocations are a Private Sector Equilibrium'

  (b) τ^* Solves Ramsey Problem
  'Best Private Sector Equilibrium'

Analysis of Ramsey Equilibrium

- Simple Utility Specification:

$$u(c, l) = c - \frac{1}{2}l^2$$

- Two Ways to Compute the Ramsey Equilibrium
  (a) Direct Way: Solve Ramsey Problem (In Practice, Hard)
  (b) Indirect Way: Solve Ramsey Allocation Problem (Can Be Easy)

Direct Approach

- Private Sector Allocation Rules:

$$u_c z(1-\tau)+u_l=0,\ c\leq(1-\tau)zl$$

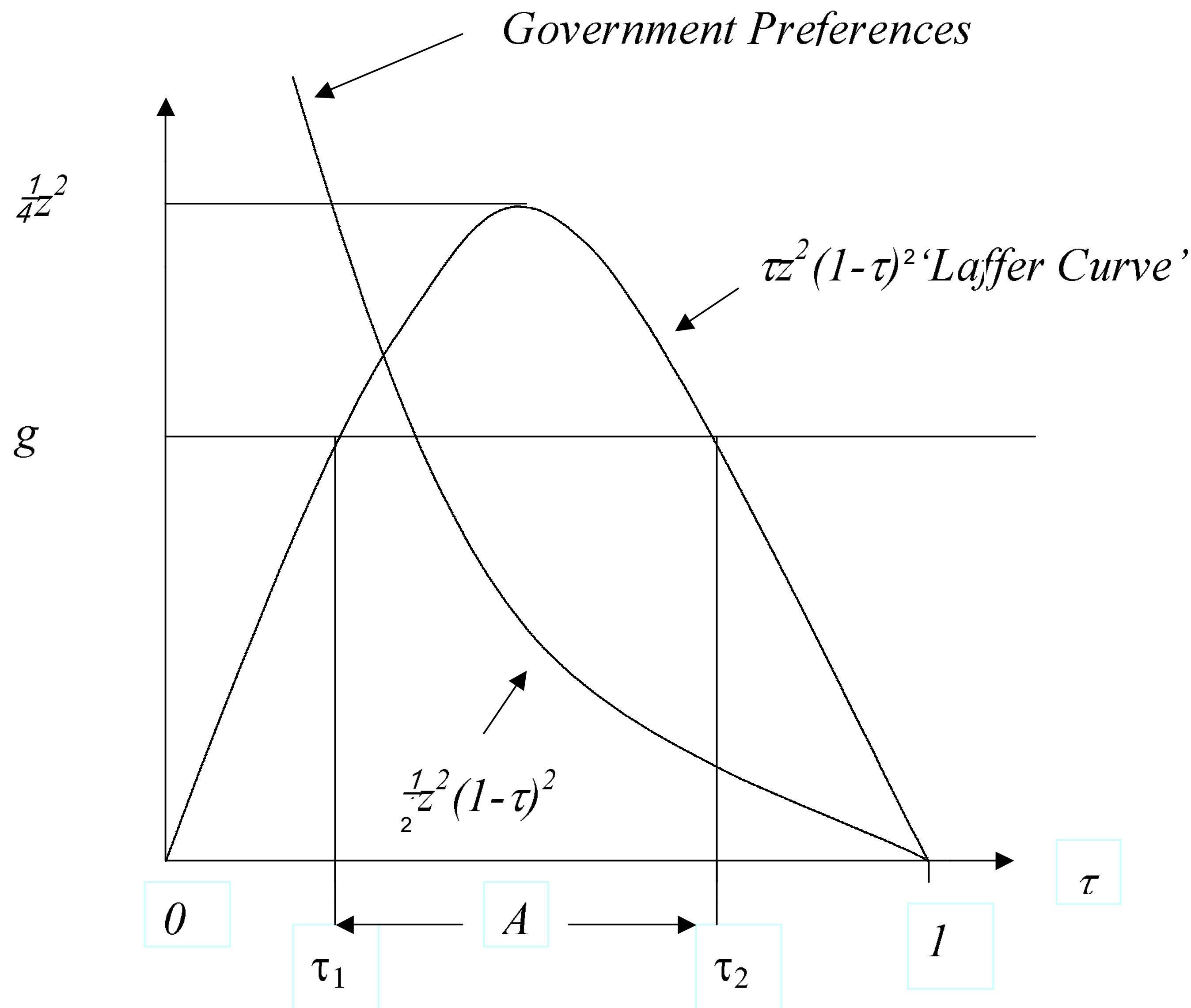
$$\Longrightarrow z(1-\tau)=l(\tau)$$

$$\Longrightarrow c(\tau)=z(1-\tau)l(\tau)=z^2(1-\tau)^2$$

- Ramsey Problem:

$$\max_{\tau}\frac{1}{2}z^2(1-\tau)^2$$

$$\text{subject to : } g\leq\tau zl(\tau)=\tau z^2(1-\tau)^2.$$



$$\tau^* = \tau_1 = \tfrac{1}{2} - \tfrac{1}{2}[\,1 - 4\,g/z^2\,]^{\frac{1}{2}} \qquad \tau_2 = \tfrac{1}{2} + \tfrac{1}{2}[\,1 - 4\,g/z^2\,]^{\frac{1}{2}}$$

$$l(\tau^*) = \tfrac{1}{2}\{\,z + [\,z - 4\,g\,]^{\frac{1}{2}}\}$$

Indirect Approach

- Approach: Solve Ramsey Allocation Problem, Then 'Inverse Map' Back into Policies
- Problem: Need a Simpler Characterization of B

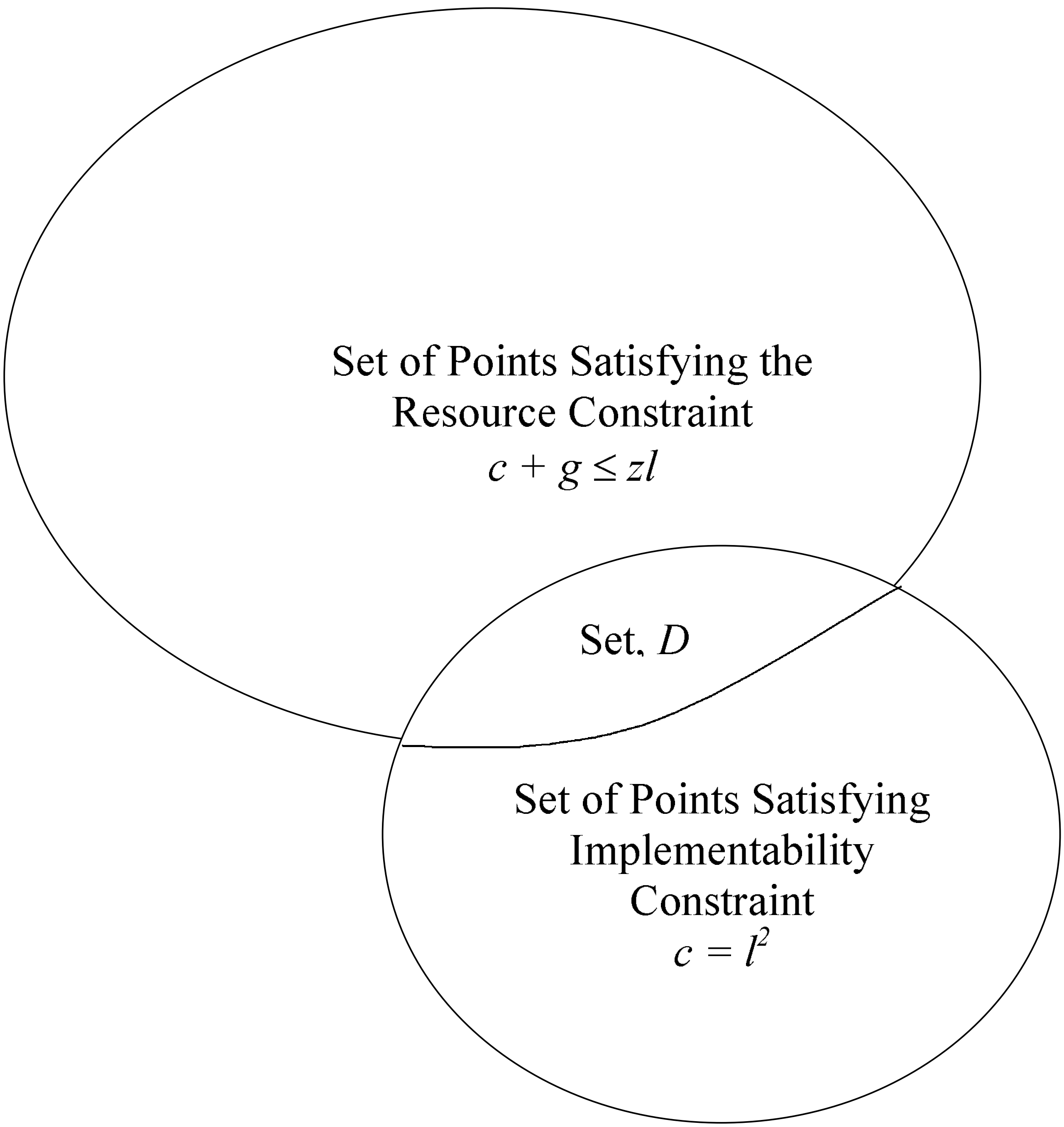
$$\begin{aligned} B &= \{(c,l) : \exists \tau \text{ s.t. } u_c(1-\tau)z + u_l l = 0, \\ c &= (1-\tau)zl,\ g \leq \tau zl\} \end{aligned}$$

- Consider the Following Set D :

$$D = \left\{ (c,l) : \quad \underbrace{c + g \leq zl}_{\text{resource constraint}} , \quad \underbrace{u_c c + u_l l = 0}_{\text{implementability constraint}} \right\}$$

- Key Result: $D = B$

Constraint Set, *D*, On Ramsey Allocation Problem



Proof of Key Result, $D = B$

Show: $(c, l) \in D \Rightarrow (c, l) \in B$

- Suppose $(c, l) \in D$, i.e., $u_c c + u_l l = 0$, $c + g \leq zl$
- Need to show: $\exists \tau$ s.t. (i) $u_c(1 - \tau)z + u_l = 0$, (ii) $c = (1 - \tau)zl$, (iii) $g \leq \tau zl$
- Set τ so that

$$1 - \tau = \frac{-u_l}{u_c z}, \text{ so (i) holds.}$$

- Multiply Both Sides by lz and rewrite:

$$(1 - \tau)\, lz = \frac{-u_l l}{u_c} = c, \text{ so (ii) holds.}$$

- (iii) follows (ii) and $c + g \leq zl$.

Show: $(c, l) \in B \Rightarrow (c, l) \in D$

- Suppose $(c, l) \in B$, i.e., $\exists \tau$ s.t. $u_c(1-\tau)z + u_l = 0$, $c = (1-\tau)zl$, $g \leq \tau zl$.
- Need to show: $(c, l) \in D$, i.e., (i) $u_c c + u_l l = 0$, (ii) $c + g \leq zl$
- Multiply by l:

$$u_c(1-\tau)zl + u_l l = 0, \text{ so (i) holds}$$

- Combine HH and Gov't Budget Constraints:

$$c + g \leq zl, \text{ so (ii) holds}$$

- Conclude:

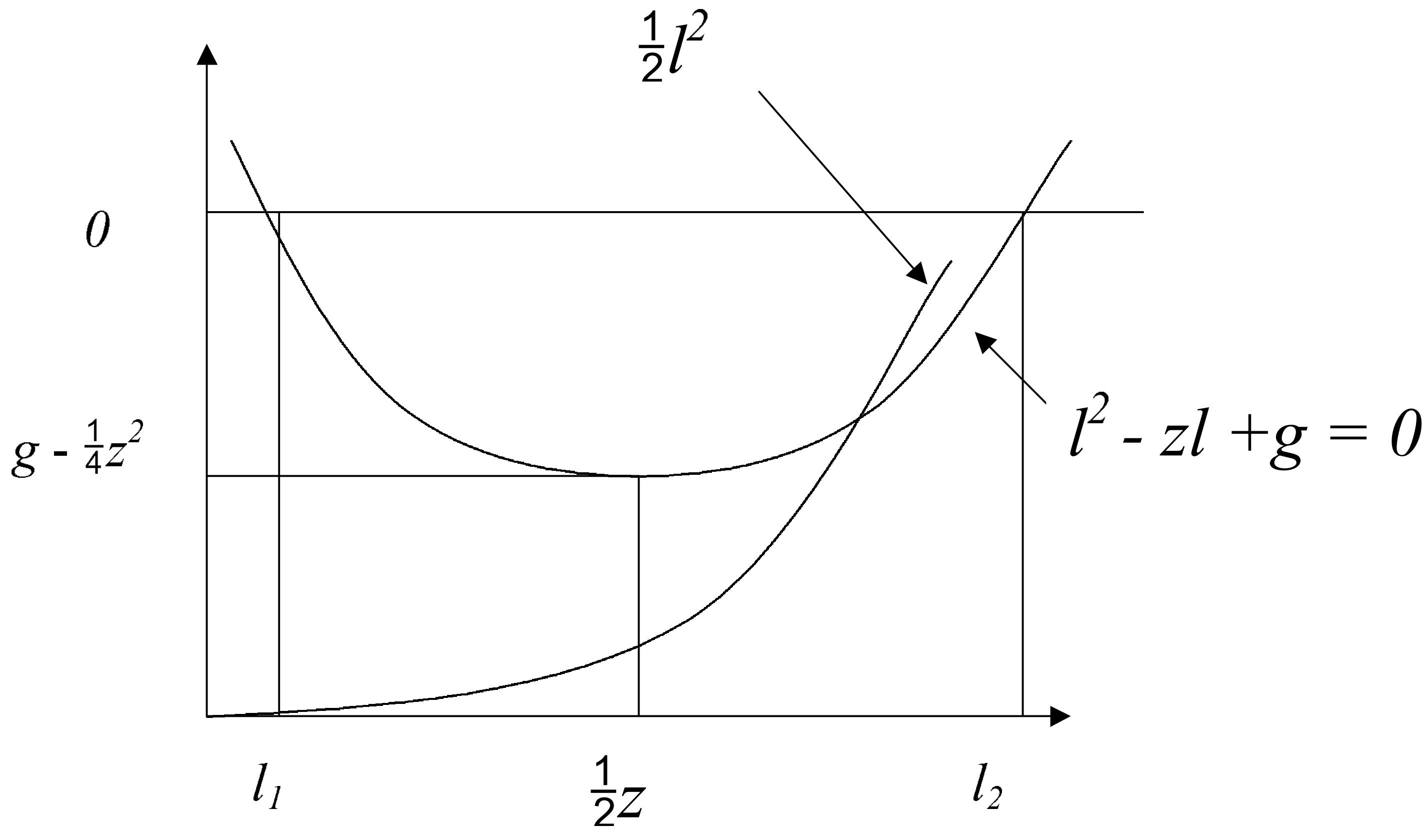
$$B = D$$

- Express Ramsey Allocation Problem:

$$\max_{c,l} u(c,l)$$
$$\text{s.t. } u_c c + u_l l \;=\; 0,\; c + g \leq zl$$

or

$$\max_{l} l^2$$
$$\text{s.t. } l^2 + g \;\leq\; zl$$



Ramsey Allocation Problem:

Max $\frac{1}{2}l^2$
Subject to $l^2 + g \leq zl$

Solution:

$l_2 = \frac{1}{2}\{ z + [z^2 - 4g]^{\frac{1}{2}} \}$

Same Result as Before!