

Ramsey allocation problem in a real economy without capital

Consider an economy populated by a continuum of identical agents, a continuum of identical firms and a government. The government must finance a stream of exogenous consumption expenditures $\{g(s^t)\}_{t \geq 0}$, where s^t denotes a history of realizations of the exogenous state s_t . It can do so by taking household's labor earnings at the proportional rate $\tau(s^t)$ and by issuing debt of varying maturities. We assume that government consumption is the only source of uncertainty, so that the state coincides with the realization of government consumption. We denote with $\mu(s^t)$ the probability of a history of realizations up to time t of the government consumption process. We characterize the Ramsey equilibrium for this economy and discuss time inconsistency of Ramsey policies.

1 Private Sector Equilibrium

We first describe the firm and household problem, formally describe the fiscal system and define a private sector equilibrium for this economy.

Firms are competitive. They hire labor to produce consumption goods, subject to the technology:

$$y(s^t) \leq n(s^t).$$

Denoting the wage $w(s^t)$, firm optimization implies that $w(s^t) = 1$. In what follows, we substitute this condition in the household and government problems.

The representative agent solves the following problem in each period:

$$\max_{\{c(s^t), n(s^t), b(s^{t+1})\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) u(c(s^t), n(s^t))$$

subject to

$$c(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) b(s^{t+1}) \leq n(s^t) (1 - \tau(s^t)) + b(s^t), \quad (1)$$

with initial debt denoted with $b(s^t)$ given. Here, $b(s^{t+1})$ denotes purchases at node s^t of state contingent government bonds, with price $q(s^{t+1}|s^t)$. Without loss of generality, we assume only one-period maturity bonds are available. This notation capture the fact that bond returns are state contingent and in each period, only one-period maturity debt is available. In addition, we assume that

u is increasing in the first and decreasing in the second argument and strictly concave.

The following conditions are jointly necessary and sufficient for household optimization:

$$\frac{-u_n(s^t)}{u_c(s^t)} = (1 - \tau(s^t)), \quad (2)$$

$$q(s^{t+1}|s^t) \mu(s^t) u_c(s^t) = \beta \mu(s^{t+1}) u_c(s^{t+1}), \quad (3)$$

$$\lim_{T \rightarrow \infty} \beta^T \mu(s^T) u_c(s^T) b(s^T) = 0. \quad (4)$$

We define government policy as sequence $\{\tau(s^t)\}$ that satisfies the government budget constraint in each period, given $\{g(s^t)\}$:

$$g(s^t) + b(s^t) - \tau(s^t) n(s^t) - \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) b(s^{t+1}) \leq 0. \quad (5)$$

A private sector equilibrium for this economy is an allocation $\{c(s^t), n(s^t)\}_{s^t, t}$, a price system $\{q(s^{t+1}|s^t), W(s^t)\}_{s^t, t}$, a sequence of bond issues $\{b(s^t)\}_{s^t, t}$, a government policy $\{\tau(s^t)\}_{s^t, t}$ and a sequence of government consumption levels $\{g(s^t)\}_{s^t, t}$ such that: i) the allocation and the bond sequence solve that household problem given the price system and government policy; ii) $W(s^t) = 1$ at all s^t, t ; iii) government policy and the bond sequence satisfy the government budget constraint at all s^t, t ; iv) markets clear.

2 Ramsey Allocation Problem

To describe the Ramsey allocation problem, we first derive the implementability constraint. To do so, multiply (1), which holds with equality at an optimum, by $u_c(s^t) \beta^t \mu(s^t)$ and use (2) to substitute for $\tau(s^t)$ and (3) to substitute for $q(s^{t+1}|s^t)$. This yields:

$$\beta^t \mu(s^t) [u_c(s^t) c(s^t) + u_n(s^t) n(s^t)] = u_c(s^t) \beta^t \mu(s^t) b(s^t) - \beta^{t+1} \sum_{s^{t+1}|s^t} \mu(s^{t+1}) u_c(s^{t+1}) b(s^{t+1}).$$

Now sum over t, s^t for $t = 0, \dots, T-1$:

$$\sum_{t=0}^{T-1} \sum_{s^t} \beta^t \mu(s^t) [u_c(s^t) c(s^t) + u_n(s^t) n(s^t)] = u_c(s^0) b_0 - \beta^T \sum_{s^T} \mu(s^T) u_c(s^T) b(s^T).$$

We adopt the convention that the government chooses policy at time 0 for all future dates after the state at time 0 has been realized, hence we can write $b(s^0) = b_0$. Using (4), the implementability constraint is:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \mu(s^t) [u_c(s^t) c(s^t) + u_n(s^t) n(s^t)] = u_c(s^0) b_0. \quad (6)$$

The Ramsey allocation problem is to maximize:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) u(c(s^t), n(s^t)),$$

by choice of $\{c(s^t), n(s^t)\}$, subject to (6) and the resource constraint:

$$c(s^t) + g(s^t) \leq n(s^t),$$

for given b_0 .

To derive properties of the solution to the Ramsey allocation problem, it is convenient to use Lagrangian methods. Denote with λ and $\xi(s^t)$ the multipliers on the implementability constraint and on the resource constraint at node s^t , respectively. Let:

$$W(c(s^t), n(s^t)) = u(c(s^t), n(s^t)) + \lambda [u_c(s^t) c(s^t) + u_n(s^t) n(s^t)].$$

The lagrangian for the Ramsey allocation problem is given by:

$$\Lambda = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) \{W(c(s^t), n(s^t)) - \xi(s^t) [c(s^t) + g(s^t) - n(s^t)]\} - \lambda u_c(s^0) b_0.$$

The first order necessary conditions for this problem are:

$$W_c(s^t) - \xi(s^t) = 0, \text{ for } t > 0, \quad (7)$$

$$W_c(s^0) - \xi(s^0) - \lambda u_{cc}(s^0) b_0 = 0, \quad (8)$$

$$W_n(s^t) + \xi(s^t) = 0, \text{ for } t > 0, \quad (9)$$

$$W_n(s^0) + \xi(s^0) - \lambda u_{cn}(s^0) b_0 = 0$$

where

$$W_c = (1 + \lambda) u_c + \lambda (u_{cc}c + u_{nc}n),$$

$$W_n = (1 + \lambda) u_n + \lambda (u_{cn}c + u_{nn}n).$$

It is impossible to know in general whether these conditions are also sufficient or whether a solution to the system of equations exists. If the optimum of the Ramsey allocation problem exists and is interior, it will satisfy this set of restrictions.

Note that for $t > 0$:

$$W_c(s^t) = -W_n(s^t), \quad (10)$$

characterize the Ramsey allocation, substituting $c(s^t)$ with the resource constraint. It follows that at each node, the optimal allocation $\{c(s^t), n(s^t)\}$, only depends on $g(s^t)$ at that node, that is it depends on s_t only. Hence, the Ramsey allocation is history independent for $t > 0$. At time 0 it depends on s^0 and on b_0 .

3 Tax Distortions and Time Consistency

We now derive the sign of the multiplier on the implementability constraint. To do so, multiply (7) and (9) by $c(s^t)$ and $n(s^t)$, respectively and sum. This obtains:

$$W_c(s^t) c(s^t) + W_n(s^t) n(s^t) - \xi(s^t) [c(s^t) - n(s^t)] = 0. \quad (11)$$

Writing this expression explicitly and dropping the dependence on s^t , this can be written as:

$$(1 + \lambda)(u_c c + u_n n) + \lambda(u_{cc} c^2 + 2u_{nc} n c + u_{nn} n^2) - \xi [c - n].$$

Note that:

$$u_{cc} c^2 + 2u_{nc} n c + u_{nn} n^2 = [c, n] \begin{bmatrix} u_{cc} & u_{nc} \\ u_{nc} & u_{nn} \end{bmatrix} \begin{bmatrix} c \\ n \end{bmatrix} < 0,$$

by strict concavity of u .

We then multiply (8) by $(c(s^0) - b_{-1})$ and sum to (9) at time 0 :

$$\begin{aligned} 0 &= [(1 + \lambda) u_c(s^0) + \lambda(u_{cc}(s^0) c(s^0) + u_{nc}(s^0) n(s^0))] (c(s^0) - b_0) - \lambda u_{cc}(s^0) b_0 (c(s^0) - b_0) \\ &\quad + [(1 + \lambda) u_n(s^0) + \lambda(u_{cn}(s^0) c(s^0) + u_{nn}(s^0) n(s^0))] n(s^0) - \lambda u_{cn}(s^0) b_0 n(s^0) \\ &\quad - \xi(s^0) [c(s^0) - b_0 - n(s^0)]. \end{aligned}$$

Collecting terms in this expression:

$$\begin{aligned} 0 &= (1 + \lambda) (u_c(s^0) c(s^0) + u_n(s^0) n(s^0)) - (1 + \lambda) u_c(s^0) b_{-1} \quad (12) \\ &\quad + \lambda \left\{ [c(s^0) - b_0, n(s^0)] \begin{bmatrix} u_{cc} & u_{nc} \\ u_{nc} & u_{nn} \end{bmatrix} \begin{bmatrix} c(s^0) - b_0 \\ n(s^0) \end{bmatrix} \right\} \\ &\quad - \xi(s^0) [c(s^0) - b_0 - n(s^0)]. \end{aligned}$$

The expression in curly brackets in the second line of this expression is negative by the strict concavity of utility.

Multiplying (11) by $\beta^t \mu(s^t)$ and summing for all $t > 0$ and s^t and adding the result to (12) obtains:

$$\begin{aligned} 0 &= (1 + \lambda) \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) [(u_c(s^t) c(s^t) + u_n(s^t) n(s^t))] - (1 + \lambda) u_c(s^0) b_0 \\ &\quad + \lambda Q_0 - \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) \xi(s^t) [c(s^t) - n(s^t)] + \xi(s^0) b_0, \end{aligned}$$

where $Q_0 < 0$ is equal to the sum of all the quadratic terms. Using the implementability constraint, the first line of this expression is equal to 0. Substituting the resource constraint, finally yields:

$$\lambda Q_0 + \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) \xi(s^t) g(s^t) + \xi(s^0) b_0 = 0. \quad (13)$$

Hence, $\lambda > 0$ if and only if $\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) \xi(s^t) g(s^t) + \xi(s^0) b_0 > 0$. That is, if the present discounted value of government spending obligations, which include government consumption at all dates and debt repayments at time 0, is positive, the multiplier on the implementability constraint is positive.

Note that $\lambda = 0$ implies:

$$\frac{-u_n}{u_c} = 1,$$

at all dates and states by (10). Allocations in a competitive equilibrium satisfy this property if the government has access to lump sum taxes, which are explicitly ruled out from the set of fiscal instruments in the Ramsey equilibrium.

Is it possible for λ to be equal to 0 in a Ramsey equilibrium for $g(s^t) > 0$ for some s^t ? Yes, if:

$$-b_0 = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) \frac{\xi(s^t)}{\xi(s^0)} g(s^t).$$

This condition says that the government has sufficient initial claims on the private sector to finance the stream of of government consumption with the resulting interest payments.

The fact that the government only has access to distortionary fiscal instruments implies that Ramsey policies are not time consistent. If the government were allowed to reoptimize over policy at some future date, it would not choose the allocations that was optimal from the standpoint of time 0.

To see this, note that in a Ramsey equilibrium at any node s^t , after the state s_t has been realized, the following equation holds:

$$\lambda Q(s^t) + \sum_{j=0}^{\infty} \beta^j \sum_{s^{t+j}|s^t} \mu(s^{t+j}|s^t) \xi(s^{t+j}) g(s^{t+j}) + \xi(s^t) b(s^t) = 0. \quad (14)$$

This is the analogue of (13) from the standpoint of node s^t , where $Q(s^t) < 0$ denotes the summation of all quadratic terms and λ is the multiplier on the implementability constraint for the time 0 Ramsey allocation problem. If the government is allowed to reset policy at s^t , the best that the government can do is to default on outstanding debt obligations in the amount necessary for (14) to be satisfied at $\lambda = 0$. Let $\delta(s^t) \in [0, 1]$ be the default rate. It satisfies:

$$-(1 - \delta(s^t)) b(s^t) = \sum_{j=0}^{\infty} \beta^j \sum_{s^{t+j}|s^t} \mu(s^{t+j}|s^t) \frac{\xi(s^{t+j})}{\xi(s^t)} g(s^{t+j}).$$

(14) can be satisfied with $\lambda = 0$ only if $b(s^t)$ is large enough. Then, the incentive to default is increasing in the size of the outstanding debt.

In an economy without capital, time inconsistency of Ramsey policies manifests itself in the incentive to default on outstanding debt obligations. In an economy with capital, the government can also increase the tax rate on capital income. Consider such an economy, where the resource constraint is given by:

$$c(s^t) + g(s^t) + k(s^t) \leq F(k(s^{t-1}), n(s^t), s_t) + (1 - \delta)k(s^{t-1}),$$

and government policy is given by sequences $\{\tau(s^t), \theta(s^t)\}$, where τ is the proportional tax rate on labor income and θ the one on capital income. The after tax return on capital is: $R_k(s^t) = 1 + (1 - \theta(s^t))(F_k(s^t) - \delta)$.

The implementability constraint for this economy is:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) [(u_c(s^t) c(s^t) + u_n(s^t) n(s^t))] = u_c(s^0) [b_0 + R_k(s^0) k_{-1}].$$

Hence, at any s^t along the Ramsey equilibrium path:

$$\lambda Q(s^t) + \sum_{j=0}^{\infty} \beta^j \sum_{s^{t+j}|s^t} \mu(s^{t+j}|s^t) \xi(s^{t+j}) g(s^{t+j}) + \xi(s^t) [b(s^t) + R_k(s^0) k(s^{t-1})] = 0.$$

The government has an incentive to set $[b(s^t) + R_k(s^t) k(s^{t-1})] = 0$ by raising $\theta(s^t)$ relative to what was optimal from the standpoint of time 0 or reducing $b(s^t)$ as discussed above. While taxation of capital in place at s^t is distortionary from the standpoint of any node $s^{\hat{t}}$ with $0 \leq \hat{t} < t$, $\theta(s^t)$ amounts to a lump sum tax at node s^t , since households cannot change $k(s^{t-1})$.

The time inconsistency of Ramsey policies requires the assumption that the government or society has access to a *commitment device*, an institution/mechanism that ensures the government will be bound to the policy chosen at time 0 at all future nodes. Absent such a mechanism, the Ramsey equilibrium is not a valid concept given our assumption of rational expectations on part of households and firms. The above reasoning illustrates how the government might deviate from a Ramsey plan but it does not characterize the optimal policy in the case in which there is no commitment, since it does not incorporate the private sector's optimal response to the government incentive to deviate from previously (at time 0) announced policies. For the case in which there is no commitment, we need to develop a different equilibrium concept.